# ON THE MODELS OF THE HOMOTHETIC SELF-SIMILAR KÁRMÁN FLOWS

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Kármán's flows on a single and between two infinite coaxial rotating disks are famous examples of self-similar solutions of the Navier–Stokes equations. Considering the shrouded two disk systems, numerical investigations of the steady axisymmetric solutions have shown the existence of a so called pseudo-similar region where the velocity profiles are homothetic. In the present work, the corresponding three parameter model is derived.

*AMS 2010 Subject Classification:* 76U05, 34L30.

*Key words:* rotating flows, Kármán flows, self-similar solutions.

## 1. **INTRODUCTION**

Rotating flows have long drawn much attention because of both technical and theoretical interests. At the beginning of the $XX^{th}$ century, the early studies have been dedicated to the deflection of the surface-layer oceanic circulation driven by the wind. Then, less than two decades later, Kármán initiated the study of self-similar flows on single and between two infinite coaxial rotating disks [14, 2]. These flows constitute an outstanding example of self-similar solutions of the Navier–Stokes equations, originally used for the study of inflexional instability in three-dimensional boundary layers [13].

Stimulated by the observations of Nansen in 1898, the theory carried out by Ekman in 1905 opened the way to the understanding of the fundamental mechanism giving rise to the steady upper-layer of the wind driven current [8] (for example, see [12]). In the framework of the Rossby similitude, the convective term is neglected with respect to the viscous term. In the rotating frame of reference attached to the Earth, the equations of motion are thus reduced to the linear balance between the Coriolis force, the pressure gradient and the eddy friction stress in the $\beta$-plan. Therefore, the velocity field is the result of the superposition of geostrophic and non geostrophic parts associated with the pressure gradient and the friction respectively.

In 1921, Kármán studied the incompressible viscous flow engendered by rotating plan which can be viewed as an infinite disk depicted in Fig. 1-a.

Assuming that, far from the disk, the flow is wholly normal to the disk, the Navier–Stokes equations degenerate to a one-parameter system of Ordinary Differential Equations (ODE) named after him. The parameter $\gamma$ is the ratio of the angular velocities

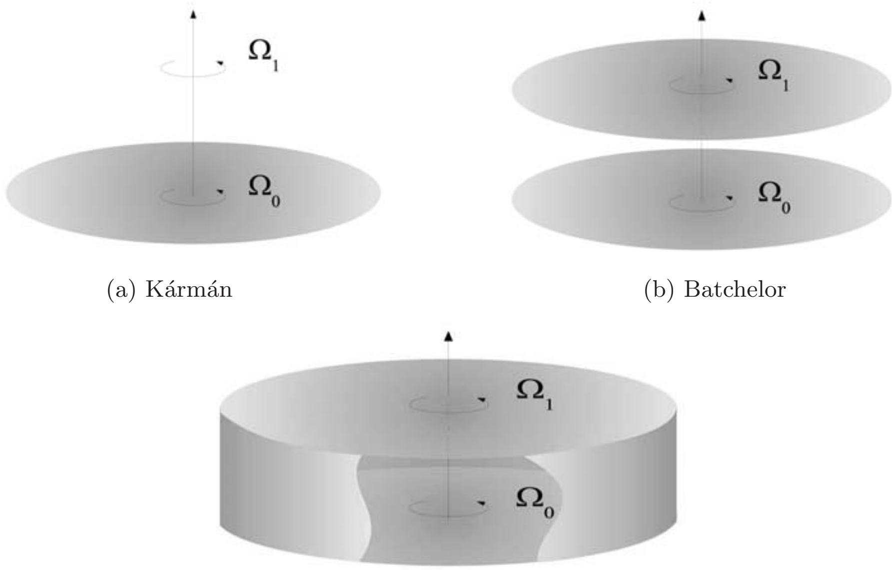
$$\gamma = \frac{\Omega_1}{\Omega_0},$$

where $\Omega_1$ and $\Omega_0 \neq 0$ are the angular velocities of the disk and of the fluid at infinity respectively or vice versa in the case of Bödewadt flows [3].

In 1951, Batchelor extended the Kármán self-similar model introducing the two-parameter ODE model of the flow powered by the differential rotation of two infinite disks sketched in Fig. 1-b. The additional parameter is the Reynolds number based of the gap height $H$

$$Re_H = \frac{UH}{\nu},$$

where $U \equiv H\Omega_0$ is the characteristic velocity. Fundamentally, Batchelor thus defined the general family of the parameterized self-similar Kármán models.



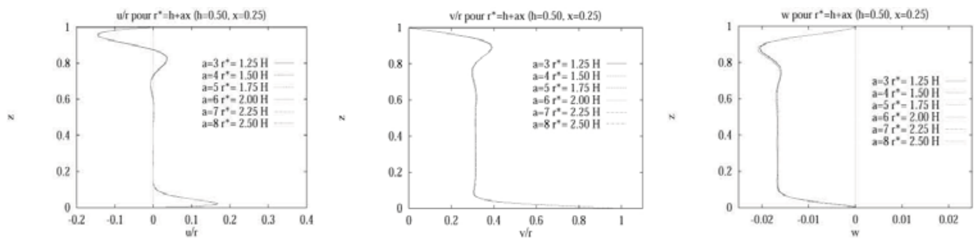

(a) Kármán

(b) Batchelor

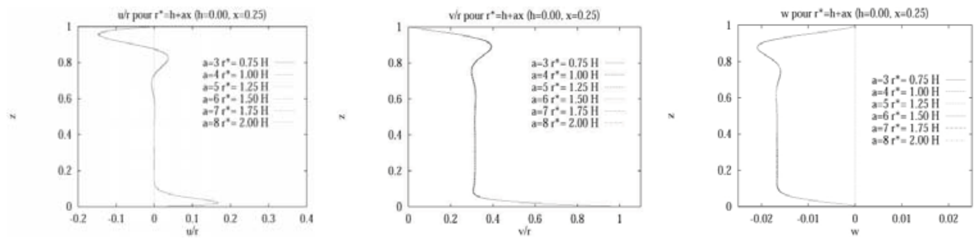
(c) Enclosed two disk system

Fig. 1. Configurations of the hierarchy of the steady rotationally-symmetric flows engendered by differential rotation fluid versus disks.

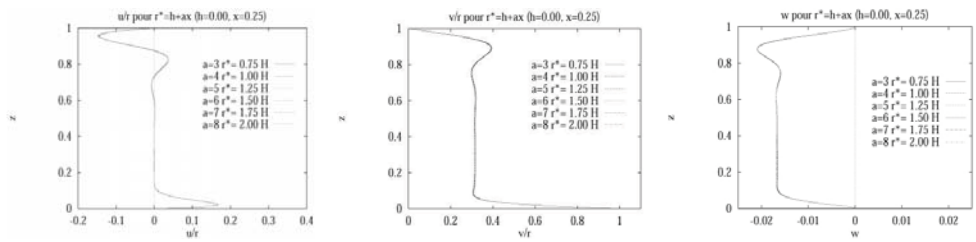
In 1996, we considered the shrouded two disk systems depicted in Fig. 1-c. In particular, the steady axisymmetric flows were investigated numerically. Adapting [10] a time-stepping pseudo-spectral code [9], the steady states were carried by mean of a Newton–Raphson's method [4]. However, the Kármán self-similar [2] does not describe the steady solution of the shrouded two disk configuration [7, 11]. In [5], it was shown that the Reynolds number based on $H$ yields a good description of the steady flows and measures the thickness of the boundary layers. In [6], a self-similar solution was shown to exist surrounding the axis, with a recirculating zone near the end-wall. Between these two zones there is a pseudo-similar region where velocity profiles are homothetic. The radial extension of the self-similar zone was studied for aspect ratios within a range [3, 10]. For a tolerance of 1% for the superposition of the velocity profile, the self-similar zone was shown to be confined in the vicinity of the rotation axis for radial positions $r \leq 1H$ where $H$ is the height of the cavity.




(a) envelop attached to the stator ($h = 0.50$)




(b) envelop attached to the rotor ($h = 0.00$)




(c) linear boundary condition between the rotor and the stator ($h = 0.00$)

Fig. 2. Homothetic zone: superposition of the velocity profiles $u/r$, $v/r$ and $w$ for various boundary conditions on the envelop [4, 6].

For radial positions greater than $H$, numerical experiments suggested that there exists a zone where the velocity profiles obey an homothetic law as a function of the aspect ratio $r = \mathbf{a}x + h$ for $r \in [h, (a-1)H]$ where $h$ depends on the boundary conditions (see Fig. 2). The so-called pseudo self-similar zone described he self-similar property of the solution on a part of the domain.

In the present work, the family of the rescaled three-parameter model is derived where the aditional parameter is the aspect ratio

$$\mathbf{a} = \frac{R}{H},$$

where $R$ is the radial extension of the disks of the enclosed two disk system.

## 2. HOMOTHETIC SELF-SIMILAR MODEL

As one member of the self-similar Kármán family of rotating flows, the key arguments merely lay on original papers of Kármán and Batchelor and can be stated as in Proposition 2.1. But, as a proof should be constructive, a self-contained proof is given.

PROPOSITION 2.1. *The homothetic steady axisymmetric solution of the generalized Kármán enclosed two disk systems solves a self-similar six order ODE governed by three parameters as follows:*

$$\mathbf{a}h'''' = -2Re_H(hh''' + gg'), \tag{1a}$$

$$\mathbf{a}g'' = 2Re_H(gh' - hg'), \tag{1b}$$

$$y = 0,\ h' = 0,\ g = 1,\ h = 0, \tag{1c}$$

$$y = 1,\ h' = 0,\ g = \gamma,\ h = 0. \tag{1d}$$

*where the dimensionless radial, tangential and axial components of the velocity satisfy*

$$w^\star = -2h(y), \quad u^\star = xh', \quad v^\star = xg(y).$$

*Proof.* Let $u$, $v$ and $w$ be the radial, tangential and axial components of the velocity field and $p$ be the pressure. In cylindrical polar coordinates and according to axisymmetry hypothesis, the steady Navier-Stokes equations are

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0, \tag{2a}$$

$$\rho\left(u\frac{\partial u}{\partial r} - \frac{v^2}{r} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left(\triangle u - \frac{u}{r^2}\right), \tag{2b}$$

$$\rho\left(u\frac{\partial v}{\partial r} + \frac{uv}{r} + w\frac{\partial v}{\partial z}\right) = \mu\left(\triangle v - \frac{v}{r^2}\right), \tag{2c}$$

(2d) $$\rho\left(u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\triangle w,$$

with
$$\triangle f=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{\partial^2 f}{\partial z^2}.$$

Let $H$ and $R$ be the characteristic length scales in $r$ and $z$ directions, respectively. If $\Omega_0\neq 0$, let

(3) $$U\equiv R\Omega_0$$

be the characteristic velocity in radial and tangential direction. One may replace $\Omega_0$ by $\Omega_1$, the following proof holds. Let $W$ be the characteristic axial velocity.

First, according to dimensional analysis (see e.g. [1]), we define the dimensionless coordinates and axial and radial components of the velocity field

(4) $$r=Rx,\quad z=Hy,\quad u=Uu^\star,\quad v=Uv^\star,\quad w=Ww^\star$$

and the continuity equation (2a) equivalently reads

(5) $$\frac{U}{\mathbf{a}W}\frac{1}{x}\frac{\partial}{\partial x}(xu^\star)+\frac{\partial\, w^\star}{\partial y}=0.$$

Hence, preserving the full tridimensionnality of the model, the order of dimensionless coefficient of the first term is one and the characteristic axial velocity is such that

(6) $$U=\mathbf{a}W.$$

According to the Kármán hypothesis, the axial velocity is of the form

(7) $$w^\star=\phi(y)\equiv -2\,h(y).$$

Thus, from the continuity equation (5), we deduce that the radial dimensionless velocity is

(8) $$u^\star=x\,h'.$$

Second, from the axial component of the momentum equation (2d), we infer that the pressure field is the superposition of radial and axial pressure fields. Let $P$ be the characteristic pressure. Let us remark that the axial component of the momentum equation (2d) equivalently reads

$$4\,hh'=-\frac{P}{\rho W^2}\frac{\partial p^\star}{\partial y}-2\,\frac{\nu}{WH}\,h''.$$

For example, the two choices $\rho W^2$ or $\rho U^2$ for the characteristic pressure are possible and respectively yields

(9) $$p^\star(x,y)=-2\left(h^2+\frac{a}{R_e}h'\right)+\Pi(x),$$

or

$$p^\star(x,y) = -\frac{2}{a^2}\left(h^2 + \frac{a}{R_e}h'\right) + \Pi(x). \tag{10}$$

Third, taking into account of (3), (4), (6), the radial momentum equation (2b) is

$$u^\star \frac{\partial u^\star}{\partial x} - \frac{v^{\star 2}}{x} - 2\,h\frac{\partial u^\star}{\partial y} = -\frac{P}{\rho\, U^2}\frac{d\Pi}{dx} + \frac{1}{a\,R_e}\left\{\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u^\star}{\partial x}\right) + \ a^2\ \frac{\partial^2 u}{\partial y^2} - \frac{u^\star}{x^2}\right\}$$

Furthermore, taking into account of (7), (8) yields

$$+\frac{P}{\rho\, U^2}\,\frac{1}{x}\frac{d\Pi}{dx} - \left(\frac{v^\star}{x}\right)^2 = \Theta(y), \tag{11}$$

where

$$\Theta(y) \equiv \frac{a}{R_e}\, h''' + 2\,h\,h'' - (h')^2.$$

The non-slip condition on the disks are (1c), (1d). Considering the boundary condition (1d) for example, the form of the tangential velocity is thus

$$v^\star = \ x\, g(y). \tag{12}$$

Taking into account of (12) and differentiating the radial momentum equation (11) with respect to $y$ yields the equation (1a).

Fourth, taking into account of (3), (4), (6) on one hand and of (7), (8), (12) on the other hand, the tangential momentum equation (2c) is (1b). This ends the proof. $\square$

## REFERENCES


[1] G.I. Barenblatt, *Scaling.* Cambridge University Press, 2003.

[2] G.K. Batchelor, *Note on a class of solutions of the Navier-Stokes equations representing steady rotationally-symmetric flow* Quart. Journ. Mech. and Applied Math., **4** (1951), 29–41.

[3] U.T. Bödewadt, *Die Drehstromung uber festem grunde.* Z. angew. Math. Mech., **20** (1940), 241–253.

[4] N.M.M. Cousin-Rittemard, *Contribution à l'étude des instabilités des écoulements axisymétriques en cavités interdisques de type rotor-stator.* Thèse de doctorat de mécanique de l'Université de Paris VI, 1996.

[5] N.M.M. Cousin-Rittemard, O. Daube and P. Le Quéré, *Description of the boundary layers of steady flows between coaxial disks in rotor-stator configuration.* C.R. mécanique, **327** (1999), 215–221.

[6] N.M.M. Cousin-Rittemard, O. Daube and P. Le Quéré, *Structure of steady flows between coaxial disks in rotor–stator configuration.* C.R. Mécanique **327** (1999), 221–226.

[7] D. Dijkstra and G.H.F. Van Heijst, *The flow between two finite rotating disks enclosed by a cylinder.* J. Fluid Mech. **128** (1983), 123–154.

[8] V. Ekman, *On the influence of the Earth's rotation on ocean currents*. Ark. Math. Astr. Fys. **2** (1905), 1–52.

[9] P. Le Quéré, *Etude de la transition à l'instationnaire des écoulements de convection naturelle en cavité différentiellement chauffée par méthodes spectrales Chebyshev*. Thèse de l'Université de Poitiers, 1987.

[10] C.K. Mamun and L.S. Tuckerman, *Asymmetry and Hopf bifurcation in spherical Couette flow*. Phys. Fluids **7** (1995), 80–91.

[11] J.M. Owen and R.H. Rogers, *Flow and heat transfer in rotating disc systems. Research Studies Press*, 1989.

[12] J. Pedlosky, *Geophysical dynamics*. Springer-Verlag, New York, 1987.

[13] J.T. Stuart, *On the non-linear mechanisms of wave disturbances in stable and unstable paralell flows*. J. Fluid Mech. **9** (1960), 353–370.

[14] T.H. Von Kármán, *Uber laminare und turbulent reibung*. Z. angew. Math. Mech. **1** (1921), 233–252.

*Received 14 July 2010*

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