A NUMERICAL INTEGRAL METHOD FOR COMPUTING THE GUIDED MODES IN AN OPTICAL HALF COUPLER IN THE SCALAR CASE

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We present a numerical integral method for computing the guided modes in an optical half coupler in scalar case. This method is based on the integral representation of the solutions of the problem. We first establish a variational formulation of our problem. In order to solve it, we introduce a boundary condition on the transversal section of the half coupler, and approach the variational problem by problems set in bounded domains. We obtain an algebraic system equivalent to the approached variational problem. At the end, we give some numerical results.

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1. INTRODUCTION

Our aim is to compute, by a numerical method based on integral representation, the propagation of guided modes in an optical half coupler. In the first part of the paper, starting with the Maxwell equations written in the scalar case, we establish a variational formulation of our problem under the weak guidance hypothesis. In the second part, using a classical method of approximation and replacing the transparent conditions used in [5, 2] by conditions of Robin type [8, 6], we obtain an algebraic system equivalent to the variational formulation of the problem. Recall that an optical half coupler is a cylindrical structure formed by a fine core surrounded by a cladding protective with one face abraded. The abraded face is placed in a liquid (see Figures 1 and 2).

To formulate the mathematical model of the problem, we assume that the optical half coupler is infinite along the propagation axis. Under the weakly guidance hypothesis, the electromagnetic field of a guided mode is transverse [13, 12]. The transverse component satisfies the two dimensional eigenvalue

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problem:

$$
(1.1)\quad \begin{cases} \text{find } \beta \in]kn_3, kn_1[\text{ and } u \in H^1(\mathbb{R}^2),\ u \neq 0, \text{ such that} \\ -\Delta u - n^2k^2u = -\beta^2 u \text{ in } \Omega_i,\ [u] = \left[\dfrac{\partial u}{\partial \nu}\right] = 0 \text{ on } \Gamma_i,\ i = 1,2,3, \end{cases}
$$

where

n_1 is the refractive index in Ω_1,
n_2 is the refractive index in Ω_2,
n_3 is the refractive index in Ω_3,
k is the wave (positive) number,
β is the propagation constant,
u is a transversal component of the electromagnetic field,
Ω_1 is the transversal section of the fiber core,
Ω_2 is the transversal section of the liquid,
Ω_3 is the transversal section of the cladding region,
$\Gamma_1 = \partial\Omega_1,\ \Gamma_2 = \partial\Omega_2,\ \Gamma_3 = \partial\Omega_3 = \Gamma_1 \cup \Gamma_2$.

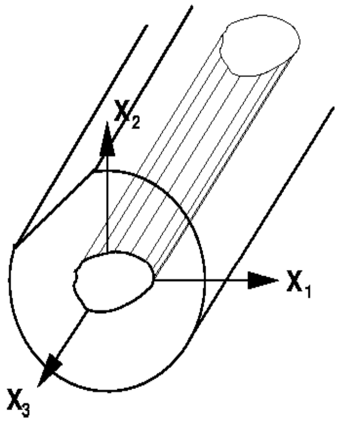


Fig. 1. An optical half coupler.

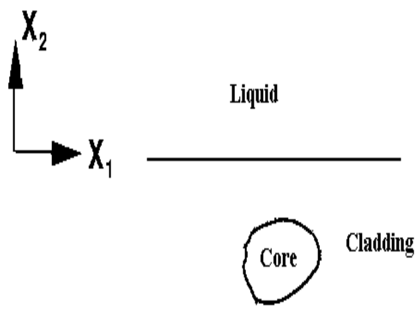


Fig. 2. Its transversal section.

Problem (1.1) is obtained from the Maxwell system when placed in mode of weak guidance [14, 7]. Solving this problem means that we find all couples (β, u), with $-\beta^2$ an eigenvalue of the operator $A_k = \Delta - k^2n^2$ and u the associated eigenfunction in $H^1(\mathbb{R}^2)$.

Denoting by $\sigma_{\text{ess}}(A_k)$ the essential spectrum of the operator A_k, we have the following result from [3, 5]:

Lemma 1.1. *For every $k > 0$ we have*

$$\sigma_{\text{ess}} = [-k^2n_3^2, +\infty[,$$

and for all λ in $]-\infty, -k^2n_1^2[$ the operator $(A_k - \lambda I)$ is invertible. Moreover, all the eigenvalues of the operator A_k are contained in the interval $]-k^2n_1^2, -k^2n_3^2[$.

2. AN INTEGRO-DIFFERENTIAL SYSTEM AND A VARIATIONAL FORMULATION

We now show that (1.1) can be written as an integro-differential system. To do so, we use the following result from [10].

PROPOSITION 2.1. *The integral representation of the solutions of problem* (1.1) *is given for all $y \in \Omega_i$ with $y \notin \Gamma_i$ by the formula*

$$u(y) = -\varepsilon_i \left\{ \int_{\Gamma_i} G_i(\beta, x, y) \frac{\partial u}{\partial \nu_x} \mathrm{d}\gamma_x \mathrm{d}x - \int_{\Gamma_i} \frac{\partial G_i}{\partial \nu_x}(\beta, x, y) u(x) \mathrm{d}\gamma_x \mathrm{d}x \right\}, \tag{2.1}$$

where $\mathrm{d}\gamma_x$ is the surface element on Γ_i and ε_i is equal to 1 (*resp.* -1) *if ν is the outside* (*resp. inside*) *unit normal vector to Ω_i while G_i, $i = 1, 2, 3$, are the Green kernels defined by*

$$G_1 = -\frac{1}{4} Y_0(\lambda_1 |x - y|), \quad G_i = \frac{1}{2\pi} K_0(\lambda_i |x - y|), \quad i = 2, 3.$$

Here $\lambda_1^2 = \beta^2 - k^2 n_1^2$, $\lambda_i^2 = k^2 n_i^2 - \beta^2$, $i = 2, 3$, and Y_0 and K_0 are the Bessel functions from [1].

In the sequel, we use the notation of [7, 5, 2], namely we set

$$\left(u|_{\Gamma_1}, \frac{\partial u}{\partial \nu}\Big|_{\Gamma_1} \right) = (j_1, m_1), \quad \left(u|_{\Gamma_2}, \frac{\partial u}{\partial \nu}\Big|_{\Gamma_2} \right) = (j_2, m_2).$$

By using the potential trace formulas of simple and double layers from [10], and writing the tangential components of the electromagnetic field along the interfaces core-cladding and cladding-liquid in the integral form, we associate with problem (1.1) the integro-differential system:

$$\begin{cases} \text{find } \beta \in]kn_3, kn_1[\text{ and } \Phi = (\Phi_1, \Phi_2) \in V_1 \times V_2 \text{ such that} \\ A_\beta \Phi = A_\beta^1 \Phi_1 + A_\beta^2 \Phi_2 = 0 \text{ in } V_1' \times V_2', \end{cases} \tag{2.2}$$

where $\Phi_1 = (j_1, m_1)$, $\quad \Phi_2 = (j_2, m_2)$. The operators A^1_β and A^2_β are defined by

$$A^1_\beta \Phi_1 = \begin{pmatrix} \displaystyle\int_{\Gamma_1} \left\{ \frac{\partial G_1}{\partial \nu_y} + \frac{\partial G_3}{\partial \nu_y} \right\} m_1 \mathrm{d}\gamma_x + \frac{\partial}{\partial \nu_y} \left(\int_{\Gamma_1} \left\{ \frac{\partial G_1}{\partial \nu_x} + \frac{\partial G_3}{\partial \nu_x} \right\} j_1(x) \mathrm{d}\gamma_x \right) \\ \displaystyle - \int_{\Gamma_2} \frac{\partial G_3}{\partial \nu_y} m_1 \mathrm{d}\gamma_x - \frac{\partial}{\partial \nu_y} \left(\int_{\Gamma_2} \frac{\partial G_3}{\partial \nu_x} j_1(x) \mathrm{d}\gamma_x \right) \\ \displaystyle\int_{\Gamma_1} \{G_1 + G_3\} m_1 \mathrm{d}\gamma_x + \left(\int_{\Gamma_1} \left\{ \frac{\partial G_1}{\partial \nu_y} + \frac{\partial G_3}{\partial \nu_y} \right\} j_1(x) \mathrm{d}\gamma_x \right) \\ \displaystyle - \int_{\Gamma_2} G_3 m_1 \mathrm{d}\gamma_x - \int_{\Gamma_2} \frac{\partial G_3}{\partial \nu_x} j_1(x) \mathrm{d}\gamma_x \end{pmatrix}$$

and

$$A^2_\beta \Phi_2 = \begin{pmatrix} \displaystyle\int_{\Gamma_2} \left\{ \frac{\partial G_2}{\partial \nu_y} + \frac{\partial G_3}{\partial \nu_y} \right\} m_2 \mathrm{d}\gamma_x + \frac{\partial}{\partial \nu_y} \left(\int_{\Gamma_2} \left\{ \frac{\partial G_2}{\partial \nu_x} + \frac{\partial G_3}{\partial \nu_x} \right\} j_2(x) \mathrm{d}\gamma_x \right) \\ \displaystyle + \int_{\Gamma_1} \frac{\partial G_3}{\partial \nu_y} m_2 \mathrm{d}\gamma_x + \frac{\partial}{\partial \nu_y} \left(\int_{\Gamma_1} \frac{\partial G_2}{\partial \nu_x} j_2(x) \mathrm{d}\gamma_x \right) \\ \displaystyle\int_{\Gamma_2} \{G_2 + G_3\} m_2 \mathrm{d}\gamma_x + \left(\int_{\Gamma_2} \left\{ \frac{\partial G_2}{\partial \nu_y} + \frac{\partial G_3}{\partial \nu_y} \right\} j_2(x) \mathrm{d}\gamma_x \right) \\ \displaystyle + \int_{\Gamma_1} G_3 m_2 \mathrm{d}\gamma_x + \int_{\Gamma_1} \frac{\partial G_3}{\partial \nu_x} j_2(x) \mathrm{d}\gamma_x \end{pmatrix}.$$

In (2.2), V_1 and V_2 are the functional spaces defined by

$$V_1 = H^{1/2}(\Gamma_1) \times H^{-1/2}(\Gamma_1), \quad V_2 = H^{1/2}(\Gamma_3) \times H^{-1/2}(\Gamma_3)$$

and V_i', is the dual space of V_i $(i = 1, 2)$.

It is classical that the operator A_β is linear and continuous from V to V', where $V = V_1 \times V_2$ and V' is the dual space of V (we refer the reader to [5, 7] for details).

We now associate with the operator A_β the real symmetric bilinear form $a_\beta(\cdot\,,\cdot)$ defined as

$$a_\beta(\cdot\,,\cdot) = \langle A_\beta \Phi, \Psi \rangle_{V' \times V}, \quad \forall \Phi \in V,\ \forall \Psi \in V,$$

where $\langle \cdot\,,\cdot \rangle$ stands for the dual product.

Remark 2.2. It is easily seen that $a_\beta(\cdot\,,\cdot)$ is symmetric and continuous on $V \times V$.

Remark 2.3. The bilinear form $a_\beta(\cdot\,,\cdot)$ represents the reaction in the sense of Rumsey [11] of the current $\Phi = (j, m)$ on the test current $\Psi = (j', m')$ (see for more details [11, 7]).

Then the following variational problem associated with (2.2) characterizes the determination of the guided modes in an optical half-coupler under weakly guidance assumptions:

$$\text{find } \beta \in]kn_3, kn_1[\text{ and } \Phi \in V \text{ such that } a_\beta(\Phi,\Psi) = 0,\ \forall \Psi \in V, \tag{2.3}$$

where for all $\Phi = ((j_1, m_1), (j_2, m_2)) \in V$ and $\Psi = ((j_1', m_1'), (j_2', m_2')) \in V$ we have

$$\begin{aligned}
a_\beta(\Phi,\Psi) = &\int_{\Gamma_1}\int_{\Gamma_1} \big(G_1 + G_2\big) m_1 m_1' \mathrm{d}\gamma_x \mathrm{d}\gamma_y + \int_{\Gamma_1}\int_{\Gamma_2} \Big(\frac{\partial G_1}{\partial \nu_x} + \frac{\partial G_2}{\partial \nu_x}\Big) j_1 m_1' \mathrm{d}\gamma_x \mathrm{d}\gamma_y \\
&- \int_{\Gamma_1}\int_{\Gamma_2} G_2\, m_2\, m_1'\, \mathrm{d}\gamma_x \mathrm{d}\gamma_y - \int_{\Gamma_1}\int_{\Gamma_2} \frac{\partial G_2}{\partial \nu_x} j_2 m_1' \mathrm{d}\gamma_x \mathrm{d}\gamma_y \\
+ &\int_{\Gamma_1}\int_{\Gamma_1} \Big(\frac{\partial G_1}{\partial \nu_y} + \frac{\partial G_2}{\partial \nu_y}\Big) m_1 j_1' \mathrm{d}\gamma_x \mathrm{d}\gamma_y + \int_{\Gamma_1} \frac{\partial}{\partial \nu_y}\Big[\int_{\Gamma_1} \Big(\frac{\partial G_1}{\partial \nu_x} + \frac{\partial G_2}{\partial \nu_x}\Big) j_1 \mathrm{d}\gamma_x\Big] j_1' \mathrm{d}\gamma_y \\
&- \int_{\Gamma_1}\int_{\Gamma_2} \frac{\partial G_2}{\partial \nu_y} m_2 j_1' \mathrm{d}\gamma_x \mathrm{d}\gamma_y - \int_{\Gamma_1} \frac{\partial}{\partial \nu_y}\Big[\int_{\Gamma_2} \frac{\partial G_2}{\partial \nu_x} j_2 \mathrm{d}\gamma_x\Big] j_1' \mathrm{d}\gamma_y \\
&- \int_{\Gamma_2}\int_{\Gamma_1} G_3 m_1 m_2' \mathrm{d}\gamma_x \mathrm{d}\gamma_y - \int_{\Gamma_2}\int_{\Gamma_1} \frac{\partial G_2}{\partial \nu_x} j_1 m_2' \mathrm{d}\gamma_x \mathrm{d}\gamma_y \\
+ &\int_{\Gamma_2}\int_{\Gamma_2} \Big(\frac{\partial G_2}{\partial \nu_y} + \frac{\partial G_3}{\partial \nu_y}\Big) m_2 j_2' \mathrm{d}\gamma_x \mathrm{d}\gamma_y + \int_{\Gamma_2} \frac{\partial}{\partial \nu_y}\Big[\int_{\Gamma_2} \Big(\frac{\partial G_2}{\partial \nu_x} + \frac{\partial G_3}{\partial \nu_x}\Big) j_2 \mathrm{d}\gamma_x\Big) j_2' \mathrm{d}\gamma_y \\
&- \int_{\Gamma_2}\int_{\Gamma_1} \frac{\partial G_2}{\partial \nu_y} m_1 j_2' \mathrm{d}\gamma_x \mathrm{d}\gamma_y - \int_{\Gamma_2} \frac{\partial}{\partial \nu_y}\Big(\int_{\Gamma_2} \frac{\partial G_2}{\partial \nu_x} j_1 \mathrm{d}\gamma_x\Big) j_2' \mathrm{d}\gamma_y.
\end{aligned}$$

Finally, in order to get rid of the hyper-singular integrals in the formula above, we use (cf. for instance, [10]) the classical formula

$$\begin{aligned}
\int_{\Gamma_i} \frac{\partial}{\partial \nu_y}\Big(\int_{\Gamma_i} \frac{\partial G_i}{\partial \nu_x} \phi_i \mathrm{d}\gamma_x\Big) \phi_i' \mathrm{d}\gamma_y = &- \int_{\Gamma_i}\int_{\Gamma_i} G_i \frac{\partial \phi}{\partial s}\frac{\partial \phi'}{\partial s} \mathrm{d}\gamma_x \mathrm{d}\gamma_y + \\
&+ \lambda_i^2 \int_{\Gamma_i}\int_{\Gamma_i} G_i \phi_i \phi_i' \mathrm{d}\gamma_x \mathrm{d}\gamma_y,
\end{aligned}$$

for all ϕ_i and ϕ_i' in $H^{1/2}(\Gamma_i)$, $i = 1, 2$.

3. NUMERICAL APPROXIMATION AND RESULTS

To solve the variational problem (2.3), we adopt a classical method of Galerkin type. This allows us to reduce the resolution of (2.3) to an algebraic system. To do so, we first approach the core-cladding interface Γ_1 by a closed polygonal curve Γ_1^h. We then truncate the liquid-cladding interface Γ_2 and approach it by a closed polygonal curve Γ_2^h. We take

$$u + \beta \frac{\partial u}{\partial \nu} = 0$$

on the extremities of Γ_2^h (see Figure 3). This procedure is inspired by [8, 6].

Second, in order to be able to make the computations, we introduce the vector spaces

$$V_i^h = W_{i,1}^h \times W_{i,2}^h, \quad i = 1, 2,$$

where $W_{i,1}^h$, $i = 1, 2$ is a vector space of finite dimension of functions defined on Γ_i^h and piecewise affine; $W_{i,2}^h$, $i = 1, 2$, is a vector space of finite dimension of functions defined on Γ_i^h and piecewise constant; Γ_1^h is the boundary of the polygon approximating Γ_1; Γ_2^h is the interface Γ_2 truncated and approximated. Obviously, V_i^h, $i = 1, 2$, is a vector subspace of V_i.

Finally, set $V^h = V_1^h \times V_2^h$ which is a vector subspace of finite dimension $4N$.

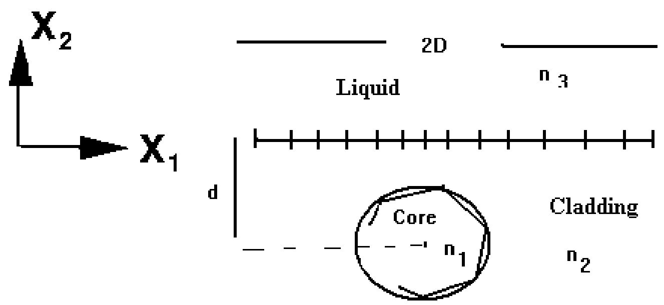


Fig. 3. Approximated interfaces of the optical half coupler.

We now associate with the variational problem (2.3) a variational problem posed in finite dimension. By using an inner Galerkin approximation method, the solutions of this finite dimensional problem will approach the solutions of the variational problem. In conclusion, we are lead to solve the algebraic system

$$(M_h) \quad \text{find } \beta \in]kn_3, kn_1[; \ I_h \in \mathbb{R}^h \text{ and } I_h \neq 0, \text{ such that } A_h(\beta_h) \cdot I_h = 0,$$

where $A_h(\beta_h)$ is a symmetric matrix.

4. NUMERICAL RESULTS

In all numerical experiences, we compute the normalized propagation constant defined by

$$b_h = \frac{\beta_h^2 - n_3^2 k^2}{(n_1^2 - n_3^2)}.$$

It is clear that $b_h \in]0,1[$.

Next, we do not give the number of waves k, but use the standardized frequency given by $\nu = \sqrt{k^2\, a\,(n_1^2 - n_3^2)}$.

To validate our method we used the data

$$(5.1) \qquad \left| \begin{array}{rl} n_1 = & 1.5085 \\ n_2 = & 1.50 \\ n_3 = & 1.50 \\ a = & 1. \\ D = & 2\Pi \\ d/a = & 1.5 \\ \nu = & 2. \\ N = & 50, \end{array} \right.$$

where a is the radius of the core of the half coupler, d the coupling distance and $2D$ the truncation distance (see Figure 3). We found $b_h = 0.416$, that is the value of the constant propagation associated with the principal mode. It is the same analytic value of the constant of propagation that the one given by Marcus [9].

In Figures 4a and 4b we present the current I_h on the half-coupler interfaces, and in Figure 5 the electromagnetic field u_h in the transversal section of the half coupler.

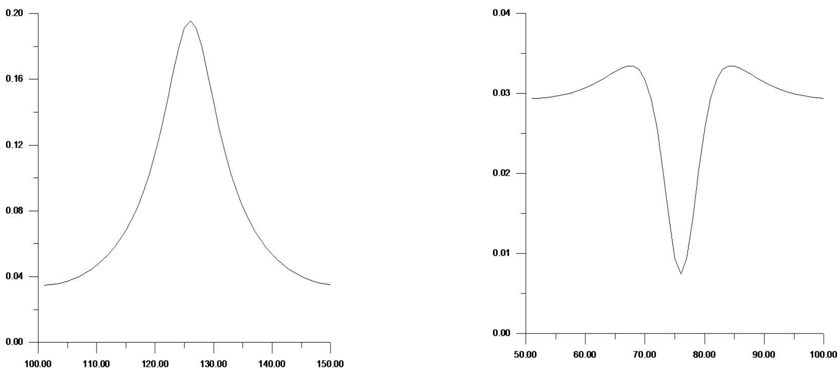


Fig. 4a. The current I_h $\left(\text{i.e., } u \text{ and } \frac{\partial u}{\partial \nu}\right)$ on Γ_h^1.

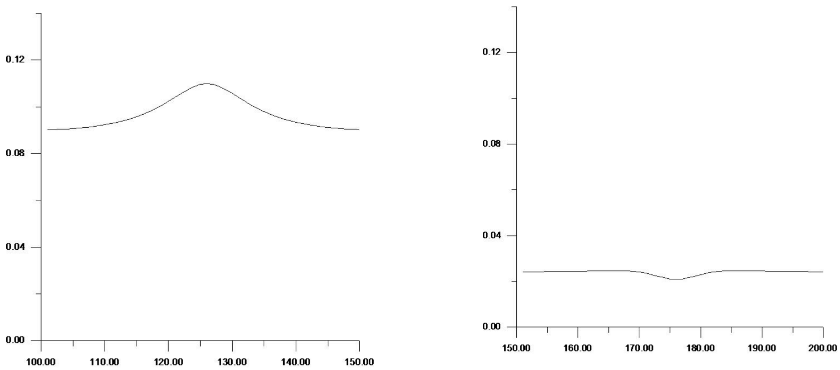


Fig. 4b. The current I_h $\left(\text{i.e., } u \text{ and } \frac{\partial u}{\partial \nu}\right)$ on Γ_h^2.

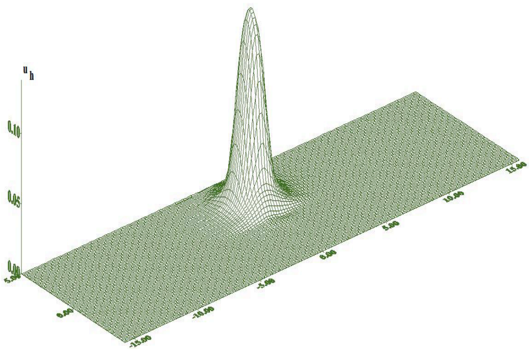
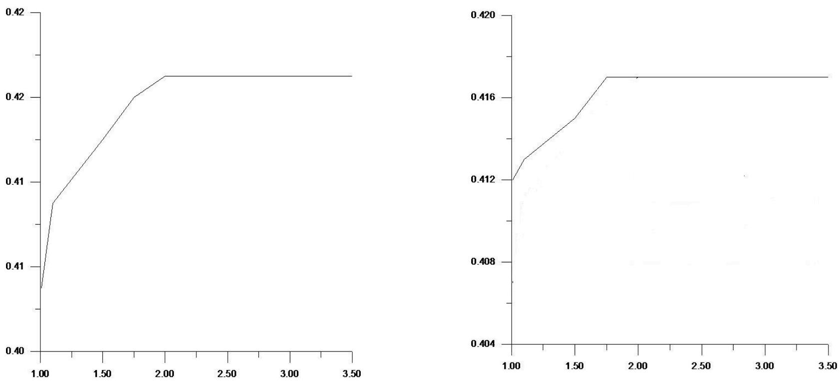


Fig. 5. The electromagnetic field u_h in the transversal section of the optical half coupler.



b_h in function of coupling d/a. b_h in function of truncation distance $2D$.

Fig. 6.

We also studied the dependence of the constant of propagation of the first mode (principal mode) on the coupling and truncation distance. The results are given in Figure 6.

REFERENCES

[1] A.S. Abramovitch and I.A. Stegun, *Handbook of Mathematical Functions.* Dover, New Work, 1974.

[2] A. Boureghda, *Calcul par méthode intégrale des modes guidés dans les coupleurs directionnels: Cas de faible guidage.* Thèse de magistère, ENS-Kouba, Alger, 1997.

[3] A.S. Bonnet and R. Djellouli, *Etude mathématique des fibres optiques, résultats complémentaires et extension au cas de couplage.* Rapport interne No. 182 du CMAP. Ecole Polytechnique, Palaiseau, 1988.

[4] A.S. Bonnet, *Analyse mathématique de la propagation des ondes guidées dans les fibres optiques.* Thèse de Doctorat es Science, Université Pierre et Marie Curie (Paris 6), 1988.

[5] A. Choutri, *Une méthode intégrale pour le calcul des modes guidés dans un demi coupleur optique.* Thèse de magistère, ENS-Kouba, Alger, 1992.

[6] A. Choutri, *Étude de l'erreur de troncature du domaine pour un probème aux valeurs propres.* C. R. Math. Acad. Sci. Paris **346** (2008), 233–237.

[7] R. Djellouli, *Contribution à l'analyse mathématique et au calcul numérique des modes guidés dans les fibres optiques.* Thèse de Doctorat es Science, Université Paris Sud-Orsay, 1988.

[8] R. Djellouli, C. Bekkey, A. Choutri and H. Rezgui, *A local boundary condition coupled to a finite element method to compute guided modes of optical fibers under weak guidance assumptions.* Math. Methods Appl. Sci. **23** (2000), 1551–1583.

[9] D. Marcus, *Theory of Dielectic Optical Waveguides.* Academic Press, New York, 1974.

[10] J.C. Nedelec, *Approximation des équations intégrales en mécanique et en physique.* Cours EDF-CEA-INRIA-CMAP, 1987.

[11] V.H. Rumsey, *Reaction concept in electromagnetic theory.* Physical Rev. (2) **94** (1954), 1483–1491.

[12] M. Reed and B. Simon, *Method of Mathematical Physics*, Vols. 2 & 4. Academic Press, New Work, 1978.

[13] S.A. Shelkunoff, *Electromagnetic Waves.* Van Nostrand, New Work, 1983.

[14] C. Vassalo, *Théorie des guides d'ondes electromagnétiques*, Tomes 1 & 2. Ed. Eyrolles et CNET-ENST, Paris, 1985.

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