A NECESSARY AND SUFFICIENT CONDITION FOR THE CONNECTIVITY OF THE ATTRACTOR OF AN INFINITE ITERATED FUNCTION SYSTEM

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The aim of the paper is to give a necessary and sufficient condition for the connectivity of the attractor of infinite iterated function system (IIFS).

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1. INTRODUCTION

We start by a short presentation of infinite iterated function systems (IIFSs) and by fixing the notation. Iterated function systems (IFS) were conceived in the present form by Hutchinson [6] and popularized by Barnsley [2] and are one of the most common and most general way to generate fractals. Many of the important examples of functions and sets with special and unusual properties from analysis turns out to be fractal sets and a great part of them are attractors of IFSs. There is a current effort to extend the classical Hutchinson's framework to more general spaces and infinite iterated function systems or, more generally, to multifunction systems and to study them and their applications (see for example [1], [3], [7], [8], [10], [12–15]). A recent such example can be found in [8] where the Lipscomb's space – which was an important example in dimension theory – can be obtain as an attractor of an IIFS defined in a very general setting, where the attractor can be a closed and bounded set in contrast with the classical theory where only compact sets are considered. Another generalization of the notion of an IFS can be found in [9, 11]. Although the fractal sets are defined with measure theory [4], [5] – being sets with noninteger Hausdorff dimension – it turns out that they have interesting topological properties as we can see from the above example [8]. One of the most important result in these direction is Theorem 1.3 below (see [17] for a proof) which states when the attractor of an IFS is a connected set. We want to extend this result to IIFSs and point out the differences between the two cases finite (IFS) and infinite (IIFS).

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The paper is divided in four parts. The first part is the introduction. The second part contains the description of the shift or code space for an IIFS. The third part contains the main results. The last one contains some examples.

For a set X, $P(X)$ denotes the subsets of X and $P^*(X) = P(X) - \{\emptyset\}$. For a subset A of $P(X)$, by A^* we mean $A - \{\emptyset\}$.

Let (X, d_X) and (Y, d_Y) be two metric spaces. By $C(X, Y)$ we will understand the set of continuous functions from X to Y.

A family of functions $(f_i)_{i\in I} \subset C(X, Y)$ is bounded if for every bounded set in $A \subset X$ the set $\bigcup_{i\in I} f_i(A)$ is bounded.

Let (X, d) be a metric space. For a nonvoid set $A \subset X$, $d(A)$ denotes the diameter of A, that is $d(A) = \sup_{x,y\in A} d(x, y)$.

Definition 1.1. Let (X, d) be a metric space. For a function $f : X \to X$ let us denote by $\mathbf{Lip}(f) \in \overline{\mathbb{R}}_+ = [0, +\infty]$ the Lipschitz constant associated to f, that is

$$\mathbf{Lip}(f) = \sup_{x,y\in X;\, x\neq y} \frac{d(f(x), f(y))}{d(x, y)}.$$

f is a Lipschitz function if $\mathbf{Lip}(f) < +\infty$ and a contraction if $\mathbf{Lip}(f) < 1$.

Let (X, d) be a metric space, $K(X)$ be the set of compact subsets of X and $B(X)$ the set of closed bounded subsets of X. It is obvious that $K(X) \subset B(X) \subset P(X)$.

Definition 1.2. On $P^*(X)$ we consider the generalized Hausdorff-Pompeiu semidistance $h : P^*(X) \times P^*(X) \to \overline{\mathbb{R}}_+$ defined by

$$h(A, B) = \max(d(A, B), d(B, A)),$$

where

$$d(A, B) = \sup_{x\in A} d(x, B) = \sup_{x\in A} \Big(\inf_{y\in B} d(x, y) \Big).$$

Concerning the Hausdorff-Pompeiu semidistance we have the following important properties:

Proposition 1.1. *Let (X, d) be a metric space. Then*

1) *If H and K are two nonempty subsets of X, then $h(H, K) = h(\overline{H}, \overline{K})$.*

2) *If $(H_i)_{i\in I}$ and $(K_i)_{i\in I}$ are two families of nonempty subsets of X, then*

$$h\Big(\bigcup_{i\in I} H_i, \bigcup_{i\in I} K_i \Big) = h\Big(\overline{\bigcup_{i\in I} H_i}, \overline{\bigcup_{i\in I} K_i} \Big) \le \sup_{i\in I} h(H_i, K_i).$$

3) *If H and K are two nonempty subsets of X and $f : X \to X$ is a Lipschitz function, then*

$$h(f(K), f(H)) \le \mathbf{Lip}(f) \cdot h(K, H).$$

Proof. See [1], [2] or [16]. □

THEOREM 1.1. *Let (X,d) be a metric space. Then $(B^*(X),h)$ is a metric space, $(K^*(X),h)$ is also a metric space and a closed subset in $B^*(X)$. $(B^*(X),h)$ is complete if (X,d) is a complete metric space and so is $(K^*(X),h)$. $(K^*(X),h)$ is compact if (X,d) is compact and $B^*(X)=K^*(X)$ in this case. $(K^*(X),h)$ is separable if (X,d) is separable.*

Proof. See [1], [2], [4], [8], and [16]. □

Definition 1.3. An infinite iterated function system (IIFS) on X consists of a bounded family of contractions $(f_i)_{i\in I}$ on X such that $\sup\limits_{i\in I}\mathbf{Lip}(f_i)<1$ and it is denoted by $\mathcal{S}=(X,(f_i)_{i\in I})$.

Definition 1.4. For an IIFS $\mathcal{S}=(X,(f_i)_{i\in I})$, $F_{\mathcal{S}}:B^*(X)\to B^*(X)$ is the function defined by $F_{\mathcal{S}}(B)=\overline{\bigcup\limits_{i\in I} f_i(B)}$.

The function $F_{\mathcal{S}}$ is a contraction with $\mathbf{Lip}(F_{\mathcal{S}})\le\sup\limits_{i\in I}\mathbf{Lip}(f_i)$.

Using the Banach contraction theorem there exists for an IIFS a unique set $A(\mathcal{S})\in B^*(X)$ such that $F_{\mathcal{S}}(A(\mathcal{S}))=A(\mathcal{S})$. More precisely we have:

THEOREM 1.2 [8]. *Let (X,d) be a complete metric space, $\mathcal{S}=(X,(f_i)_{i\in I})$ an* IIFS *and $c=\sup\limits_{i\in I}\mathbf{Lip}(f_i)<1$. Then there exists a unique $A(\mathcal{S})\in B^*(X)$ such that $F_{\mathcal{S}}(A(\mathcal{S}))=A(\mathcal{S})$. Moreover, for any $H_0\in B^*(X)$ the sequence $(H_n)_{n\ge0}$ defined by $H_{n+1}=F_{\mathcal{S}}(H_n)$ is convergent to $A(\mathcal{S})$. For the speed of the convergence we have the following estimation*

$$h(H_n,A(\mathcal{S}))\le\frac{c^n}{1-c}h(H_0,H_1).$$

Definition 1.5. The invariant set, $A(\mathcal{S})$, is called the attractor of the IIFS.

Definition 1.6. Let (X,d) be a metric space and $(A_i)_{i\in I}$ a family of nonvoid subset of X. The family $(A_i)_{i\in I}$ is said to be strongly-connected if for every $i,j\in I$ there exists $(i_k)_{k=\overline{1,n}}\subset I$ such that $i_1=i$, $i_n=j$ and $A_k\cap A_{k+1}\neq\emptyset$ for every $k\in\{1,2,\dots,n-1\}$.

The following result is well-known.

LEMMA 1.1. *Let (X,d) a metric space and $(A_i)_{i\in I}$ be a strongly-connected family of connected subsets of X. Then $\bigcup\limits_{i\in I} A_i$ is connected.*

Definition 1.7. A metric space (X,d) is arcwise connected if for every $x,y\in X$ there exists a continuous function $\varphi:[0,1]\to X$ such that $\varphi(0)=x$ and $\varphi(1)=y$.

Concerning the connectivity of the attractor of an IFS we have the following theorem (see [17]):

THEOREM 1.3. *Let* (X, d) *be a complete metric space,* $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ *an* IFS *with* $c = \max_{k=\overline{1,n}} \mathbf{Lip}(f_k) < 1$ *and* $A(\mathcal{S})$ *the attractor of* $\mathcal{S}$*. The following are equivalent*

1) *the family* $(A_i)_{i=\overline{1,n}}$ *is strongly-connected, where* $A_i = f_i(A(\mathcal{S}))$;
2) $A(\mathcal{S})$ *is arcwise connected*;
3) $A(\mathcal{S})$ *is connected.*

We want to obtain a similar result for IIFS. This theorem is not true for IIFSs as we can see from Example 4.1 and 4.2. The attractor of an IIFS can be connected but not arcwise connected. Also, from Examples 4.1 and 4.2 we see that if the attractor of an IIFS $\mathcal{S} = (X, (f_i)_{i\in I})$ is connected, then it does not results that the family $(A_i)_{i\in I}$ is stronly-connected, where $A_i = f_i(A(\mathcal{S}))$.

2. THE SHIFT SPACE FOR AN IIFS

In this section we present the shift or code space of an IIFS. The shift space for an IIFS is a generalization of the shift space for an IFS.

We start with notation. $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}^* = \mathbb{N} - \{0\}$, $\mathbb{N}_n^* = \{1, 2, \ldots, n\}$. For two sets A and B, B^A denotes the set of functions from A to B.

By $\Lambda = \Lambda(B)$ we will understand the set $B^{\mathbb{N}^*}$ and by $\Lambda_n = \Lambda_n(B)$ we will understand the set $B^{\mathbb{N}_n^*}$. The elements of $\Lambda = \Lambda(B) = B^{\mathbb{N}^*}$ will be written as infinite words $\omega = \omega_1\omega_2\ldots\omega_m\omega_{m+1}\ldots$, where $\omega_m \in B$ and the elements of $\Lambda_n = \Lambda_n(B) = B^{\mathbb{N}_n^*}$ will be written as words $\omega = \omega_1\omega_2\ldots\omega_n$. $\Lambda(B)$ is the set of infinite words with letters from the alphabet B and $\Lambda_n(B)$ is the set of words of length n. By $\Lambda^* = \Lambda^*(B)$ we will understand the set of all finite words $\Lambda^* = \Lambda^*(B) = \bigcup_{n\geq 1} \Lambda_n(B)$. If $\omega = \omega_1\omega_2\ldots\omega_m\omega_{m+1}\ldots$ or if $\omega = \omega_1\omega_2\ldots\omega_n$ and $n \geq m$ we define $[\omega]_m = \omega_1\omega_2\ldots\omega_m$.

For two words $\alpha \in \Lambda_n(B)$ and $\beta \in \Lambda_m(B)$ or $\beta \in \Lambda(B)$ by $\alpha\beta$ we will understand the concatenation of the words α and β, namely, $\alpha\beta = \alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_m$ and, respectively, $\alpha\beta = \alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_m\beta_{m+1}\ldots$.

Let I be a nonvoid set. On $\Lambda = \Lambda(I) = (I)^{\mathbb{N}^*}$ we consider the metric $d_s(\alpha, \beta) = \sum_{k=1}^{\infty} \frac{1-\delta_{\alpha_k}^{\beta_k}}{3^k}$, where $\delta_x^y = \begin{cases} 1 \text{ if } x = y \\ 0 \text{ if } x \neq y \end{cases}$.

Definition 2.1. The pair $(\Lambda(I) = (I)^{\mathbb{N}^*}, d_s)$ is a metric space and it is caled the shift space associated with an IIFS whose functions are indexed by the set I.

Remark 2.1. The convergence in the metric space $(\Lambda(I), d_s)$ is the convergence on components. $(\Lambda(I), d_s)$ is a complete metric space.

Let $F_i : \Lambda(I) \to \Lambda(I)$ be defined by $F_i(\omega) = i\omega$ for $i \in I$. The functions F_i are contractions and are named the right shift functions. We have

$$d_s(F_i(\alpha), F_i(\beta)) = \frac{d_s(\alpha, \beta)}{3}.$$

The function $R : \Lambda(I) \to \Lambda(I)$ defined by $R(\omega) = \omega_2\omega_3 \dots \omega_m\omega_{m+1} \dots$ is also continuous and is named the left shift function, where $\omega = \omega_1\omega_2 \dots \omega_m\omega_{m+1} \dots$. We have

$$d_s(R(\alpha), R(\beta)) = 3d_s(\alpha, \beta) - (1 - \delta_{\alpha_1}^{\beta_1}) \le 3d_s(\alpha, \beta).$$

Remark 2.2. With the above notations we have

1) $R \circ F_i(\omega) = \omega$ and $F_i \circ R(\omega) = i\omega_2\omega_3 \dots \omega_m\omega_{m+1} \dots$;

2) $\Lambda(I) = \bigcup\limits_{i \in I} F_i(\Lambda(I))$ and so $\Lambda(I)$ is the attractor of the IFS $\mathcal{S} = (\Lambda(I), (F_i)_{i \in I})$.

Notation 2.1. Let (X, d) be a metric space, $\mathcal{S} = (X, (f_i)_{i \in I})$ an IIFS on X and $A = A(\mathcal{S})$ the attractor of the IIFS $\mathcal{S}$. For $\omega = \omega_1\omega_2 \dots \omega_m \in \Lambda_m(I)$ $f_\omega = f_{\omega_1} \circ f_{\omega_2} \circ \dots . \circ f_{\omega_m}$ and $H_\omega = f_\omega(H)$ for a set $H \subset X$, in particular $A_\omega = f_\omega(A)$.

Notation 2.2. Let $f : X \to X$ be a contraction. We denote by e_f the fixed point of f. If $f = f_\omega$ then we denote by e_{f_ω} or by e_ω the fixed point of $f = f_\omega$.

The main results concerning the relation between the attractor of an IIFS and the shift space is contained in the following theorem:

Theorem 2.1 [10]. *If $A = A(\mathcal{S})$ is the attractor of the* IIFS $\mathcal{S} = (X, (f_i)_{i \in I})$ *and* $c = \sup\limits_{i \in I} \mathbf{Lip}(f_i) < 1$, *then*

1) *for $\omega \in \Lambda = \Lambda(I)$, $A_{[\omega]_{m+1}} \subset A_{[\omega]_m}$ and $d(A_{[\omega]_m}) \to 0$ as $m \to \infty$, more precisely*

$$d(A_{[\omega]_m}) \le c^m d(A);$$

2) *if a_ω is defined by $\{a_\omega\} = \bigcap\limits_{m \ge 1} \overline{A_{[\omega]_m}}$, then $d(e_{[\omega]_m}, a_\omega) \to 0$ as $m \to \infty$;*

3) *$A = A(\mathcal{S}) = \overline{\bigcup\limits_{\omega \in \Lambda} \{a_\omega\}}$, $A_\alpha = \overline{\bigcup\limits_{\omega \in \Lambda} \{a_{\alpha\omega}\}}$ for every $\alpha \in \Lambda^*$; if $A = \bigcup\limits_{i \in I} f_i(A)$ then $A = A(\mathcal{S}) = \bigcup\limits_{\omega \in \Lambda} \{a_\omega\}$ and $A_\alpha = \bigcup\limits_{\omega \in \Lambda} \{a_{\alpha\omega}\}$ for every $\alpha \in \Lambda^*$;*

4) *the set $\{e_{[\omega]_m} / \omega \in \Lambda$ and $m \in \mathbb{N}^*\}$ is dense in A;*

5) *the function $\pi : \Lambda \to X$ defined by $\pi(\omega) = a_\omega$ is continuous such that $\pi(\Lambda) \subset A$ and $\pi(\Lambda) = A$ if $A = \bigcup\limits_{i \in I} f_i(A)$;*

6) *$\pi \circ F_i = f_i \circ \pi$ for every $i \in I$.*

Definition 2.3. The function $\pi : \Lambda \to X$ from the above theorem is named the canonical projection from the shift space on the attractor of the IIFS.

3. THE MAIN RESULTS

For the main results we have to a define the connectivity for an infinite family of sets. We give two such definitions.

Definition 3.1. Let (X, d) be a metric space and $(A_i)_{i\in I}$ a family of nonvoid subsets of X. The family $(A_i)_{i\in I}$ is said to be disconnected if there exists $J, J' \subset I$ such that $J \cup J' = I$, $J \neq \emptyset$, $J' \neq \emptyset$ and $\overline{A_J} \cap A_{J'} = A_J \cap \overline{A_{J'}} = \emptyset$, where $A_J = \bigcup_{i\in J} A_i$. The family $(A_i)_{i\in I}$ is said to be connected if it is not disconnected.

Definition 3.2. Let (X, d) be a metric space and $(A_i)_{i\in I}$ a family of nonvoid subsets of X. The family $(A_i)_{i\in I}$ is said to be strongly-disconnected if there exists $J, J' \subset I$ such that $J \cup J' = I$, $J \neq \emptyset$, $J' \neq \emptyset$ and $\overline{A_J} \cap \overline{A_{J'}} = \emptyset$. The family $(A_i)_{i\in I}$ is said to be weakly connected if it is not strong-disconnected.

Remark 3.1. Let (X, d) be a metric space and $(A_i)_{i\in I}$ a family of nonvoid subset of X. We have

1) if $(A_i)_{i\in I}$ is strongly-disconnected it is also disconnected;
2) if $(A_i)_{i\in I}$ is connected it is also weakly-connected;
3) if $(A_i)_{i\in I}$ is strongly-connected it is also connected;
4) there are connected families of sets which are not strongly-connected and also there are weakly-connected families of sets which are not connected.

LEMMA 3.1. *Let (X, d_X) and (Y, d_Y) be two metric spaces, $f : X \to Y$ a continuous function and $(A_i)_{i\in I}$ a family of nonvoid subsets of X. If the family $(A_i)_{i\in I}$ is connected, then the family $(f(A_i))_{i\in I}$ is also connected. If the family $(A_i)_{i\in I}$ is weakly-connected, then the family $(f(A_i))_{i\in I}$ is also weakly-connected.*

Proof. We first consider the case where the family $(A_i)_{i\in I}$ is connected. Let $J, J' \subset I$ be such that $J \cup J' = I$, $J \cap J' = \emptyset$, $J \neq \emptyset$ and $J' \neq \emptyset$. Because the family $(A_i)_{i\in I}$ is connected, we have $\overline{A_J} \cap A_{J'} \neq \emptyset$ or $A_J \cap \overline{A_{J'}} \neq \emptyset$. By symmetry we can suppose that $\overline{A_J} \cap A_{J'} \neq \emptyset$. Let $a \in \overline{A_J} \cap A_{J'}$. Then

$$f(a) \in f(\overline{A_J}) \cap f(A_{J'}) \subset \overline{f(A_J)} \cap f(A_{J'}) = \overline{\bigcup_{i\in J} f(A_i)} \cap \Big(\bigcup_{i\in J'} f(A_i)\Big).$$

This proves that the family $(f(A_i))_{i\in I}$ is also connected. The case when the family $(A_i)_{i\in I}$ is weakly-connected is similar. □

LEMMA 3.2. *Let (X, d) be a complete metric space, $\mathcal{S} = (X, (f_i)_{i\in I}))$ an* IIFS *with $c = \sup_{i\in I} \mathbf{Lip}(f_i) < 1$. If the family $(A_i)_{i\in I}$ is weakly-connected, then the families $(A_w)_{w\in\Lambda_p}$ for $p \in \mathbb{N}^*$ are weakly-connected, where $\Lambda_p = \Lambda_p(I)$.*

Proof. The proof will be made by induction. The first step $p = 1$ is the hypothesis. For the induction step we suppose that $(A_w)_{w\in\Lambda_p}$ is weakly-connected. We want to prove that the family $(A_w)_{w\in\Lambda_{p+1}}$ is weakly-connected. Let us suppose by reduction ad absurdum that the family $(A_w)_{w\in\Lambda_{p+1}}$ is not weakly-connected. Then there exists $J, J' \subset \Lambda_{p+1}$ such that $J \cup J' = \Lambda_{p+1}$, $J \neq \emptyset$, $J' \neq \emptyset$ and $\overline{A_J} \cap \overline{A_{J'}} = \emptyset$. For $\omega \in \Lambda_p$, set $J_\omega = \{w \in J \mid [w]_p = \omega\}$ and $J'_\omega = \{w \in J' \mid [w]_p = \omega\}$. We have $J_\omega = \emptyset$ or $J'_\omega = \emptyset$. Indeed $J_\omega \subset J$, $J'_\omega \subset J'$ and so $\overline{A_{J_\omega}} \cap \overline{A_{J'_\omega}} \subset \overline{A_J} \cap \overline{A_{J'}} = \emptyset$. Since the family $(A_i)_{i\in I}$ is weakly-connected from hypothesis, from Lemma 3.1 we deduce that the family $(A_w)_{w\in J_\omega \cup J'_\omega} = (A_w)_{w\in\Lambda_{p+1};[w]_p=\omega} = (f_\omega(A_i))_{i\in I}$ is weakly-connected. So, we have $J_\omega = \emptyset$ or $J'_\omega = \emptyset$. Let $L = \{[w]_p \mid w \in J\}$ and $L' = \{[w]_p \mid w \in J'\}$. It follows that $L \cap L' = \emptyset$, $L \neq \emptyset$, $L' \neq \emptyset$, $L \cup L' = \Lambda_p$, $\overline{A_J} = \overline{A_L}$, $\overline{A_{J'}} = \overline{A_{L'}}$ and $\overline{A_L} \cap \overline{A_{L'}} = \emptyset$. This imply that the family $(A_w)_{w\in\Lambda_p}$ is not weakly-connected, which is a contradiction. □

LEMMA 3.3. *Let (X, d) be a complete metric space, $\mathcal{S} = (X, (f_i)_{i\in I}))$ an* IIFS *with $c = \sup_{i\in I} \mathbf{Lip}(f_i) < 1$ and $A(\mathcal{S})$ the attractor of* IIFS *$\mathcal{S}$. If the family $(A_i)_{i\in I}$ is weakly-connected and $A(\mathcal{S}) = \bigcup_{i\in I} f_i(A(\mathcal{S}))$, then the family $(A_i)_{i\in I}$ is connected, where $A_i = f_i(A(\mathcal{S}))$.*

Proof. Let us suppose by reduction ad absurdum that the family $(A_i)_{i\in I}$ is not connected. Then it is disconnected. This means that there exists $J, J' \subset I$ such that $J \cup J' = I$, $J \neq \emptyset$, $J' \neq \emptyset$ and $\overline{A_J} \cap A_{J'} = A_J \cap \overline{A_{J'}} = \emptyset$. Because the family $(A_i)_{i\in I}$ is weakly-connected we have $\overline{A_J} \cap \overline{A_{J'}} \neq \emptyset$. Let $x \in \overline{A_J} \cap \overline{A_{J'}}$. But $A_J \cup A_{J'} = \bigcup_{j\in I} A_j = A = \overline{\bigcup_{j\in I} A_j} = \overline{A_J} \cup \overline{A_{J'}}$ and so $x \in A_J$ or $x \in A_{J'}$. Then $A_J \cap \overline{A_{J'}} \neq \emptyset$ or $\overline{A_J} \cap A_{J'} \neq \emptyset$. This ends the proof. □

Definition 3.2. Let (X, d) be a metric space. A set $A \subset X$ is said to be decomposable if there exists two nonvoid subsets B and C of A such that $B \cup C = A$ and $\delta(B, C) = \inf_{x\in B, y\in C} d(x, y) > 0$.

Remark 3.2. Let (X, d) be a metric space. Then

1) every decomposable set is not connected;

2) a compact set $A \subset X$ is decomposable if and only if it is not a connected set.

Let A be a disconnected compact set. Then there exists two compact nonvoid sets B and C such that $B \cap C = \emptyset$ and $B \cup C = A$. Let $\varepsilon = \delta(B, C) =$

$\inf_{x\in B,\, y\in C} d(x,y)$. Since B and C are compact sets, we have $\varepsilon > 0$ and so A is decomposable.

THEOREM 3.1. *Let (X,d) be a complete metric space and $\mathcal{S} = (X, (f_i)_{i\in I}))$ an* IIFS *with $c = \sup_{i\in I} \mathbf{Lip}(f_i) < 1$. Then*

1) *if $A(\mathcal{S})$ is a connected set, then the family $(A_i)_{i\in I}$ is weakly-connected where $A_i = f_i(A(\mathcal{S}))$;*

2) *if $A(\mathcal{S})$ is a connected set and $A(\mathcal{S}) = \bigcup_{i\in I} f_i(A(\mathcal{S}))$, then the family $(A_i)_{i\in I}$ is connected;*

3) *if the family $(A_i)_{i\in I}$ is weakly-connected, then $A(\mathcal{S})$ is not decomposable.*

Proof. We remark first that A_ω for $\omega \in \Lambda_p$ and $p \in \mathbb{N}^*$ are nonvoid sets, because $A(\mathcal{S})$ is nonvoid.

1) Let us suppose by reduction ad absurdum that the family $(A_i)_{i\in I}$ is not weak by connected. Then there exists $J, J' \subset I$ such that $J \cup J' = I$, $J \neq \emptyset$, $J' \neq \emptyset$ and $\overline{A_J} \cap \overline{A_{J'}} = \emptyset$. Because A_i are nonvoid, A_J and $A_{J'}$ are nonvoid. Also $A = \overline{\bigcup_{j\in I} A_j} = \overline{A_J} \cup \overline{A_{J'}}$. This imply that A is not a connected set, which is a contradiction.

2) It results from point 1) and Lemma 3.3.

3) Let us suppose by reduction ad absurdum that A is a decomposable set. Then there exists two sets such that $B \cup C = A$ and $\delta(B,C) > 0$. Since A is closed B and C are closed. Let $\varepsilon = \delta(B,C)$. Let m be a natural number such that $c^m d(A) < \varepsilon$. We consider the family $(A_w = f_w(A))_{w\in\Lambda_m}$. Then $d(A_w) \leq c^m d(A) < \varepsilon$ for every $w \in \Lambda_m$. It is not possible that $A_w \cap B \neq \emptyset$ and $A_w \cap C \neq \emptyset$ for a $w \in \Lambda_m$, because in this case we should have $\varepsilon > d(A_w) \geq \delta(B,C) = \varepsilon$. Because $A_w \subset A = B \cup C$ we have $A_w \subset B$ or $A_w \subset C$. Let $J = \{w \in \Lambda_m \mid A_w \subset B\}$ and $J' = \{w \in \Lambda_m \mid A_w \subset C\}$. Then $J \cup J' = \Lambda_m$, $J \neq \emptyset$, $J' \neq \emptyset$, $\overline{A_J} \subset B$ and $\overline{A_{J'}} \subset C$. Since $A = \overline{A_J} \cup \overline{A_{J'}} = B \cup C$ and $B \cap C = \emptyset$ we have $\overline{A_J} = B$ and $\overline{A_{J'}} = C$. This means that the family $(A_w = f_w(A))_{w\in\Lambda_m}$ is strongly-disconnected. This contradicts with Lemma 3.1. □

Taking into account Remark 3.2, for IIFSs with compact attractors Theorem 3.1 becomes

COROLLARY 3.1. *Let (X,d) be a complete metric space, $\mathcal{S} = (X, (f_i)_{i\in I}))$ an* IIFS *with $c = \sup_{i\in I} \mathbf{Lip}(f_i) < 1$. We suppose that $A = A(\mathcal{S})$, the attractor of $\mathcal{S}$, is compact. The following are equivalent:*

1) *the family $(A_i)_{i\in I}$ is weakly-connected where $A_i = f_i(A(\mathcal{S}))$;*

2) *$A(\mathcal{S})$ is connected.*

Taking into account in addition Remark 3.1.2, one can obtain the following result.

COROLLARY 3.2. *Let (X, d) be a complete metric space and $\mathcal{S} = (X, (f_i)_{i\in I}))$ an* IIFS *with $c = \sup_{i\in I} \mathbf{Lip}(f_i) < 1$. We suppose that $A = A(\mathcal{S})$, the attractor of $\mathcal{S}$, is compact. If $A = \bigcup_{i\in I} f_i(A)$ the following are equivalent:*

1) *the family $(A_i)_{i\in I}$ is weakly-connected where $A_i = f_i(A(\mathcal{S}))$;*
2) *the family $(A_i)_{i\in I}$ is connected where $A_i = f_i(A(\mathcal{S}))$;*
3) *$A(\mathcal{S})$ is connected.*

COROLLARY 3.3. *Let (X, d) be a complete metric space and $\mathcal{S} = (X, (f_i)_{i\in I})$ be an* IIFS *such that $A(\mathcal{S})$, the attractor of* IIFS *$\mathcal{S}$, is compact. If the family $(A_i)_{i\in I}$ is strongly-connected, then $A(\mathcal{S})$ is connected, where $A_i = f_i(A(\mathcal{S}))$.*

Proof. It results from Corollary 3.1 and Remark 3.1.3. □

4. EXAMPLES AND REMARKS

The hypothesis that $A = A(\mathcal{S})$, the attractor of $\mathcal{S}$, is compact has an important role in the proof of the implication 2) ⇒ 1) from Corollary 3.1. It is an open question that Corollary 3.1 remains valid if we drop the hypothesis that $A = A(\mathcal{S})$, the attractor of $\mathcal{S}$, is compact. We think that the result is not true in this case but it is difficult to give a concrete example.

Example 4.1 (a generalization of [15]). Let (X, d) be a complete metric space and $A \in B^*(X)$. For an element $a \in X$, f_a will denote the constant function with value a, that is $f_a : X \to X$ and $f_a(x) = a$ for every $x \in X$. Then A is the attractor of the IIFS $\mathcal{S} = (X, (f_a)_{a\in A})$ if A is infinite or IFS $\mathcal{S} = (X, (f_a)_{a\in A})$ if A is finite. Also, A is the attractor of the IIFS $\mathcal{S}_B = (X, (f_a)_{a\in B})$ for any dense set B in A. If A is separable and B is a countable dense set in A, then A is the attractor of the CIFS (countable IFS) $\mathcal{S}_B = (X, (f_a)_{a\in B})$. This happens, in particular, for any compact set A. Since a closed set could be connected but not arcwise connected, conditions 2) and 3) from Theorem 1.3 are not equivalent for a IIFS. Also, because the family of sets $(A_a = f_a(A) = \{a\})_{a\in A}$ is not strongly-connected for every set A (in fact $A_a \cap A_b = \emptyset$ for $a, b \in A$ with $a \neq b$), points 1) and 2) (and also 1) and 3)) from Theorem 1.3 are not equivalent for an IIFS.

Remark 4.1. 1) If the set A from the previous example is connected and compact and we consider the IIFS $\mathcal{S} = (X, (f_a)_{a\in A})$, then we are in the case of Corollary 3.2.

2) If the set A from the previous example is connected and compact and we consider the IIFS $\mathcal{S}_B = (X, (f_a)_{a\in B})$, then we are (in general) in the case of Corollary 3.1.

Example 4.2. Let $X = \mathbb{R}^2$ endowed with the Euclidean distance. For a line d, $\pi_d : \mathbb{R}^2 \to d$ denotes the projection on d. For two different points $A, B \in \mathbb{R}^2$, AB denotes the line which passes trough A and B and $[A, B]$ denotes the segment with ends in A and B.

Let $A, B, C \in \mathbb{R}^2$ be such that $A \neq B$. Then there exist an unique real number x_C such that $\pi_d(C) = A + x_C(B - A)$.

Also, for two different points $A, B \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$ let $f^{\alpha}_{A,B} : \mathbb{R}^2 \to AB$ be the function defined by

$$f^{\alpha}_{A,B}(C) = \begin{cases} A & \text{if } x_C < 0, \\ A + \alpha x_C(B - A) & \text{if } x_C \in [0, 1), \\ A + \alpha(B - A) & \text{if } x_C \geq 1. \end{cases}$$

$f^{\alpha}_{A,B}$ is a Lipschitz function with $\mathbf{Lip}(f^{\alpha}_{A,B}) = |\alpha|$. We remark that $f^{1/2}_{A,B}(\mathbb{R}^2) = \left[A, \frac{A+B}{2}\right]$ and that the attractor of the IFS $(\mathbb{R}^2, (f^{1/2}_{A,B}, f^{1/2}_{B,A}))$ is $[A, B]$, because

$$f^{1/2}_{A,B}([A, B]) \cup f^{1/2}_{B,A}([A, B]) = \left[A, \frac{A + B}{2}\right] \cup \left[\frac{A + B}{2}, B\right] = [A, B].$$

We consider the points $A_n = \left(\frac{1}{2^n}, \frac{1+(-1)^{n-1}}{2}\right)$ for $n \in \mathbb{N}$, $A_{-1} = (0, 0)$ and $A_{-2} = (0, 1)$, and the functions $f_n, f_{-1}, f_{-2} : \mathbb{R}^2 \to \mathbb{R}^2$, for $n \in \mathbb{N}$, defined by $f_{2n} = f^{1/2}_{A_n,A_{n+1}}$, $f_{2n+1} = f^{1/2}_{A_{n+1},A_n}$, $f_{-1} = f^{1/2}_{A_{-1},A_{-2}}$ and $f_{-2} = f^{1/2}_{A_{-2},A_{-1}}$. Let $\mathcal{S} = (\mathbb{R}^2, (f_n)_{n\geq 0})$ and $\mathcal{S}_1 = (\mathbb{R}^2, (f_n)_{n\geq -2})$ be two IIFSs. Then $A(\mathcal{S}) = A(\mathcal{S}_1) = [A_{-1}, A_{-2}] \cup \left(\bigcup_{n\geq 0} [A_n, A_{n+1}]\right)$. This results from the facts that $\overline{\bigcup_{n\geq 0} [A_n, A_{n+1}]} = [A_{-1}, A_{-2}] \cup \left(\bigcup_{n\geq 0} [A_n, A_{n+1}]\right)$, for $n \geq 0$, the attractor of the IFS $\mathcal{S}'_n = (\mathbb{R}^2, (f_{2n}, f_{2n+1}))$ is $[A_n, A_{n+1}]$ and $f_{2n}(\mathbb{R}^2), f_{2n+1}(\mathbb{R}^2) \subset [A_n, A_{n+1}]$ and for $n = -1$ the attractor is $[A_{-1}, A_{-2}]$ and $f_{-1}(\mathbb{R}^2), f_{-2}(\mathbb{R}^2) \subset [A_{-1}, A_{-2}]$. We remark first that the set $A(\mathcal{S}) = A(\mathcal{S}_1)$ is connected by not arcwise connected. Also, the family of sets $(f_n(A(\mathcal{S})))_{n\geq 0}$ is strong connected but the family of sets $(f_n(A(\mathcal{S}_1)))_{n\geq -2})$ is connected but not strong connected.

Other interesting examples of connected attractors of IIFSs can be found in [13, 14, 16].

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