# DUALITY FOR NONLINEAR FRACTIONAL PROGRAMMING INVOLVING GENERALIZED $\rho$-SEMILOCALLY $b$-PREINVEX FUNCTIONS

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We consider a nonlinear fractional programming problem with inequality constraints, where the functions involved are $\rho$-semilocally $b$-preinvex, $\rho$-semilocally explicitly $b$-preinvex, $\rho$-semilocally quasi $b$-preinvex, $\rho$-semilocally pseudo $b$-preinvex and $\rho$-semi-locally strongly pseudo $b$-preinvex functions. Necessary optimality conditions are obtained in terms of the right derivative of a function along one direction. Wolfe and Mond-Weir type duals are associated, and weak, direct and strict converse duality are established. Our results generalize those obtained by Lyall, Suneja and Aggarwal [7], Patel [9], Stancu-Minasian [14], [16] and Stancu-Minasian and Andreea Mădălina Stancu [17], [18].

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## 1. INTRODUCTION

The convexity notion plays an important role in the mathematical programming field. Various generalizations of convex functions have appeared in literature.

In the last three decades, generalized convex functions received more attention since these functions are able to describe and treat better models of the real world from economics, decision science, management science, stochastics, applied mathematics, engineering, mechanics and other applied sciences. Therefore, researchers working in various fields may be interested in the subject of generalized convex functions. Among them we recall the class of semilocally convex functions introduced by Ewing [2] by reducing the width of the segment part and used by him to obtain sufficient optimality conditions in variational and control problems.

Kaul and Kaur [5] defined semilocally quasiconvex and semilocally pseudoconvex functions and obtained sufficient optimality conditions for a class of nonlinear programming problems involving such functions.

Gupta and Vartak [3] defined $\rho$-semilocally convex, $\rho$-semilocally quasiconvex and $\rho$-semilocally pseudoconvex functions. Sufficient optimality criteria were obtained by Gupta and Vartak [3] and Kaul and Kaur [4] for a nonlinear programming problem involving these functions. Kaur [6] obtained necessary optimality conditions and duality results by taking the objective and constraint function to be semillocaly convex and their right differentials at a point to be lower semicontinuous. Mathematical programming problems with semilocally convex ($\rho$-semilocal convex) objective functions were considered by Suneja and Gupta [20], [22], and Lyall, Suneja and Aggarwal [7].

Preda, Stancu-Minasian and Bătătorescu [10] introduced the concepts of semilocally preinvex, semilocally quasi-preinvex and semilocally pseudo-preinvex functions. Fritz John and Kuhn-Tucker necessary optimality conditions and sufficient optimality conditions were given and duality results stated for Wolfe and Mond-Weir types duals using these concepts. Stancu-Minasian [14] (see also [13]) considered a nonlinear fractional programming problem where the functions are $\eta$-semidifferentiable, semilocally preinvex, semilocally quasi-preinvex and semilocally pseudo-preinvex, and obtained necessary and sufficient optimality conditions. A dual was formulated and duality results were proved.

Another class of functions, more general than the class of semilocal preinvex functions is that of semilocally $b$-vex, semilocally quasi $b$-vex and semilocally pseudo $b$-vex functions introduced by Suneja and Gupta [19]. These authors established relationships between them and sufficient optimality criteria [21] for a class of nonlinear programming problems together with Wolfe and Mond-Weir types duals. Also, Patel [9] defined $\rho$-semilocally $b$-vex, $\rho$ -semilocally quasi $b$-vex, $\rho$-semilocally pseudo $b$-vex functions, and considered a nonlinear fractional programming problem involving these functions, associated a Wolfe dual and proved duality theorems.

In this paper, a nonlinear fractional programming problem with inequality constraints is considered, where the functions involved are $\rho$-semilocally $b$-preinvex, $\rho$-semilocally explicitly $b$-preinvex, $\rho$-semilocally quasi $b$-preinvex, $\rho$-semilocally pseudo $b$-preinvex and $\rho$-semilocally strongly pseudo $b$-preinvex. These functions were introduced by Stancu-Minasian and Andreea Mădălina Stancu [18], who obtained Fritz John and Karush-Kuhn-Tucker types necessary optimality conditions and Wolfe and Mond-Weir types duality results for a nonlinear programming problem. Here, necessary optimality conditions are obtained in terms of the right derivative of a function along one direction. Wolfe and Mond-Weir type duals are associated, and weak, direct and strict converse duality are established. Our results generalize those obtained by Lyall, Suneja and Aggarwal [7], Patel [9], Stancu-Minasian [14], [16] and Stancu-Minasian and Andreea Mădălina Stancu [17], [18].

## 2. DEFINITIONS AND PRELIMINARIES

In this section we introduce the notation and definitions which are used throughout the paper.

Let $\mathbf{R}^n$ be the $n$-dimensional Euclidean space and $\mathbf{R}^n_+$ its nonnegative orthant, i.e., $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n,\ x_j \geqq 0,\ j = 1, \dots, n\}$.

For $x, y \in \mathbf{R}^n$, by $x \leqq y$ we mean $x_i \leqq y_i$ for all $i$, $x \leq y$ means $x_i \leqq y_i$ for all $i$ and $x_j < y_j$ for at least one $j$, $1 \leqq j \leqq n$. By $x < y$ we mean $x_i < y_i$ for all $i$, and by $x \nleq y$ we mean the negation of $x \leq y$.

Throughout the paper, all definitions, theorems, lemmas, corollaries, remarks are numbered consecutively in a single numeration system in each section.

Let $X^0 \subseteq \mathbf{R}^n$ be a set and $\eta : X^0 \times X^0 \to \mathbf{R}^n$ a vector function.

*Definition* 2.1. A set $X^0 \subseteq \mathbf{R}^n$ is said to be **$\eta$-vex** at $\overline{x} \in X^0$ (with respect to the chosen fixed function $\eta$) if $\overline{x} + \lambda\eta\,(x, \overline{x}) \in X^0$ for all $x \in X^0$ and $\lambda \in [0, 1]$. The set $X^0$ is said to be $\eta$-vex if is $\eta$-vex at any $x \in X^0$.

*Definition* 2.2 ([1]). Let $\eta : \mathbf{R}^n \times \mathbf{R}^n$ be a vector function and $X^0$ a nonempty $\eta$-vex set. A function $f : X^0 \to \mathbf{R}$ is said to be **preinvex** on $X^0$ ($f$ is $\eta$-vex, for short) if we have

$$f\,(u + \lambda\eta\,(x, u)) \leqq \lambda f\,(x) + (1 - \lambda) f(u), \quad \forall x, u \in X^0,\ \lambda \in [0, 1].$$

*Definition* 2.3. A set $X^0$ is said to be an **$\eta$-locally starshaped set** at $\bar{x}$ ($\bar{x} \in X^0$) if for any $x \in X^0$ there exists a positive number $a_\eta(x, \bar{x})$, with $0 < a_\eta(x, \bar{x}) \leqq 1$ such that $\bar{x} + \lambda\eta(x, \bar{x}) \in X^0$ for any $\lambda \in [0, a_\eta(x, \bar{x})]$. The set $X^0$ is said to be $\eta$-locally starshaped if is $\eta$-locally starshaped at any $x \in X^0$.

*Definition* 2.4. Let $\eta : \mathbf{R}^n \times \mathbf{R}^n$ be a vector function and $X^0$ $\eta$-locally starshaped set at $\overline{x} \in X^0$, with the corresponding maximum positive number $a_\eta(x, \overline{x})$ satisfying the required conditions. Also let $\rho \in \mathbf{R}$ and $d\,(\cdot, \cdot) : X^0 \times X^0 \to \mathbf{R}_+$ such that $d\,(x, \overline{x}) \neq 0$ for $x \neq \overline{x}$. A function $f : X^0 \to \mathbf{R}$ is said to be:

($\mathrm{i}_1$) **$\rho$-semilocally $b$-preinvex** ($\rho$-sl$b$-preinvex) at $\overline{x}$ if for any $x \in X^0$, there exist a positive number $d_\eta\,(x, \overline{x}) \leqq a_\eta(x, \overline{x})$ and a function $b : X^0 \times X^0 \times [0, 1] \to \mathbf{R}_+$ such that

$$f\,(\overline{x} + \lambda\eta(x, \overline{x})) \leqq \lambda b(x, \bar{x}, \lambda) f(x) + (1 - \lambda b(x, \bar{x}, \lambda)) f(\overline{x}) - \rho\lambda d\,(x, \overline{x})$$

for $0 < \lambda < d_\eta(x, \overline{x}), \lambda b(x, \bar{x}, \lambda) \leqq 1$; if $f$ is $\rho$-semilocally $b$-preinvex at each $\overline{x} \in X^0$ for the same $\rho$ and $b$, then $f$ is said to be $\rho$-semilocally $b$-preinvex on $X^0$;

($\mathrm{i}_2$) **$\rho$-semilocally quasi $b$-preinvex** ($\rho$-slq$b$-preinvex) at $\overline{x}$ if for any $x \in X^0$, there exist a positive number $d_\eta\,(x, \overline{x}) \leqq a_\eta(x, \overline{x})$ and a function

$b : X^0 \times X^0 \times [0,1] \to \mathbf{R}_+$ such that

$$\left.\begin{array}{r} f(x) \leqq f(\overline{x}) \\ 0<\lambda<d_\eta(x,\overline{x}) \\ \lambda b(x,\bar{x},\lambda) \leqq 1 \end{array}\right\} \Rightarrow b\,(x,\overline{x},\lambda)\, f\,[\overline{x}+\lambda\eta(x,\overline{x})] \leqq b\,(x,\overline{x},\lambda)\, f(\overline{x})-\rho\lambda d(x,\overline{x});$$

if $f$ is $\rho$-semilocally quasi $b$-preinvex at each $\overline{x} \in X^0$ for the same $\rho$ and $b$, then $f$ is said to be $\rho$-semilocally quasi $b$-preinvex on $X^0$.

We remark that although the cases $\eta(x,\overline{x}) \equiv 0$ or $b\,(x,\overline{x},\lambda) \equiv 0$ are not excluded, in these definitions we agree not to take them into account.

*Definition* 2.5 ([10]). Let $f : X^0 \to \mathbf{R}$ be a function, where $X^0$ is an $\eta$-locally starshaped set at $\overline{x} \in X^0$. We say that $f$ is **$\eta$-semidifferentiable** at $\overline{x}$ if $(\mathrm{d}f)^+(\overline{x},\eta(x,\overline{x}))$ exists for each $x \in X^0$, where

$$(\mathrm{d}f)^+(\overline{x},\eta\,(x,\overline{x})) = \lim_{\lambda\to 0^+} \frac{1}{\lambda}\,[f(\overline{x}+\lambda\eta(x,\overline{x})) - f\,(\overline{x})]$$

(the right derivative at $\overline{x}$ along the direction $\eta(x,\overline{x})$). If $f$ is $\eta$-semidifferentiable at any $\overline{x} \in X^0$, then $f$ is said to be $\eta$-semidifferentiable on $X^0$.

*Definition* 2.6. Let $f : X^0 \to \mathbf{R}$ be an $\eta$-semidifferentiable function on $X^0$. We say that $f$ is **$\rho$-semilocally pseudo $b$-preinvex** ($\rho$-slp$b$-preinvex) at $\overline{x} \in X^0$ if

$$(\mathrm{d}f)^+(\overline{x},\eta\,(x,\overline{x})) \geqq -\rho d(x,\overline{x}) \Rightarrow b\,(x,\overline{x},\lambda)\, f(x) \geqq b\,(x,\overline{x},\lambda)\, f(\overline{x}).$$

If $f$ is $\rho$-semilocally pseudo $b$-preinvex at each $\overline{x} \in X^0$ for the same $\rho$ and $b$, then $f$ is said to be $\rho$-semilocally pseudo $b$-preinvex on $X^0$.

*Definition* 2.7. Let $f : X^0 \to \mathbf{R}$ be an $\eta$-semidifferentiable function on $X^0$. We say that $f$ is **$\rho$-semilocally explicitly $b$-preinvex** ($\rho$-sle$b$-preinvex)at $\overline{x} \in X^0$ if

$$\overline{b}(x,\overline{x})\,[f(x) - f\,(\overline{x})] > (\mathrm{d}f)^+(\overline{x},\eta(x,\overline{x})) + \rho d(x,\overline{x})$$

for each $x \in X^0$, $x \neq \overline{x}$, where

$$\overline{b}(x,\overline{x}) = \lim_{\lambda\to 0^+} b\,(x,\overline{x},\lambda) \quad \text{and} \quad \lambda b\,(x,\overline{x},\lambda) \leqq 1. \tag{2.1}$$

*Definition* 2.8. Let $f : X^0 \to \mathbf{R}$ be an $\eta$-semidifferentiable function on $X^0$. We say that $f$ is **$\rho$-semilocally strongly pseudo $b$-preinvex** ($\rho$-slsp$b$-preinvex) at $\bar{x} \in X^0$ if

$$\overline{b}(x,\overline{x})(\mathrm{d}f)^+(\bar{x},\eta(x,\bar{x})) \geqq -\rho d(x,\overline{x}) \Rightarrow f(x) \geqq f(\overline{x}),$$

where $\overline{b}(x,\overline{x})$ is defined by (2.1). If $f$ is $\rho$-slsp$b$-preinvex at each $\bar{x} \in X^0$ for the same $\rho$ and $b$, then $f$ is said to be $\rho$-slsp$b$-preinvex on $X^0$.

An $m$-dimensional vector-valued function $\varphi : X^0 \to \mathbf{R}^m$, is said to be $\rho$-sl$b$ $b$-preinvex ($\rho$-slq $b$-preinvex, $\rho$-slpb-preinvex, $\rho$-slspb-preinvex, $\rho$-sleb-preinvex) on $X^0$ if each of its components is $\rho$-sl$b$-preinvex ($\rho$-slq$b$-preinvex, $\rho$-slpb-preinvex, $\rho$-slspb-preinvex, $\rho$-sleb-preinvex) on $X^0$.

*Remark* 2.9. These definitions reduce to those of $\rho$ -semilocally convex, $\rho$-semilocally quasiconvex, $\rho$-semilocally pseudoconvex considered by Gupta and Vartak [3] (when $b(x,\bar{x},\lambda) \equiv 1$, $\eta(x,\overline{x}) = x - \overline{x}$ and $d(x,\overline{x}) = \|x-\overline{x}\|^2$), to those of $\rho$-semilocally $b$-vex, $\rho$ -semilocally quasi $b$-vex, $\rho$-semilocally pseudo $b$-vex considered by Patel [9] (when $\eta(x,\overline{x}) = x - \overline{x}$ and $d(x,\overline{x}) = \|x-\overline{x}\|^2$), to those of semilocally preinvex, semilocally quasi-preinvex, semilocally pseudo-preinvex considered by Preda, Stancu-Minasian and Bătătorescu [10] (when $b(x,\bar{x},\lambda) \equiv 1$ and $\rho = 0$), to those of semilocally $b$-preinvex, semilocally quasi $b$-preinvex, semilocally pseudo $b$-preinvex, semilocally explicitly $b$-preinvex and semilocally strongly pseudo $b$-preinvex considered by Stancu-Minasian [16] (when $\rho = 0$).

If $f$ is $\eta$-semidifferentiable, an alternate and equivalent definition of $\rho$-semilocally $b$-preinvex and $\rho$-semilocally quasi $b$-preinvex functions is given in Theorem 2.10. These results were proved in [17] and follow from the above definitions.

THEOREM 2.10. *Let $f : X^0 \to \mathbf{R}$ be an $\eta$-semidifferentiable function on an $\eta$-locally starshaped set $X^0$.*

(a) *If $f$ is $\rho$-sl$b$-preinvex at $\bar{x} \in X^0$, then*

$$\overline{b}(x,\overline{x})[f(x) - f(\overline{x})] \geqq (\mathrm{d}f)^+(\bar{x},\eta(x,\bar{x})) + \rho d(x,\overline{x}), \quad \forall x \in X^0. \tag{2.2}$$

(b) *If $f$ is $\rho$-slq$b$-preinvex at $\bar{x} \in X^0$, then*

$$f(x) \leqq f(\overline{x}) \Rightarrow \overline{b}(x,\overline{x})(\mathrm{d}f)^+(\bar{x},\eta(x,\bar{x})) \leqq -\rho d(x,\overline{x}), \quad \forall x \in X^0,$$

*where* $\overline{b}(x,\overline{x}) = \lim\limits_{\lambda\to 0^+} b(x,\overline{x},\lambda)$ *and* $\lambda b\,(x,\bar{x},\lambda) \leqq 1$.

*Remark* 2.11. In Definitions 2.1 and 2.2 of $\eta$-vex set and $\eta$-functions we do not used the classical phrase "... if there exists a function $\eta(\cdot)$ such that ...", which was originally used by various Australian authors. The problem is that using this "existence" approach, the $\eta$'s of the domain sets and of the preinvex functions may differ, thus able to lead to errors.

## 3. NECESSARY OPTIMALITY CRITERIA

Consider the nonlinear fractional programming problem

$$\text{(P)} \qquad \text{Minimize } q(x) = \frac{f(x)}{g(x)}$$

subject to

$$h(x) \leqq 0, \quad x \in X^0,$$

where

(i) $X^0 \subseteq \mathbf{R^n}$ is a nonempty $\eta$-locally starshaped set;

(ii) $f : X^0 \to \mathbf{R},\ f(x) \geqq 0,\ \forall\, x \in X^0$;

(iii) $g : X^0 \to \mathbf{R},\ g(x) > 0,\ \forall\, x \in X^0$;

(iv) $h = (h_i)_{1 \leqq i \leqq m} : X^0 \to \mathbf{R}^m$;

(v) the right differentials of $f, g$ and $h_j,\ j = 1, \ldots, m$, at $\bar{x}$, along the direction $\eta\,(x, \overline{x})$ do exist.

Let $X = \{x \in X^0 \mid h(x) \ \leqq 0\}$ be the set of all feasible solutions to (P), and

$$N_\varepsilon\,(\bar{x}) = \{x \in \mathbf{R}^n \mid \|x - \overline{x}\| < \varepsilon\}.$$

*Definition* 3.1. (a) $\bar{x}$ is said to be a *local minimum solution* to Problem (P) if $\bar{x} \in X$ and there exists $\varepsilon > 0$ such that $x \in N_\varepsilon(\overline{x}) \cap X \Rightarrow f(\bar{x}) \leqq f(x)$.

(b) $\bar{x}$ is said to be the *minimum solution* to problem (P) if $\bar{x} \in X$ and $f\,(\bar{x}) = \min\limits_{x \in X} f(x)$.

For $\bar{x} \in X$ we define the sets of subscripts $I = I\,(\bar{x}) = \{i \mid h_i\,(\bar{x}) = 0\}$, $J = J\,(\bar{x}) = \{i \mid h_i\,(\bar{x}) < 0\}$, and denote $h_I = (h_i)_{i \in I}$ . Obviously, $I \cup J = \{1, 2, \ldots, m\}$ .

Now, we introduce

*Definition* 3.2. We say that the function $h$ satisfies the generalized Slater's constraint qualification (GSQ) at $\bar{x} \in X$, if $h_i$ is $\rho$-semilocally strongly pseudo $b$-preinvex at $\bar{x}$ with $\rho \geqq 0$, and there exists $\hat{x} \in X$ such that $h_i(\hat{x}) < 0$ for $i \in I$.

In what follows we need the following theorem of the alternative stated by Weir and Mond [22].

Theorem 3.3 ([22], Theorem 2.1). *Let $S$ be a nonempty set in $\mathbf{R}^n$ and $f : S \to \mathbf{R}^p$, a preinvex function on $X^0$ (with respect to $\eta$). Then either*

*$f(x) < 0$ has a solution $x \in S$*

*or*

*$\lambda^t f(x) \geqq 0$ for all $x \in S$, for some $\lambda \in \mathbf{R}^p,\ \lambda \geq 0$,*
*but both alternatives are never true simultaneously.*

The following result can be proved on the lines of Mangasarian [8] and Stancu-Minasian [14].

Lemma 3.4. *Let $\bar{x} \in X$ be a (local) minimum solution for* (P). *Assume that $h_i$ is continuous at $\bar{x}$ for any $i \in J$, and that $f, g$ and $h_I$ are*

*$\eta$-semidifferentiable at $\overline{x}$. Then the system*

$$(3.1)\quad (\mathrm{d}f)^{+}(\overline{x},\eta(x,\overline{x}))<0,\quad (\mathrm{d}g)^{+}(\overline{x},\eta(x,\overline{x}))>0,\quad (\mathrm{d}h_I)^{+}(\overline{x},\eta(x,\overline{x}))<0,$$

*has no solution $x \in X^0$.*

Now, we consider the parametric problem

$$(\mathrm{P}_\lambda)\qquad \text{Min } f(x)-\lambda g(x),\quad \lambda \in \mathbf{R}\ (\lambda \text{ a parameter})$$

subject to $h(x) \leqq 0,\ x \in X^0$.

It is well known that $(\mathrm{P}_\lambda)$ is closely related to problem (P).

The next result is well known in fractional programming [12] and establishes a connection between the fractional programming problem (P) and a certain parametric programming problem $(\mathrm{P}_\lambda)$.

LEMMA 3.5. *$\bar{x}$ is an optimal solution to Problem* (P) *if and only if it is optimal solution to Problem* $(\mathrm{P}_\lambda)$ *with* $\bar{\lambda} = f(\bar{x})/g(\bar{x})$.

We shall use Karush-Kuhn-Tucker necessary optimality conditions for $(\mathrm{P}_\lambda)$ as given by Stancu-Minasian and Andreea Mădălina Stancu [18].

THEOREM 3.6 (Karush-Kuhn-Tucker Necessary Optimality Conditions). *Let $\bar{x}$ be a* (*local*) *minimum solution to Problem* (P) *and let $h_i$ be continuous at $\bar{x}$ for $i \in J$. Assume also that $\left(\mathrm{d}(f-\bar{\lambda}g)\right)^{+}(\overline{x},\eta(x,\overline{x}))$ and $(\mathrm{d}g_I)^{+}(\overline{x},\eta(x,\overline{x}))$ are preinvex functions of $x$ on $X^0$, which is an $\eta$-locally starshaped set. If $h$ satisfies* GSQ *at $\bar{x}$ with $\overline{b}(x,\overline{x}) = \lim_{\lambda\to 0^+} b(x,\overline{x},\lambda) > 0$, then there exists $\bar{y} \in \mathbf{R}^m$ such that*

$$(3.2)\qquad \left(\mathrm{d}(f-\bar{\lambda}g)\right)^{+}(\bar{x},\eta(x,\overline{x})) + \bar{y}^t(\mathrm{d}h)^{+}(\bar{x},\eta(x,\overline{x})) \geqq 0,\quad \forall x \in X^0,$$

$$(3.3)\qquad f(\bar{x})-\bar{\lambda}g(\bar{x})=0,$$

$$(3.4)\qquad \bar{y}^t h(\bar{x})=0,$$

$$(3.5)\qquad \bar{y} \geqq 0.$$

*Proof.* Let $\bar{x} \in X$ be a (local) optimal solution to Problem (P). By Lemma 3.5, $\bar{x}$ is also a (local) optimal solution to Problem $(\mathrm{P}_{\bar{\lambda}})$, where $\bar{\lambda} = f(\bar{x})/g(\bar{x})$. Since the conditions of Lemma 3.4 are satisfied, system (3.1) has no solution $x \in X^0$. Also the system

$$(3.6)\quad (\mathrm{d}f)^{+}(\overline{x},\eta(x,\overline{x}))-\bar{\lambda}(\mathrm{d}g)^{+}(\overline{x},\eta(x,\overline{x}))<0,\quad (\mathrm{d}h_I)^{+}(\overline{x},\eta(x,\overline{x}))<0$$

has no solution $x \in X^0$. Therefore, by Theorem 3.3 there exist $u_0^* \in \mathbf{R}$, $u_i^* \in \mathbf{R}$, $i \in I(\bar{x})$, such that

$$(3.7)\qquad \begin{aligned} &u_0^*\left[(\mathrm{d}f)^{+}(\bar{x},\eta(x,\overline{x})) - \bar{\lambda}(\mathrm{d}g)^{+}(\bar{x},\eta(x,\overline{x}))\right]+\\ &+u_I^{*t}(\mathrm{d}h_I)^{+}(\bar{x},\eta(x,\overline{x})) \geqq 0,\quad \forall x \in X^0 \end{aligned}$$

$$h\left(\bar{x}\right) \leqq 0 \tag{3.8}$$

$$\left(u_0^*, u_I^*\right) \geq 0. \tag{3.9}$$

If we define $u_J^* = 0$, by (3.7) we get

$$\begin{aligned} &u_0^* \left[ (\mathrm{d}f)^+ \left(\bar{x}, \eta\left(x, \overline{x}\right)\right) - \bar{\lambda}\left(\mathrm{d}g\right)^+ \left(\bar{x}, \eta(x, \overline{x})\right) \right] + \\ &\quad + u^{*t} \left(\mathrm{d}h\right)^+ \left(\bar{x}, \eta\left(x, \overline{x}\right)\right) \geqq 0, \quad \forall x \in X^0 \end{aligned} \tag{3.10}$$

and from (3.9), we have

$$\left(u_0^*, u^*\right) \geq 0, \tag{3.11}$$

where $u^* = \left(u_I^*, u_J^*\right)$.

Since $h_I\left(\bar{x}\right) = 0$, for $u^* = \left(u_I^*, u_J^*\right)$ we have

$$u^* h\left(\bar{x}\right) = 0. \tag{3.12}$$

Now, suppose that $u_0^* = 0$. Then (3.8), (3.10), (3.11), and (3.12) reduce to

$$u^{*t} \left(\mathrm{d}h\right)^+ \left(\bar{x}, \eta\left(x, \overline{x}\right)\right) \geqq 0, \quad \forall x \in X^0 \tag{3.13}$$

$$u^* h\left(\bar{x}\right) = 0, \tag{3.14}$$

$$h\left(\bar{x}\right) \leqq 0, \tag{3.15}$$

$$u^* \geq 0. \tag{3.16}$$

Using the generalized Slater's constraint qualification at $\bar{x}$ of $h$, we deduce that $h_i\left(\cdot\right)$ is $\rho$-semilocally strongly pseudo $b$-preinvex at $\bar{x}$ and there exists $\widehat{x} \in X$ such that

$$h_I\left(\widehat{x}\right) < 0. \tag{3.17}$$

Since $u_J^* = 0$, (3.13) becomes

$$u_I^{*t} \left(\mathrm{d}h_I\right)^+ \left(\bar{x}, \eta\left(x, \overline{x}\right)\right) \geqq 0. \tag{3.18}$$

As $h_I\left(\bar{x}\right) = 0$, by (3.17) we obtain $h_I\left(\widehat{x}\right) < h_I\left(\bar{x}\right)$.

Now, by $\rho$-slsp $b$-preinvexity of $h_I$ at $\bar{x}$, since $\rho \geqq 0$ and $d\left(\bar{x}, \widehat{x}\right) \geqq 0$, by Definition 2.8, we have

$$\overline{b}\left(\widehat{x}, \overline{x}\right) \left(\mathrm{d}h_I\right)^+ \left(\bar{x}, \eta\left(\widehat{x}, \overline{x}\right)\right) < -\rho d\left(\bar{x}, \widehat{x}\right) \leqq 0. \tag{3.19}$$

Combining (3.16) with (3.19), as $\overline{b}\left(\widehat{x}, \overline{x}\right) > 0$, we obtain

$$u_I^{*t} \left(\mathrm{d}h_I\right)^+ \left(\bar{x}, \eta\left(\widehat{x}, \overline{x}\right)\right) < 0,$$

which contradicts (3.18). Thus, $u_0^* \neq 0$.

If we put $\bar{y} = \frac{u^*}{u_0^*}$ from (3.10)–(3.12) we get the conditions in the statement of the theorem, and the proof is complete. □

## 4. **WOLFE DUALITY**

With (P) we associate the Wolfe dual stated as

$$\text{(D)} \qquad \max \varphi(u,v,y) = (f(u) - vg(u)) + y^t h(u)$$

subject to

$$(\mathrm{d}(f-vg))^+ (u, \eta(u,x)) + y^t (\mathrm{d}h)^+ (u, \eta(u,x)) \geqq 0, \quad \forall x \in X^0,$$

$$f(u) - vg(u) \geqq 0, \quad y \geqq 0;$$

$u \in X^0,\ y \in \mathbf{R}^m$. Let $W$ denote the set of all feasible solutions to Problem (D).

The next theorems show that Problem (D) is a dual problem to (P).

Theorem 4.1 (Weak Duality). *Let $x^0 \in X$ and $(u^0, v^0, y^0) \in W$. Assume that*

a$_1$) *$f - vg$ is $\rho$-slb-preinvex and $h_j,\ j = 1,\dots,m$, are $\rho_j$-slb-preinvex on $X^0$ with $\overline{b}(x^0,u^0) = \lim_{\lambda\to 0^+} b(x^0,u^0,\lambda) > 0$;*

a$_2$) $\rho + \sum_{j=1}^m \rho_j y_j^0 \geqq 0$.

*Then*

$$f(x^0) - v^0 g(x^0) \geqq \varphi(u^0, v^0, y^0).$$

*Proof.* By condition (a$_1$) and Theorem 2.10 we have

$$\begin{aligned} \overline{b}(x^0,u^0)\left\{\left[f(x^0) - v^0 g(x^0)\right] - \left[f(u^0) - v^0 g(u^0)\right]\right\} &\geqq \\ \geqq (\mathrm{d}(f-vg))^+ \left(u^0, \eta(x^0,u^0)\right) + \rho d(x^0,u^0) \end{aligned} \tag{4.1}$$

and

$$\overline{b}(x^0,u^0)\left[h_j(x^0) - h_j(u^0)\right] \geqq (\mathrm{d}h_j)^+ \left(u^0, \eta(x^0,u^0)\right) + \rho_j d(x^0,u^0), \tag{4.2}$$

for all $j = 1,\dots,m$.

From (4.1) and $(u^0,v^0,y^0) \in W$ we have

$$\overline{b}(x^0,u^0)\left\{\left[f(x^0) - v^0 g(x^0)\right] - \left[f(u^0) - v^0 g(u^0)\right]\right\} \geqq$$

$$\geqq -y^{0t} (\mathrm{d}h)^+ \left(u^0, \eta(x^0,u^0)\right) + \rho d(x^0,u^0) \geqq$$

$$\geqq -\sum_{j=1}^m y_j^0 (\mathrm{d}h_j)^+ \left(u^0, \eta(x^0,u^0)\right) - \sum_{j=1}^m \rho_j y_j^0 d(x^0,u^0) \ \left(\text{as } \rho + \sum_{j=1}^m \rho_j y_j^0 \geqq 0\right) =$$

$$= -\sum_{j=1}^m y_j^0 \left[(\mathrm{d}h_j)^+ u^0, \eta(x^0,u^0) + \rho_j d(x^0,u^0)\right]$$

$$\geqq -\sum_{j=1}^m y_j^0 \left(h_j(x^0) - h_j(u^0)\right) \ (\text{by (4.2)}) =$$

$$= -\sum_{j=1}^{m} y_j^0 h_j(x^0) + \sum_{j=1}^{m} y_j^0 h_j(u^0) \geqq \sum_{j=1}^{m} y_j^0 h_j(u^0) \text{ (as } h_j\left(x^0\right) \leqq 0 \text{ and } y_j^0 \geqq 0).$$

So, as $\overline{b}\left(x^0, u^0\right) > 0$, it follows that

$$f(x^0) - v^0 g(x^0) \geqq f(u^0) - v^0 g(u^0) + y^{0t} h(u^0) = \varphi\left(u^0, v^0, y^0\right)$$

and the proof is complete. □

Corollary 4.2. *Let* $x^0 \in X$ *and* $\left(u^0, v^0, y^0\right) \in W$ *such that* $f(x^0) - v^0 g(x^0) = \varphi\left(u^0, v^0, y^0\right)$. *If the hypotheses of Theorem* 4.1 *are satisfied, then* $x^0$ *is an optimal solution to Problem* (P) *and* $\left(u^0, v^0, y^0\right)$ *is an optimal solution to Problem* (D).

Theorem 4.3 (Direct Duality). *Let* $x^0$ *be a* (*local*) *optimal solution to Problem* (P) *such that*

($\mathrm{b}_1$) *for each* $i \in J$, $h_i$ *is continuous at* $x^0$;

($\mathrm{b}_2$) $(\mathrm{d}(f - vg))^+\left(x^0, \eta(x, x^0)\right)$ *and* $(\mathrm{d}h)^+\left(x^0, \eta(x, x^0)\right)$ *are preinvex functions of* $x \in X^0$;

($\mathrm{b}_3$) $h$ *satisfies* GSQ *at* $x^0$;

($\mathrm{b}_4$) $f - vg$ *is* $\rho$*-slb-preinvex on* $X^0$, *with* $\overline{b}(x, x^0) = \lim\limits_{\lambda \to 0^+} b\left(x, x^0, \lambda\right) > 0$;

($\mathrm{b}_5$) *for each* $j = 1, 2, \ldots, m$, $h_j$ *is* $\rho_j$*-slb-preinvex on* $X^0$.

*Then there exist* $v^0 \in \mathbf{R}$ *and* $y^0 \in \mathbf{R}^m$ *such that* $\left(x^0, v^0, y^0\right)$ *is an optimal solution to Problem* (D) *and* $f(x^0) - v^0 g\left(x^0\right) = \varphi\left(x^0, v^0, y^0\right)$ *provided* $\rho + \sum\limits_{j=1}^{m} \rho_j y_j^0 \geqq 0$ *and* $\rho_j \geqq 0$ *for* $j \in I$.

*Proof.* Since $\rho_j \geqq 0$ for $j \in I$, and $h_I$ is $\rho$-sl$b$-preinvex for all $j = 1, \ldots, m$, $\rho$-slp$b$-preinvex. Thus, $x^0$ satisfies the conditions of Theorem 3.6 and there exist $v^0 \in \mathbf{R}$ and $y^0 \in \mathbf{R}^m$ such that $\left(x^0, v^0, y^0\right)$ is feasible to Problem (D) and $y^{0t} h\left(x^0\right) = 0$. Therefore, by Corollary 4.2, $\left(x^0, v^0, y^0\right)$ is an optimal solution to Problem (D) and

$$f(x^0) - v^0 g(x^0) = \varphi\left(x^0, v^0, y^0\right). \quad \square$$

Theorem 4.4 (Strict Converse Duality). *Let* $x^0$ *and* $(x^*, v^*, y^*)$ *be a* (*local*) *optimal solution to* (P) *and* (D), *respectively, such that*

($\mathrm{c}_1$) *for each* $i \in J$, $h_i$ *is continuous at* $x^0$;

($\mathrm{c}_2$) $(\mathrm{d}(f - vg))^+\left(x^0, \eta(x, x^0)\right)$ *and* $(\mathrm{d}h)^+\left(x^0, \eta(x, x^0)\right)$ *are preinvex functions of* $x \in X^0$;

($\mathrm{c}_3$) $h$ *satisfies* GSQ *at* $x^0$;

($\mathrm{c}_4$) $f - vg$ *is* $\rho$*-sleb-preinvex on* $X^0$, *with* $\overline{b}\left(x^0, x^*\right) = \lim\limits_{\lambda \to 0^+} b\left(x^0, x^*, \lambda\right) > 0$;

($\mathrm{c}_5$) *for each* $j = 1, 2, \ldots, m$, $h_j$ *is* $\rho_j$-slb-*preinvex on* $X^0$;

(c$_6$) $\rho + \sum_{j=1}^{m} \rho_j y_j^* \geqq 0.$

*Then* $x^* = x^0$ *and* $f(x^0) - v^0 g\left(x^0\right) = \rho\left(x^0, v^0, y^0\right).$

*Proof.* Assume for a contradiction that $x^* \neq x^0$. Since $x^0$ is an optimal solution to (P), by Lemma 3.5 it also is an optimal solution to (P$_{v^0}$). It then follows from Theorem 4.3 that there exist $v^0 \in \mathbf{R}$ and $y^0 \in \mathbf{R}^m$ such that $\left(x^0, v^0, y^0\right)$is a feasible solution to (D) and $y^{0t}h(x^0) = 0$ i.e. $f(x^0) - v^0 g\left(x^0\right) = \varphi\left(x^0, v^0, y^0\right)$. Since $(x^*, v^*, y^*)$ is an optimal solution to Problem (D) and $x^0 \in X$, we have

$$(4.3) \qquad (\mathrm{d}\,(f - v^* g))^+ \left(x^*, \eta\left(x^0, x^*\right)\right) + y^{*t}\,(\mathrm{d}h)^+ \left(x^*, \eta\left(x^0, x^*\right)\right) \geqq 0,$$

From the $\rho$-sl$b$-preinvex assumption of $f - vg$ and $x^* \neq x^0$, by Theorem 2.5 we have

$$\overline{b}\left(x^0, x^*\right)\left\{\left[f\left(x^0\right) - v^0 g(x^0)\right] - \left[f\left(x^*\right) - v^* g\left(x^*\right)\right]\right\} >$$
$$> (\mathrm{d}(f - vg))^+ \left(x^*, \eta\left(x^0, x^*\right)\right) + \rho d(x^0, x^*)$$

while from (c$_5$) and keeping in mind that $y^* \geqq 0$, we have

$$\overline{b}(x^0, x^*) y^{*t}\left(h(x^0) - h(x^*)\right) \geqq \sum_{j=1}^{m} y_j^* (\mathrm{d}h_j)^+ \left(x^*, \eta(x^0, x^*)\right) + \sum_{j=1}^{m} \rho_j y_j^* d(x^0, x^*).$$

Now, adding these inequalities, by (c$_6$) and (4.3) we obtain

$$\overline{b}\left(x^0, x^*\right)\left\{\left[f\left(x^0\right) - v^0 g(x^0)\right] - \left[f\left(x^*\right) - v^* g\left(x^*\right)\right] + y^{*t}\left(h(x^0) - h\left(x^*\right)\right)\right\} >$$
$$> (\mathrm{d}\,(f - vg))^t \left(x^*, \eta\left(x^0, x^*\right)\right) +$$
$$+ \sum_{j=1}^{m} y_j^*\,(\mathrm{d}h_j)^+ \left(x^*, \eta(x^0, x^*)\right) + \left(\rho + \sum_{j=1}^{m} \rho_j y_j^*\right) d(x^0, x^*) \geqq 0,$$

i.e.,

$$(4.4) \qquad \varphi\left(x^0, v^0, y^*\right) > \varphi\left(x^*, v^*, y^*\right).$$

We also have $\varphi\left(x^*, v^*, y^*\right) \geqq \varphi\left(x^0, v^0, y^0\right)$ because $(x^*, v^*, y^*)$ is an optimal solution to Problem (D). Hence, by (4.4),

$$(4.5) \qquad \varphi\left(x^0, v^0, y^*\right) > \varphi\left(x^0, v^0, y^0\right).$$

Since $y^{0t}h(x^0) = 0$, from (4.5) we have

$$(4.6) \qquad y^{*t}h(x^0) > 0.$$

But $h(x^0) \leqq 0$ and $y^* \geqq 0$ yield $y^{*t}h(x^0) \leqq 0$ which contradicts (4.6). Therefore, $x^* = x^0$ and $f(x^0) - v^0 g(x^0) = \varphi(x^0, v^0, y^*)$, and the proof is complete. $\square$

*Remark* 4.5. (a) If $g(x) \equiv$ constant for all $x \in X^0$ we obtain the Wolfe dual considered by Stancu-Minasian and Andreea Mădălina Stancu [17],

(b) If $g(x) \equiv$ constant for all $x \in X^0$ and $\rho = 0$, we obtain the Wolfe dual considered by Stancu-Minasian [16],

(c) If $\rho = 0$, $\eta(x, x^0) = x - x^0$ and $d(x, x^0) = \|x - x^0\|^2$ we obtain the Wolfe dual considered by Patel [9].

## 5. MOND-WEIR DUALITY

For Problem (P) we consider the Mond-Weir dual problem

$$(\mathrm{D}_1) \qquad \min v(\lambda) = \lambda$$

subject to

$$(5.1) \quad (\mathrm{d}f)^+(y, \eta(x, y)) - \lambda(\mathrm{d}g)^+(y, \eta(x, y)) + u^t(\mathrm{d}h)^+(y, \eta(x, y)) \geqq 0, \ \forall x \in X,$$

$$(5.2) \qquad f(y) - \lambda g(y) \geqq 0,$$

$$(5.3) \qquad u^t h(y) \geqq 0,$$

$$(5.4) \qquad u \geqq 0, \ y \in X^0, \ u \in \mathbf{R}^m, \ \lambda \in \mathbf{R}, \ \lambda \geqq 0.$$

Let $T$ denote the set of all feasible solutions to Problem $(\mathrm{D}_1)$.

THEOREM 5.1 (Weak Duality). *Let $x^0 \in X$ and $(y^0, \lambda^0, u^0) \in T$. Assume that*

($a_1$) *$f$ is $\sigma$-slb$_1$-preinvex at $x^0$, $-g$ is $\tau$-slb$_1$-preinvex at $x^0$, and $h_j$ is $\rho_j$-slb$_2$-preinvex at $x^0$, $j = 1, \ldots, m$, with $\overline{b}_i(x^0, y^0) = \lim_{\lambda \to 0^+} b_i(x^0, y^0, \lambda) > 0$, $i = 1, 2$;*

($a_2$) $\sigma + \lambda^0 \tau + \sum_{j=1}^{m} \rho_j u_j^0 \geqq 0$.

*Then* $\frac{f(x^0)}{g(x^0)} \geqq v(\lambda^0)$.

*Proof.* Assumption ($a_1$) yields

$$(5.5) \qquad \overline{b}_1(x^0, y^0)[f(x^0) - f(y^0)] \geqq (\mathrm{d}f)^+(y^0, \eta(x^0, y^0)) + \sigma d(x^0, y^0)$$

and

$$(5.6) \qquad \overline{b}_1(x^0, y^0)[-g(x^0) + g(y^0)] \geqq -(\mathrm{d}g)^+(y^0, \eta(x^0, y^0)) + \tau d(x^0, y^0).$$

For each $j = 1, \ldots, m$ the function $h_j$ is $\rho_j$-slb$_2$-preinvex at $x^0$ and, therefore, by Theorem 2.10 we have

$$\overline{b}_2(x^0, y^0)\left[h_j(x^0) - h_j(y^0)\right] \geqq (\mathrm{d}h_j)^+\left(y^0, \eta(x^0, y^0)\right) + \rho_j d(x^0, y^0),$$

for all $j = 1, \dots, m$. Multiplying by $u_j^0 \geqq 0$, and adding these inequalities yield

(5.7)
$$\overline{b}_2(x^0,y^0)\sum_{j=1}^{m}\overline{u}_j\left[h_j(x^0)-h_j(y^0)\right]\geqq u^{0t}(\mathrm{d}h)^+\left(y^0,\eta\left(x^0,y^0\right)\right)+\sum_{j=1}^{m}u_j^0\rho_j d\left(x^0,y^0\right).$$

From (5.5)–(5.7) and (5.1) we obtain

$$\overline{b}_1(x^0,y^0)\left[f(x^0)-f(y^0)\right]\geqq(\mathrm{d}f)^+\left(y^0,\eta(x^0,y^0)\right)+\sigma d\left(x^0,y^0\right)\geqq$$

$$\geqq\lambda^0\,(\mathrm{d}g)^+\left(y^0,\eta(x^0,y^0)\right)-u^{0t}\,(\mathrm{d}h)^+\left(y^0,\eta(x^0,y^0)\right)+\sigma d\left(x^0,y^0\right)\geqq$$

$$\geqq\lambda^0\overline{b}_1(x^0,y^0)\left(g\left(x^0\right)-g(y^0)\right)-\overline{b}_2\left(x^0,y^0\right)\sum_{j=1}^{m}u_j^0\left[h_j\left(x^0\right)-h_j(y^0)\right]+$$

$$+\left(\sigma+\lambda^0\sigma+\sum_{j=1}^{m}\rho_j u_j^0\right)d(x^0,y^0)=$$

$$=\lambda^0\overline{b}_1(x^0,y^0)\left(g\left(x^0\right)-g(y^0)\right)-\overline{b}_2\left(x^0,y^0\right)u^{0t}h(x^0)+\overline{b}_2\left(x^0,y^0\right)u^{0t}h(y^0)+$$

$$+\left(\sigma+\lambda^0\sigma+\sum_{j=1}^{m}u_j^0\rho_j\right)d(x^0,y^0).$$

As $u^0 \geqq 0$, $h(x^0) \leqq 0$, $\overline{b}_2(x^0,y^0) \geqq 0$ and $\sigma+\lambda^0\sigma+\sum_{j=1}^{m}\rho_j u_j^0 \geqq 0$, this along with (5.3) and ($a_3$), yields

$$\overline{b}_1(x^0,y^0)[f(x^0)-f(y^0)]\geqq\lambda^0\overline{b}_1(x^0,y^0)[g(x^0)-g(y^0)].$$

Since $\overline{b}_1(x^0,y^0) > 0$, we have $f(x^0)-f(y^0) \geqq \lambda^0\left(g(x^0)-g(y^0)\right)$ or $f(x^0)-\lambda^0 g(x^0) \geqq f(y^0)-\lambda^0 g(y^0) \geqq 0$ (using 5.2).

Hence, $\frac{f(x^0)}{g(x^0)} \geqq \lambda^0$, thus completing the proof. □

Theorem 5.2 (Weak Duality). *Let $x^0 \in X$ and $\left(y^0,\lambda^0,u^0\right) \in T$. Assume that*

($b_1$) $\left(f-\lambda^0 g\right)$ *is* $\rho$-slb$_1$-*preinvex at* $x^0$, *and* $h_j$ *is* $\rho_j$-slb$_2$-*preinvex at* $x^0$, $j = 1,\dots,m$, *with* $\overline{b}_i(x^0,y^0) = \lim_{\lambda\to 0^+} b_i\left(x^0,y^0,\lambda\right) > 0$, $i = 1,2$;

($b_2$) $\rho+\sum_{j=1}^{m}\rho_j u_j^0 \geqq 0$.

*Then* $\frac{f(x^0)}{g(x^0)} \geqq v\left(\lambda^0\right)$.

*Proof.* From ($b_1$), $\left(y^0,\lambda^0,u^0\right) \in T$ and Theorem 2.10 we have

(5.8)
$$\overline{b}_1(x^0,y^0)\left[f(x^0)-\lambda^0 g(x^0)-\left(f(y^0)-\lambda^0 g(y^0)\right)\right]\geqq$$

$$\geqq \left(\mathrm{d}(f-\lambda^0 g)\right)^+ \left(y^0,\eta(x^0,y^0)\right)+\rho d(x^0,y^0)\geqq$$
$$\geqq -u^{0t}\left(\mathrm{d}h\right)^+\left(y^0,\eta(x^0,y^0)\right)+\rho d(x^0,y^0)\geqq$$
$$\geqq -\overline{b}_2(x^0,y^0)u^{0t}h(x^0)+\overline{b}_2(x^0,y^0)u^{0t}h(y^0)+\left(\rho+\sum_{j=1}^{m}\rho_j u_j^0\right)d(x^0,y^0).$$

Since $u^0 \geqq 0$, $h(x^0)\leqq 0$, $\overline{b}_2(x^0,y^0)\geqq 0$, from $\left(y^0,\lambda^0,u^0\right)\in T$ and (b$_2$) we have

$$\overline{b}_1(x^0,y^0)\left[f(x^0)-\lambda^0 g(x^0)-\left(f(y^0)-\lambda^0 g(y^0)\right)\right]\geqq 0,$$

whence, as $\overline{b}_1(x^0,y^0)>0$, we also have $f(x^0)-f(y^0)\geqq \lambda^0\left(g(x^0)-g(y^0)\right)$, or $f(x^0)-\lambda^0 g(x^0)\geqq f(y^0)-\lambda^0 g(y^0)\geqq 0$ (using 5.2).

Hence, $\frac{f(x^0)}{g(x^0)}\geqq \lambda^0$ thus completing the proof. □

Theorem 5.3 (Weak Duality). *Let* $x^0\in X$ *and* $\left(y^0,\lambda^0,u^0\right)\in T$. *Assume that:*

(c$_1$) $\left(f-\lambda^0 g+u^{0t}h\right)$ *is* $\rho$*-semilocally strongly pseudo b-preinvex at* $x^0$, *with* $\overline{b}(x^0,y^0)=\lim\limits_{\lambda\to 0^+} b\left(x^0,y^0,\lambda\right)>0$;

(c$_2$) $\rho\geqq 0$.

*Then* $\frac{f(x^0)}{g(x^0)}\geqq v\left(\lambda^0\right)$.

*Proof.* Let $x^0\in X$ and $\left(y^0,\lambda^0,u^0\right)\in T$. Condition (5.1) can be written as

$$\left(\mathrm{d}(f-\lambda^0 g+u^{0t}h)\right)^+\left(y^0,\eta(x^0,y^0)\right)\geqq 0.$$

So, as $\overline{b}(x^0,y^0)>0$ and $\rho\geqq 0$, we have

$$\overline{b}(x^0,y^0)(d(f-\lambda g+u^t h))^+\left(y^0,\eta(x^0,y^0)\right)\geqq 0\geqq -\rho d(x^0,y^0).$$

This inequality and assumption (c$_1$) yield

$$f(x^0)-\lambda^0 g(x^0)+u^{0t}h(x^0)\geqq f(y^0)-\lambda^0 g(y^0)+u^{0t}h(y^0)$$

or

$$f(x^0)-\lambda^0 g(x^0)\geqq f(y^0)-\lambda^0 g(y^0)-u^{0t}h(x^0)+u^{0t}h(y^0). \tag{5.9}$$

But, for $x^0\in X$ and $u^0\geqq 0$ we have $u^{0t}h(x^0)\leqq 0$ while for $\left(y^0,\lambda^0,u^0\right)\in T$ we have $f(y^0)-\lambda^0 g(y^0)\geqq 0$ and $u^{0t}h(y^0)\geqq 0$. Thus, inequality (5.9) becomes $f(x^0)-\lambda^0 g(x^0)\geqq 0$, i.e.,

$$\frac{f(x^0)}{g(x^0)}\geqq \lambda^0.$$

The proof is complete. □

Corollary 5.4. *Let* $x^0\in X$ *and* $\left(x^0,\lambda^0,u^0\right)\in T$ *such that* $q\left(x^0\right)=v(\lambda^0)$. *If the hypotheses of either Theorems* 5.1, 5.2 *or* 5.3 *are satisfied, then* $x^0$ *is an optimal solution to* (P) *and* $\left(x^0,\lambda^0,u^0\right)$ *is an optimal solution to* (D$_1$).

*Proof.* According to Theorems 5.1, 5.2 and 5.3 , for each $x \in X$ we have $q(x) \geqq v\left(\lambda^0\right) = q(x^0)$ hence $x^0$ is an optimal solution to problem (P). Also if $\left(x^0, \lambda^0, u^0\right) \in T$ then according to Theorems 5.1, 5.2 and 5.3, we have $v(\lambda) \leqq q(x^0) = v\left(\lambda^0\right)$, hence $\left(x^0, \lambda^0, u^0\right)$ is an optimal solution to problem ($D_1$). □

THEOREM 5.5 (Direct Duality). *Let $x^0$ be a (local) optimal solution for (P). Let $h_i$, $i \in J$ be continuous at $x^0$ and let $-(\mathrm{d}f)^+\left(x^0, \eta(x, x^0)\right)$, $(\mathrm{d}g)^+\left(x^0, \eta(x, x^0)\right)$, $(\mathrm{d}h)^+\left(x^0, \eta\left(x, x^0\right)\right)$ be $\eta$-vex functions of $x$ on $X^0$ (a $\eta$-vex set at $x^0$) with respect to a function $\eta$. If $h$ satisfies GSQ at $x^0$, then there exists $\left(x^0, \lambda^0, u^0\right) \in T$ such that $q(x^0) = v\left(\lambda^0\right)$. Moreover, if either of hypotheses ($d_1$)–($d_3$) below holds, then $\left(x^0, \lambda^0, u^0\right)$ is an optimal solution to ($D_1$):*

($d_1$) *$f$ is $\sigma$-slb$_1$-preinvex at $x^0$, $-g$ is $\tau$-slb$_1$-preinvex at $x^0$, and $h_j$, $j = 1, \ldots, m$, is $\rho_j$-slb$_2$-preinvex at $x^0$, with $\overline{b}_i\left(x^0, y^0\right) = \lim_{\lambda \to 0^+} b_i\left(x^0, y^0, \lambda\right) > 0$, $i = 1, 2$, and $\sigma + \lambda^0 \tau + \sum_{j=1}^{m} \rho_j u_j^0 \geqq 0$;*

($d_2$) *$\left(f - \lambda^0 g\right)$ is $\rho$-slb$_1$-preinvex at $x^0$, and $h_j$ is $\rho_j$-slb$_2$-preinvex at $x^0$, $j = 1, \ldots, m$, with $\overline{b}_i(x^0, y^0) = \lim_{\lambda \to 0^+} b_i\left(x^0, y^0, \lambda\right) > 0$, $i = 1, 2$, and $\rho + \sum_{j=1}^{m} \rho_j u_j^0 \geqq 0$;*

($d_3$) *$\left(f - \lambda^0 g + u^{0t} h\right)$ is $\rho$-semilocally strongly pseudo $b$-preinvex at $x^0$, with $\overline{b}(x^0, y^0) = \lim_{\lambda \to 0^+} b\left(x^0, y^0, \lambda\right) > 0$, and $\rho \geqq 0$,*

*Proof.* Since $x^0$ satisfies the conditions of Theorem 3.6, there exist $\lambda^0 \in \mathbf{R}$, $u^0 \in \mathbf{R}^m$ such that $\left(x^0, \lambda^0, u^0\right)$ is feasible for (D) and $q(x^0) = v(\lambda^0)$. Hence, by Corollary 5.4, $(x^0, \lambda^0, u^0)$ is optimal for ($D_1$). □

*Remark* 5.6. We can formulate a more general Mond-Weir dual for Problem (P) by using a partitioning scheme of the constraints. Let $\{J_1, \ldots, J_r\}$ be a partition of the subscript set $M = \{1, 2, \ldots, m\}$, i.e., $J_s \subset M$ for each $s \in \{1, \ldots, r\}$, $J_r \cap J_s = \emptyset$ for each $r, s \in \{1, \ldots, n\}$ with $r \neq s$, and $\bigcup_{s=1}^{m} J_s = M$. Instead of constraint (5.3) we can use a constraint of the form

$$u_{J_k}^t h_{J_k}(y) \geqq 0, \quad \text{for } k = 1, 2, \ldots, r,$$

and then prove weak, strong and strict converse duality theorem under generalized $\rho$-semilocally $b$-preinvexity conditions.

*Remark* 5.7. (a) If $\rho = 0$, we obtain the Mond-Weir dual considered by Stancu-Minasian and Andreea Mădălina Stancu [17].

(b) If $\rho = 0$, $b\left(x, x^0, \lambda\right) \equiv 1$ and $\eta\left(x, x^0\right) = x - x^0$ we obtain the Mond-Weir dual considered by Lyall, Suneja and Aggarwal [7].

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REFERENCES

[1] A. Ben-Israel and B. Mond, *What is invexity*? J. Austral. Mat. Soc. Ser. B **28** (1986), 1–9.
[2] G.M. Ewing, *Sufficient conditions for global minima of suitable convex functional from variational and control theory.* SIAM Rev. **19** (1977), *2*, 202–222.
[3] I. Gupta and M.N. Vartak, *Kuhn-Tucker and Fritz John type sufficient optimality conditions for generalized semilocally convex programs.* Opsearch **26** (1989), *1*, 11–27.
[4] R.N. Kaul and S. Kaur, *Sufficient optimality conditions using generalized convex functions.* Opsearch **19** (1982), *4*, 212–224.
[5] R.N. Kaul and S. Kaur, *Generalizations of convex and related functions.* European J. Oper. Res. **9** (1982), *4*, 369–377.
[6] S. Kaur, *Theoretical Studies in Mathematical Programming.* Ph.D. Thesis, University of Delhi, India, 1984.
[7] V. Lyall, S. Suneja and S. Aggarwal, *Optimality and duality in fractional programming involving semilocally convex and related functions.* Optimization **41** (1997), *3*, 237–255.
[8] O.L. Mangasarian, *Nonlinear Programming.* McGraw-Hill, New York, 1969.
[9] R.B. Patel, *Duality for nonlinear fractional programming involving generalized semilocally b-vex functions.* J. Indian Math. Soc. **68** (2001), *1-4*, 41–48.
[10] V. Preda, I.M. Stancu-Minasian and A. Bătătorescu, *Optimality and duality in nonlinear programming involving semilocally preinvex and related functions.* J. Inform. Optim. Sci. **17** (1996), *3*, 585–596.
[11] Andreea Mădălina Stancu, *Optimality and duality in nonlinear programming involving ρ-semilocally b-preinvex functions.* In: Elisabetta Allevi, Marida Bertocchi, Adriana Gnudi and Igor V. Konnov (Eds.), *Nonlinear Analysis with Applications in Economics, Energy and Transportation*, pp. 255–271, Bergamo University Press, Sestante Edizioni, 2007.
[12] I.M. Stancu-Minasian, *Fractional Programming. Theory, Methods and Applications.* Kluwer, Dordrecht, 1997.
[13] I.M. Stancu-Minasian, *Fractional programming with semilocally preinvex and related functions.* Proc. Romanian Acad. Ser. *A* **1** (2000) *1*, 21–24.
[14] I.M. Stancu-Minasian, *Optimality conditions and duality in fractional programming involving semilocally preinvex and related functions.* Report no. 140, Dipartimento di Statistica e Matematica Applicata All'Economia, Università di Pisa, Italy, Maggio 1999. Also in: J. Inform. Optim. Sci. **23** (2002), *1*, 185–201.
[15] I.M. Stancu-Minasian, *Nonlinear programming with semilocally b-preinvex and related functions.* Proc. Romanian Acad. Ser. *A* **4** (2003) *1*, 9–14.
[16] I.M. Stancu-Minasian, *Optimality and duality in nonlinear programming involving semilocally b-preinvex and related functions.* European J. Oper. Res. **173** (2006), *1,* 47–58.

[17] I.M. Stancu-Minasian and Andreea Mădălina Stancu, *Optimality conditions and duality in fractional programming involving semilocally b-preinvex and related functions.* Math. Reports **6(56)** (2004), *3*, 305–317.

[18] I.M. Stancu-Minasian and Andreea Mădălina Stancu, *Optimality and duality in nonlinear programming involving $\rho$-semilocally b-preinvex and related functions.* Math. Reports **8(58)** (2006), *4*, 459–474.

[19] S.K. Suneja and Sudha Gupta, *Semilocally b-vex and related functions.* Opsearch **29** (1992), *2*, 136–146.

[20] S.K. Suneja and Sudha Gupta, *Duality in nonlinear programming involving semilocally convex and related functions.* Optimization **28** (1993), *1*, 17–29.

[21] S.K. Suneja and Sudha Gupta, *Duality in nonlinear programming involving semilocally b-vex and related functions.* J. Inform. Optim. Sci. **15** (1994), *1*, 137–151.

[22] S.K. Suneja, C. Singh and C.R. Bector, *Generalization of preinvex and b-vex functions.* J. Optim. Theory Appl. **76** (1993), *3*, 577–587.

[23] T. Weir and B. Mond, *Preinvex functions in multiple objective optimization.* J. Math. Anal. Appl. **136** (1988), *1*, 29–38.

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