MULTIPARAMETER MARKOV PROCESSES: GENERATORS AND ASSOCIATED MARTINGALES

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We propose a notion of a generator for multiparameter Feller semigroups and summarise a calculus for these generators. Furthermore for multiparameter Feller processes a martingale is associated with the generator.

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1. INTRODUCTION

The subject of this article is multiparameter stochastic processes with independent increments, in literature also referred to as additive multiparameter processes. Among the vast research into two- or multiparameter processes the closest to our merely analytic approach are the papers [5] by E.B. Dynkin who solved a boundary value problem for the operator $(-1)^N \Delta_1 \cdots \Delta_N$ acting on functions defined on $G_1 \times \cdots \times G_N$, $G_j \subset \mathbb{R}^{n_j}$, where Δ_j acts on the variables of G_j only, see also [4]. Dynkin's approach was taken up by Mazziotto [11] for elliptic operators of second order and the associated diffusions.

For the predominantly probabilistic parts we mention [2] where the Lévy-Khinchin theorem as well as the Lévy-Itô decomposition for Lévy processes indexed by $[0,1]^N$ are developed. For multivariate subordination, i.e., subordination (in the sense of Bochner) for multiparameter Lévy processes we refer the reader to Barndorff-Nielsen et al. [3]. Moreover we want to mention Khoshnevisan [10] as a standard reference.

In what follows we introduce an operator which is in some sense the infinitesimal generator A of multiparameter (Feller) semigroups $(T_{(t_1,\ldots,t_N)})_{(t_1,\ldots,t_N)\in\mathbb{R}_+^N}$ as it resembles the generator of a one-parameter semigroup in the associated (partial) differential equation. Moreover making use of this generator we introduce a martingale for the canonical processes associated with a Feller semigroup $(T_{(t_1,\ldots,t_N)})_{(t_1,\ldots,t_N)\in\mathbb{R}_+^N}$.

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It seems that our analytic point of view is new and relates partly to a special type of multiparameter processes. In Section 2 we define multiparameter convolution semigroups and, more generally, multiparameter semigroups of operators. We define their generator and offer the necessary calculus. Finally, Section 3 is devoted to a martingale for multiparameter processes, here we refer to P. Imkeller [7] as a reliable source for two-parameter martingales. For more background material on the analytic part of our theory we refer to [12] as well as our joint paper [9] with A. Potrykus.

2. MULTIPARAMETER CONVOLUTION SEMIGROUPS AND FELLER SEMIGROUPS

In this section we introduce N-parameter convolution semigroups of probability measures and multiparameter Feller semigroups.

For an arbitrarily fixed natural number $N \in \mathbb{N}$ a family of probability measures $(\mu_{(t_1,\dots,t_N)})_{t\in\mathbb{R}_+^N}$ indexed by non-negative real N-dimensional vectors, which satisfies for all $s, t \in \mathbb{R}_+^N$ the conditions

1. $\mu_t(\mathbb{R}^n) = 1$;
2. $\mu_s * \mu_t = \mu_{s+t}$;
3. $\mu_t \to \mu_0$ vaguely for $t \to 0$ and $\mu_0 = \varepsilon_0$

is called a **multiparameter convolution semigroup of probability measures**.

A first example of a two-parameter convolution semigroup $(\eta_{(s,t)})_{(s,t)\in\mathbb{R}_+^2}$ can be constructed as the product of two one-parameter convolution semigroups $(\mu_s)_{s\geq 0}$ and $(\nu_t)_{t\geq 0}$, by defining

$$\eta_{(s,t)} := \mu_s \otimes \nu_t \quad \text{for all } (s,t) \in \mathbb{R}_+^2.$$

Then $(\eta_{(s,t)})_{(s,t)\in\mathbb{R}_+^2}$ is called the **product semigroup** of $(\mu_s)_{s\geq 0}$ and $(\nu_t)_{t\geq 0}$.

Multiparameter convolution semigroups feature the following decomposition property:

THEOREM 2.1. *For an N-parameter convolution semigroup $(\mu_t)_{t\in\mathbb{R}_+^N}$ on $\mathbb{R}^n$ there exist continuous negative definite functions $\psi_1, \psi_2, \dots, \psi_N : \mathbb{R}^n \to \mathbb{C}$ such that*

$$\widehat{\mu}_t(\xi) = (2\pi)^{\frac{n}{2}}\, \mathrm{e}^{-t_1\psi_1(\xi)-t_2\psi_2(\xi)-\dots-t_N\psi_N(\xi)} \tag{1}$$

holds for all $\xi \in \mathbb{R}^n$ and $t \succeq 0$, i.e., $t_j \geq 0$ for $j = 1, \dots, N$.

Equation (1) exhibits that every N-parameter convolution semigroup can be decomposed into the convolution of N one-parameter semigroups.

The proof of Theorem 2.1 uses the continuity of the mapping $t \mapsto \mu_t$ and results about generalised Cauchy functional equations, see [1], p. 226. More details are given in [12].

Now we consider N-parameter semigroups of strongly continuous operators on a real or complex Banach space $(X, \|\cdot\|_X)$.

Definition 2.2. **A.** An N-parameter family $(T_t)_{t\succeq 0}$, $t \in \mathbb{R}_+^N$, of bounded linear operators $T_t : X \to X$ is called an **N-parameter semigroup of operators**, if $T_0 = \mathrm{id}$ and for all $s, t \in \mathbb{R}_+^N$ we have

$$T_{s+t} = T_s \circ T_t. \tag{2}$$

B. We call $(T_t)_{t\succeq 0}$ **strongly continuous** if for all $x \in X$ we have

$$\lim_{t\to 0} \|T_t u - u\|_X = 0.$$

C. The semigroup $(T_t)_{t\succeq 0}$ is a **contraction** semigroup, if

$$\|T_t\| \leq 1$$

for all $t \succeq 0$, i.e., each operator T_t is a contraction. Here $\|\cdot\|$ denotes the operator norm $\|\cdot\|_{X,X}$.

D. A strongly continuous contraction N-parameter semigroup $(T_t)_{t\succeq 0}$ on $(C_\infty(\mathbb{R}^n), \|\cdot\|_\infty)$ which is positivity preserving is called an **N-parameter Feller semigroup**.

Multiparameter semigroups feature the commuting property, which turns out to be very useful when introducing the generator but also for the construction of associated stochastic processes. A family of operators $(T_t)_{t\in\mathbb{R}_+^N}$ is said to fulfill the **commuting property**, if for all $s, t \in \mathbb{R}_+^N$ we have

$$[T_t, T_s] := T_t \circ T_s - T_s \circ T_t = 0. \tag{3}$$

This is a direct consequence of (2), since $T_t \circ T_s = T_{t+s} = T_s \circ T_t$, thus the commuting property is fulfilled.

The following construction establishes the connection with convolution semigroups.

Example 2.3. Any arbitrary N-parameter convolution semigroup $(\mu_t)_{t\in\mathbb{R}_+^N}$ gives rise to an N-parameter operator semigroup by

$$T_t u(x) := \int_{\mathbb{R}^n} u(x-y)\ \mu_t(\mathrm{d}y), \tag{4}$$

for all $t \in \mathbb{R}_+^N$ and $u \in C_\infty(\mathbb{R}^n)$. Indeed $(T_t)_{t\succeq 0}$ is a strongly continuous contraction semigroup on $\mathbb{R}^n$ which is positivity preserving. Hence it is a Feller semigroup.

Now, we want to introduce the generator for multiparameter Feller semigroups. For all $j = 1, \dots, N$, let $A^{(j)}$ denote the generator of the jth (one-parameter) marginal semigroup $(T^{(j)}_{t_j})_{t_j \geq 0}$, i.e., $T^{(j)}_{t_j} = T_{t_j \cdot e_j}$ for all $t_j \in \mathbb{R}_+$, where e_j is the jth canonical basis vector in $\mathbb{R}^N$. Then we define the operator A called the **infinitesimal generator** of $(T_t)_{t \in \mathbb{R}^N_+}$ by

$$A = A^{(1)} \circ \ldots \circ A^{(N)}. \tag{5}$$

With this newly defined generator A we associate the partial differential equation

$$\frac{\partial^N}{\partial t_1 \cdot \ldots \cdot \partial t_N} u(t,x) = A\, u(t,x) \tag{6}$$

which is solved by $u(t,x) = T_t f(x)$ for all $t \in \mathbb{R}^N_+$, $x \in \mathbb{R}^n$, and $f \in \mathcal{S}(\mathbb{R}^n)$. This is a generalization of the differential equation associated to the generator in the one-parameter case, see [8], and is one strong motivation for our notion of A.

For the generator we now develop a calculus, which is a powerful tool when handling multiparameter operator semigroups and associated stochastic processes. This calculus much resembles the calculus available for generators of one-parameter semigroups, see [6] by Ethier and Kurtz or [8], and is the second motivation for the introduction of the operator A as the generator of multiparameter semigroups.

Theorem 2.4 (Calculus for the generator). *Let A be the generator of an N-parameter Feller semigroup $(T_{(t_1,\dots,t_N)})_{(t_1,\dots,t_N) \in \mathbb{R}^N_+}$.*

A. *If $u \in D(A)$ then $T_t u \in D(A)$, i.e., $D(A)$ is invariant under T_t for all $t \in \mathbb{R}^N_+$.*

B. *Every marginal semigroup commutes with its own generator and the generator of every other marginal semigroup, moreover the generators of the marginal semigroup commute mutually, i.e., for all $i, j \in 1, \dots, N$ we have*

$$[T^{(i)}, A^{(j)}] = 0 \quad \text{and} \quad [A^{(i)}, A^{(j)}] = 0. \tag{7}$$

C. *For all $u \in C_\infty(\mathbb{R}^N)$ and arbitrary $(t_1, \dots, t_N) \in \mathbb{R}^N_+$ the following integration rules*

$$A \int_{(0,\dots,0)}^{(t_1,\dots,t_N)} T_{(s_1,\dots,s_N)}\, u \; \mathrm{d}(s_1,\dots,s_N) = \int_{(0,\dots,0)}^{(t_1,\dots,t_N)} A T_{(s_1,\dots,s_N)}\, u \; \mathrm{d}(s_1,\dots,s_N)$$
$$= \sum_{s_j \in \{0,t_j\},\, j \in \{1,\dots,N\}} (-1)^N \prod_{j=1}^{N} (-1)^{s_j} T_{(s_1,\dots,s_N)}\, u.$$

We refer the reader to [12] for the proof of Theorem 2.4. Moreover in [12] a detailed and comprehensive investigation of the properties of $(A, D(A))$ is given.

3. A MARTINGALE ASSOCIATED WITH MULTIPARAMETER FELLER SEMIGROUPS

We restrict ourselves now for simplicity to the 2-parameter case and we prove that using A as well as $A^{(1)}$ and $A^{(2)}$ we can associate a martingale with a 2-parameter Feller process extending the 1-parameter case in a natural way.

We define the filtration we will be working with. Let $(F^{(j)}_{t_j})_{t_j\geq 0}$ be the natural filtration of the marginal processes $(X^{(j)}_{t_j})_{t_j\geq 0}$, then for all $t\in\mathbb{R}^N_+$ define

$$\mathcal{F}_t := \bigvee_{j=1}^{N} \mathcal{F}^{(j)}_{t_j} := \sigma\Big(\bigcup_{j=1}^{N} \mathcal{F}^{(j)}_{t_j}\Big). \tag{8}$$

For the existence càdlàg-modification which we invoke in the following theorem we refer the reader to Theorem 2.1 in [11].

THEOREM 3.1. *Let* $(T_{(t_1,t_2)})_{(t_1,t_2)\in\mathbb{R}^2_+}$ *be a Feller semigroup with generator* $(A, D(A))$ *and associated càdlàg-modification process* $\big((X_{(t_1,t_2)})_{(t_1,t_2)\in\mathbb{R}^2_+}, (\mathcal{F}_{(t_1,t_2)})_{(t_1,t_2)\in\mathbb{R}^2_+}\big)$. *Then for every* $u \in D(A)$ *we have*

$$\begin{aligned} M^u_{(t_1,t_2)} &:= u(X_{(t_1,t_2)}) - u(X_{(t_1,0)}) - u(X_{(0,t_2)}) + u(X_{0,0}) - \\ &- \int_0^{t_1} A^{(1)}\big(u(X_{(r_1,t_2)}) - u(X_{(r_1,0)})\big)\,\mathrm{d}r_1 - \\ &- \int_0^{t_2} A^{(2)}\big(u(X_{(t_1,r_2)}) - u(X_{(0,r_2)})\big)\,\mathrm{d}r_2 + \int_0^{t_1}\int_0^{t_2} Au(X_{(r_1,r_2)})\,\mathrm{d}r_1\mathrm{d}r_2 \end{aligned} \tag{9}$$

is an $\{\mathcal{F}_t\}_{t\in\mathbb{R}^2_+}$*-martingale, i.e., the equality*

$$\mathbb{E}\big[M^u_t \big| \mathcal{F}_s\big] = M^u_s$$

holds for all $s \preceq t,\ s,t \in \mathbb{R}^2_+$.

Proof. For $u\in D(A)$ we have $Au\in C_\infty(\mathbb{R}^n)$, and especially $A^{(1)}u, A^{(2)}u\in C_\infty(\mathbb{R}^n)$ such that the integrals in (3.1) are well-defined. For $0 \preceq s \preceq t$,

$s,t\in\mathbb{R}^2_+$ we find

$$\mathbb{E}\big[M^u_{(t_1,t_2)}-M^u_{(s_1,s_2)}\big|\mathcal{F}_{(s_1,s_2)}\big]=\mathbb{E}\Bigg[u(X_{(t_1,t_2)})-u(X_{(t_1,0)})-u(X_{(0,t_2)})+u(X_{(0,0)})$$
$$-\int_0^{t_1}A^{(1)}\big(u(X_{(r_1,t_2)})-u(X_{(r_1,0)})\big)\,\mathrm{d}r_1-\int_0^{t_2}A^{(2)}\big(u(X_{(t_1,r_2)})-u(X_{(0,r_2)})\big)\,\mathrm{d}r_2$$
$$+\int_0^{t_1}\int_0^{t_2}Au(X_{(r_1,r_2)})\,\mathrm{d}r_1\mathrm{d}r_2-u(X_{(s_1,s_2)})+u(X_{(s_1,0)})+u(X_{(0,s_2)})-u(X_{(0,0)})$$
$$+\int_0^{s_1}A^{(1)}\big(u(X_{(r_1,t_2)})-u(X_{(r_1,0)})\big)\,\mathrm{d}r_1+\int_0^{s_2}A^{(2)}\big(u(X_{(t_1,r_2)})-u(X_{(0,r_2)})\big)\,\mathrm{d}r_2$$
$$-\int_0^{s_1}\int_0^{s_2}Au(X_{(r_1,r_2)})\,\mathrm{d}r_1\mathrm{d}r_2\Bigg|\mathcal{F}_{(s_1,s_2)}\Bigg].$$

By the Markov Property, we get

$$=T_{(t_1-s_1,t_2-s_2)}u(X_{(s_1,s_2)})-T^{(1)}_{t_1-s_1}u(X_{(s_1,0)})-T^{(2)}_{t_2-s_2}u(X_{(0,s_2)})$$
$$-\int_0^{s_1}A^{(1)}\big(T^{(2)}_{t_2-s_2}u(X_{(r_1,s_2)})u(X_{(r_1,s_2)})-u(X_{(r_1,0)})\big)\,\mathrm{d}r_1$$
$$-\int_{s_1}^{t_1}A^{(1)}\big(T_{(r_1-s_1,t_2-s_2)}u(X_{(s_1,s_2)})-T^{(1)}_{r_1-s_1}u(X_{(s_1,0)})\big)\,\mathrm{d}r_1$$
$$-\int_0^{s_2}A^{(2)}\big(T^{(1)}_{t_1-s_1}u(X_{(s_1,r_2)})-u(X_{(0,r_2)})\big)\,\mathrm{d}r_2$$
$$-\int_{s_2}^{t_2}A^{(2)}\big(T_{(t_1-s_1,r_2-s_2)}u(X_{(s_1,r_2)})-T^{(2)}_{t_2-s_2}u(X_{(0,s_2)})\big)\,\mathrm{d}r_2$$
$$+\int_{s_1}^{t_1}\int_{s_2}^{t_2}AT_{(r_1-s_1,r_2-s_2)}u(X_{(s_1,s_2)})\,\mathrm{d}r_1\mathrm{d}r_2+\int_{s_1}^{t_1}\int_0^{s_2}AT^{(1)}_{r_1-s_1}u(X_{(s_1,r_2)})\,\mathrm{d}r_1\mathrm{d}r_2$$
$$+\int_0^{s_1}\int_{s_2}^{t_2}AT^{(2)}_{r_2-s_2}u(X_{(r_1,s_2)})\,\mathrm{d}r_1\mathrm{d}r_2-u(X_{(s_1,s_2)})+u(X_{(s_1,0)})+u(X_{(0,s_2)})$$
$$+\int_0^{s_1}A^{(1)}\big(u(X_{(r_1,t_2)})-u(X_{(r_1,0)})\big)\,\mathrm{d}r_1+\int_0^{s_2}A^{(2)}\big(u(X_{(t_1,r_2)})-u(X_{(0,r_2)})\big)\,\mathrm{d}r_2$$
$$=T_{(t_1-s_1,t_2-s_2)}u(X_{(s_1,s_2)})-T^{(1)}_{t_1-s_1}u(X_{(s_1,0)})-T^{(2)}_{t_2-s_2}u(X_{(0,s_2)})$$
$$-T^{(2)}_{t_2-s_2}\int_0^{s_1}A^{(1)}u(X_{(r_1,s_2)})\,\mathrm{d}r_1+\int_0^{s_1}A^{(1)}u(X_{(r_1,0)})\,\mathrm{d}r_1$$
$$-T_{(t_1-s_1,t_2-s_2)}u(X_{(s_1,s_2)})+T^{(2)}_{t_2-s_2}u(X_{(s_1,s_2)})+T^{(1)}_{(t_1-s_1)}u(X_{(s_1,0)})$$

$$
\begin{aligned}
&-u(X_{(s_1,0)}) - T^{(1)}_{t_1-s_1}\int_0^{(2)} A^{(2)}u(X_{(s_1,r_2)})\,\mathrm{d}r_2\\
&+\int_0^{s_2} A^{(2)}u(X_{(0,r_2)})\,\mathrm{d}r_2 - T_{(t_1-s_1,t_2-s_2)}u(X_{(s_1,s_2)}) + T^{(1)}_{(t_1-s_1)}u(X_{(s_1,s_2)})\\
&+T^{(2)}_{t_2-s_2}u(X_{(0,s_2)}) + u(X_{(0,s_2)}) + T_{(t_1-s_1,t_2-s_2)}u(X_{(s_1,s_2)})\\
&-T^{(1)}_{(t_1-s_1)}u(X_{(s_1,s_2)}) - T^{(2)}_{(t_2,s_2)}u(X_{(s_1,s_2)}) + u(X_{(s_1,s_2)})\\
&+T^{(1)}_{t_1-s_1}\int_0^{s_2} A^{(2)}u(X_{(s_1,r_2)})\,\mathrm{d}r_2 - \int_0^{s_2} A^{(2)}u(X_{(s_1,r_2)})\,\mathrm{d}r_2\\
&+T^{(2)}_{t_2-s_2}\int_0^{s_1} A^{(1)}u(X_{(r_1,s_2)})\,\mathrm{d}r_1 - \int_0^{s_1} A^{(1)}u(X_{(r_1,s_2)})\,\mathrm{d}r_1 - u(X_{(s_1,s_2)})\\
&+u(X_{(s_1,0)}) + u(X_{(0,s_2)}) + \int_0^{s_1} A^{(1)}u(X_{(r_1,s_2)})\,\mathrm{d}r_1 - \int_0^{s_1} A^{(1)}u(X_{(r_1,0)})\,\mathrm{d}r_1\\
&+\int_0^{s_2} A^{(2)}u(X_{(s_1,r_2)})\,\mathrm{d}r_2 - \int_0^{s_2} A^{(2)}u(X_{(0,r_2)})\,\mathrm{d}r_2 = 0. \quad \square
\end{aligned}
$$

The extension of this martingale to stochastic processes depending on three or more parameters does not make any problem.

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