

CONTINUATION METHODS AND DISJOINT EQUILIBRIA

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Continuation methods are efficient to trace branches of fixed point solutions in the parameter space as long as these branches are connected. However, the computation of isolated branches of fixed points is a crucial issue and require ad-hoc techniques. We suggest a modification of the standard continuation methods to determine these isolated branches more systematically. The so-called residue continuation method is a global homotopy starting from an arbitrary disjoint initial guess. Explicit conditions ensuring the quadratic convergence of the underlying Newton-Raphson process are derived and illustrated by several examples.

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1. INTRODUCTION

The models of nonlinear physical phenomena depend on parameters and, in many cases, the transitions in the model behavior are found as values of a particular parameter are changed. The dynamical systems theory studies the common features of transitions in these nonlinear systems. This theory basically comprises bifurcation theory and the theory of ergodic systems. One of the important basic issues of the bifurcation theory is the determination of fixed points (or steady states) of the system under investigation.

For example, in applications in fluid mechanics, the flows are described by a set of partial differential non-linear equations, i.e., the Navier–Stokes equations. The first step in the bifurcation analysis is the discretization of the governing equations leading to a system of algebraic-differential equations of the form

$$(1) \quad \frac{\partial \mathbf{u}}{\partial t} = \mathbf{A}(\mathbf{u}, \mu),$$

where $\mathbf{u} \in \mathbb{R}^n$ is the vector of the unknown quantities at the gridpoints, $\mu \in \mathbb{R}^p$ is the vector of parameters, $\mathbf{A} : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$ is a nonlinear operator.

The fixed point solutions are determined by $\mathbf{A}(\mathbf{u}, \mu) = \mathbf{0}$ while branches of steady states are usually computed versus a control parameter using the so-called continuation methods [12, 19]. These continuation methods are very efficient when the branches are connected in the parameter space, as techniques exist to switch between the branches in the vicinity of the bifurcation points. There is a rich literature on the application of these techniques in the fields of fluid or structural dynamics, and mathematical analysis of the methodology. See, e.g., [21, 4, 8, 17, 14, 1, 19, 15]. We refer to [15] for an up-to-date state of art.

One of the basic continuation methods is the *natural* continuation method. A branch of steady states of (1) is computed through infinitesimal increments of a control parameter $\mu \in \{\mu_1, \mu_2, \dots, \mu_p\}$. Given a previously determined solution (\mathbf{u}_0, μ_0) , the solution $(\mathbf{u}, \mu_0 + \delta\mu)$ is determined by the Newton-Raphson method through

$$(2) \quad \mathbf{D}_{\mathbf{u}}\mathbf{A}^k \delta\mathbf{u}^k = -\mathbf{A}^k, \quad \mathbf{u}^k \leftarrow \mathbf{u}^k + \delta\mathbf{u}^k,$$

where $\delta\mu$ is a small increment with respect to μ_0 , $\delta\mathbf{u}^k = \mathbf{u}^{k+1} - \mathbf{u}^k$, $\mathbf{D}_{\mathbf{u}}\mathbf{A}^k$ is the Jacobian matrix of \mathbf{A}^k with respect to \mathbf{u} at the k^{th} estimate $(\mathbf{u}^k, \mu_0 + \delta\mu)$. At regular points one has

$$(3) \quad \text{rank}(\mathbf{D}_{\mathbf{u}}\mathbf{A}^k) = n.$$

Hence, the system arising after the Newton linearization can be solved. At the saddle-node bifurcation points, however, the Jacobian is singular and the natural continuation method cannot go around these bifurcation points.

To accomplish this, the pseudo-arclength continuation [12] was suggested. In this method, an arclength parametrization of the solution branch of the form $(\mathbf{u}(s), \mu(s))$ is introduced, where s is the arclength parameter. Because an additional degree of freedom is introduced, a scalar normalization of the tangent along the branch is introduced, too. It has the form

$$(4) \quad \mathcal{N}(\mathbf{u}(s), \mu(s)) \equiv \mathbf{D}_{\mathbf{u}}\mathbf{N} \cdot \delta\mathbf{u} + \mathbf{D}_{\mu}\mathbf{N} \delta\mu - \delta s = 0,$$

where $\mathbf{D}_{\mu}\mathbf{N}$ and $\mathbf{D}_{\mathbf{u}}\mathbf{N}$ are the derivatives of the operator \mathbf{N} with respect to μ and \mathbf{u} , respectively.

Given a solution $(\mathbf{u}(s_0), \mu(s_0))$, the solution $(\mathbf{u}(s_0 + \delta s), \mu(s_0 + \delta s))$ is determined again by the Newton-Raphson method from the system

$$(5) \quad \begin{pmatrix} \mathbf{D}_{\mathbf{u}}\mathbf{A}^k & \mathbf{D}_{\mu}\mathbf{A}^k \\ \mathbf{D}_{\mathbf{u}}\mathbf{N}^{kT} & \mathbf{D}_{\mu}\mathbf{N}^k \end{pmatrix} \begin{pmatrix} \delta\mathbf{u}^k \\ \delta\mu^k \end{pmatrix} = - \begin{pmatrix} \mathbf{A}^k \\ \mathcal{N}^k \end{pmatrix},$$

$$\begin{pmatrix} \mathbf{u}^k \\ \mu^k \end{pmatrix} \leftarrow \begin{pmatrix} \mathbf{u}^k + \delta\mathbf{u}^k \\ \mu^k + \delta\mu^k \end{pmatrix},$$

where $\mathbf{D}_\mu \mathbf{A}^k$, $\mathbf{D}_\mu \mathbf{N}^k$ and $\mathbf{D}_u \mathbf{N}^k$ are the derivatives of the operators at the current estimate (\mathbf{u}^k, μ^k) . In practice, both natural continuation and pseudo-arclength continuation are implemented and a switch between the two schemes is based on a value of the slope $\mathbf{D}_\mu \mathbf{N}$ [5, 6, 7, 10]. Usually, one starts the continuation from a known trivial (or analytical) solution (\mathbf{u}_0, μ_0) . In many applications, however, there exist branches of steady state solutions disconnected from the branch containing a trivial starting solution. These branches are the so-called *isolated* branches. Bifurcation theory in many cases may a priori indicate that there are isolated (or disjoint) branches of solutions. A typical example is the case of an imperfect pitchfork bifurcation, for example, that occurring in the wind-driven ocean flows [3, 20].

There are basically three methods to compute these isolated branches. But it is not guaranteed that one will find all branches by either of these methods. Two of them are more or less trial and error while in the latter method a more systematic approach is followed:

- (i) Transient integration.

In this approach, a set of initial conditions is chosen and a transient computation is started. If lucky, one of the initial conditions is in the attraction basin of a steady state on the isolated branch. Once found, one can continue tracing this branch using the continuation methods.

- (ii) Isolated Newton-Raphson search.

One can also start a Newton-Raphson process uncoupled from the pseudo-arclength continuation from several chosen starting points. Since the convergence of the Newton-Raphson process is only quadratic in the vicinity of the steady state, this method may not be very efficient. But again, if very lucky, an isolated branch might be found.

- (iii) Two-parameter continuation.

In several cases, a second parameter can be varied such that the targeted branch connects to an already known branch. An important example is where there are values of the second parameter for which the dynamical system has a particular symmetry and pitchfork bifurcations are present. Once the connection is present, the isolated branch can be computed by restoring the second parameter to its original value.

As is very important to determine isolated branches, there is a need for more systematic methodology to find them. In this paper, we propose a modification of the continuation methods which enables the determination of the isolated branches in Section 2. Explicit conditions ensuring the quadratic convergence of the Newton-Raphson process in the case of natural and pseudo-arclength continuation are derived in Section 3. The capabilities and efficiency of the method are illustrated in Section 4 by several examples.

2. THE RESIDUE CONTINUATION METHOD

The Residue Continuation Method is based on the global homotopy idea as pioneered by Keller [13]. Let \mathbf{u}^* be an initial guess for a fixed point of (1). Keller considered the modified set of equations

$$\mathbf{A}(\mathbf{u}) - e^{-\alpha(\mathbf{u})t} \mathbf{A}(\mathbf{u}^*) = 0,$$

where $\alpha(\mathbf{u}) > 0$ and $0 < t < \infty$. As $t \rightarrow \infty$, the solution $\mathbf{u}(t)$, if it exists, approaches a fixed point of (1). Keller showed that the choice of $\alpha(\mathbf{u})$ is crucial and clarified the sharp conditions that this operator must satisfy on a tubular neighborhood of the path for the existence of $\mathbf{u}(t)$. As the proof is constructive, the choice of a particular $\alpha(\mathbf{u})$ for a given operator is clearly indicated.

In the present paper, a systematic approach is suggested. Let (\mathbf{u}^*, μ^*) be an initial guess for an isolated steady state (\mathbf{u}_0, μ_0) of (1). The idea of the residue continuation method is to solve the global homotopy

$$(6) \quad \begin{aligned} \mathcal{H}(\mathbf{u}, \mu, \alpha) &\equiv \mathbf{A}(\mathbf{u}, \mu) - \alpha \mathbf{r} = 0, \\ \mathcal{K}(\mathbf{u}, \mu) &= k_{\mathbf{u}} \cdot \mathbf{u} + k_{\mu} \mu = 0, \end{aligned}$$

where $\mathbf{r} = \mathbf{A}(\mathbf{u}^*, \mu^*)$ is the *residue*, α is the *residue parameter*. Furthermore, the operators $\mathcal{H} : \mathbb{R}^n \times \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}^n$ and $\mathcal{K} : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ are introduced to define the total system of equations. For a given residue \mathbf{r} , assuming that $k_{\mu} \neq 0$, it follows from the Implicit Function Theorem that (6) can be written as

$$(7) \quad \begin{aligned} \mathcal{H}(\mathbf{u}(\alpha)) &\equiv \mathbf{A}(\mathbf{u}(\alpha), \mu(\mathbf{u}(\alpha))) - \alpha \mathbf{r} = 0, \\ \mathcal{K}(\mathbf{u}(\alpha)) &\equiv k_{\mathbf{u}} \cdot \mathbf{u}(\alpha) + k_{\mu} \mu(\mathbf{u}(\alpha)) = 0. \end{aligned}$$

Let $(\alpha_{\nu})_{\nu}$ be a real sequence such that $\alpha_{\nu} \in I \equiv [a, b] \subset \mathbb{R}$. For $(\mathbf{u}_{\nu-1}, \mu_{\nu-1})$ solution of the homotopy (7), denote

$$(8) \quad \mathbf{r}_{\nu-1} = \mathbf{A}(\mathbf{u}_{\nu-1}, \mu_{\nu-1}).$$

The Newton-Raphson method is used to solve the system of equations. For the natural continuation, the scheme can be written as

$$(9) \quad \mathbf{D}_{\mathbf{u}} \mathcal{H}_{\nu}^k \delta \mathbf{u}_{\nu}^k = -\mathcal{H}_{\nu}^k, \quad \mathbf{u}_{\nu}^k \leftarrow \mathbf{u}_{\nu}^k + \delta \mathbf{u}_{\nu}^k$$

while for the pseudo-arclength continuation it becomes

$$(10) \quad \begin{pmatrix} \mathbf{D}_{\mathbf{u}} \mathcal{H}_{\nu}^k & D_{\alpha} \mathcal{H}_{\nu}^k \\ \mathbf{D}_{\mathbf{u}} \mathcal{N}_{\nu}^{kT} & D_{\alpha} \mathcal{N}_{\nu}^k \end{pmatrix} \begin{pmatrix} \delta \mathbf{u}_{\nu}^k \\ \delta \alpha_{\nu}^k \end{pmatrix} = - \begin{pmatrix} \mathcal{H}_{\nu}^k \\ \mathcal{N}_{\nu}^k \end{pmatrix}, \\ \begin{pmatrix} \mathbf{u}_{\nu}^k \\ \alpha_{\nu}^k \end{pmatrix} \leftarrow \begin{pmatrix} \mathbf{u}_{\nu}^k + \delta \mathbf{u}_{\nu}^k \\ \alpha_{\nu}^k + \delta \alpha_{\nu}^k \end{pmatrix}.$$

Here, μ_ν is such that $k_{\mathbf{u}} \cdot \mathbf{u}_\nu + k_\mu \mu_\nu = 0$ and $\mathcal{H}_\nu^k \equiv \mathbf{A}_\nu^k - \alpha_\nu \mathbf{r}_{\nu-1}$. Furthermore, $\mathbf{D}_{\mathbf{u}} \mathcal{H}_\nu^k$ and $D_\alpha \mathcal{H}_\nu^k$ are the Jacobian with respect to \mathbf{u} and the residue parameter, respectively, at the current estimate $(\mathbf{u}_\nu^k, \alpha_\nu^k)$ of the solution $(\mathbf{u}_\nu, \alpha_\nu)$ of (7).

It follows from (7a) and (8) that

$$(11) \quad |\alpha_\nu| = \frac{\|\mathbf{r}_\nu\|}{\|\mathbf{r}_{\nu-1}\|}.$$

Hence the residue parameter α_ν may be seen as the control parameter of the norm of the residue. The residue increases (respectively decreases) as long as $|\alpha| > 1$ (respectively $|\alpha| < 1$) and $\alpha = 1$ is a critical value corresponding to an extremum of the norm of the residue.

3. CONVERGENCE AND ESTIMATE

In this section, a priori estimations of the convergence radius for the Newton–Raphson method are derived for the residue continuation scheme for both natural continuation (Subsection 3.1) and pseudo-arclength continuation (Subsection 3.2).

3.1. NATURAL CONTINUATION

For any division $(\alpha_\nu)_\nu$ of $I \subset \mathbb{R}$, \mathbf{u}_ν denotes the solution $\mathbf{u}(\alpha_\nu)$. Given an initial guess of the solution of homotopy (7) ($\mathbf{u}^0 \equiv \mathbf{u}_{\nu-1}$, $\mu_\nu^0 \equiv \mu_{\nu-1}$) with $\alpha \equiv \alpha_{\nu-1}$ and $\mathbf{r} \equiv \mathbf{r}_{\nu-2}$, the Newton–Raphson scheme is written in the equivalent form below. For $\nu = 1, \dots, N$ and $k = 0, \dots, p_\nu - 1$,

$$\mathbf{D}_{\mathbf{u}} \mathbf{A}_\nu^k (\mathbf{u}_\nu^{k+1} - \mathbf{u}_\nu^k) = -\mathbf{A}_\nu^k + \alpha_\nu \mathbf{r}_{\nu-1}, \quad \mathbf{u}_\nu^k \leftarrow \mathbf{u}_\nu^{k+1},$$

with the corresponding value of the control parameter μ_ν such as

$$(12) \quad k_{\mathbf{u}} \cdot (\mathbf{u}_\nu - \mathbf{u}_{\nu-1}) + k_\mu (\mu_\nu - \mu_{\nu-1}) = 0.$$

Assuming that $\mathbf{D}_{\mathbf{u}} \mathbf{A}$ is nonsingular for every $\mathbf{u} \in \mathbb{R}^n$, the operator $\mathbf{A} : \mathbb{R}^n \mapsto \mathbb{R}^n$ (hence also $\mathcal{H} : \mathbb{R}^n \mapsto \mathbb{R}^n$) is a homeomorphism. Therefore, for any α in some compact range $I_\nu \subset I \subset \mathbb{R}$ with extremities α_ν and $\alpha_{\nu-1}$, equation (7) admits a unique solution

$$(13) \quad \mathbf{u}(\alpha) = \mathbf{A}^{-1}(\phi_\nu(\alpha) \mathbf{r}_{\nu-1}),$$

$$\phi_\nu(\alpha) \equiv \frac{(1 - \alpha_\nu)\alpha - (1 - \alpha_{\nu-1})\alpha_\nu}{\alpha_{\nu-1} - \alpha_\nu} \mathbf{r}_{\nu-1}.$$

As \mathbf{A}^{-1} is continuously differentiable, it follows that $\alpha \mapsto \mathbf{u}(\alpha)$ is continuous and piecewise C^1 while $\alpha \mapsto \mathbf{D}_{\mathbf{u}} \mathbf{A}(\mathbf{u}(\alpha))^{-1}$ is continuous on each

I_ν . Therefore, assuming that the sequence $(\|\mathbf{r}_\nu\|)_\nu$ is bounded, there exists a constant $c > 0$ such that

$$(14) \quad \|\mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u}(\alpha))^{-1}\| \leq c, \quad \forall \alpha \in I_\nu. \subset I \subset \mathbb{R}$$

It follows that

$$(15) \quad \mathbf{u}'(\alpha) = \mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u}(\alpha))^{-1} \left(\frac{(1 - \alpha_\nu)}{\alpha_{\nu-1} - \alpha_\nu} \mathbf{r}_{\nu-1} \right), \quad \alpha \in I_\nu.$$

Denote by \mathcal{C} the limit curve of the Newton-Raphson process, defined as

$$(16) \quad \mathcal{C} \equiv \{\mathbf{u}(\alpha) \in \mathbb{R}^n, \alpha \in I \subset \mathbb{R}\}.$$

As I is compact and convex and \mathbf{u} is piecewise C^1 and continuous, there exists a compact convex set $D \subset \mathbb{R}^n$ such that $\mathcal{C} \subset D$.

PROPOSITION 3.1. *For some constant $c' > 0$ we have*

$$(17) \quad \forall \mathbf{u} \in D : \|\mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u})^{-1}\| \leq c'.$$

Proof. As $\mathbf{D}_{\mathbf{uu}}\mathbf{A}$ is continuous, it is also bounded on D and we define the constant $\kappa > 0$ such that

$$(18) \quad \|\mathbf{D}_{\mathbf{uu}}\mathbf{A}\| \leq \kappa, \quad \forall \mathbf{u} \in D.$$

Because $\mathbf{D}_\mathbf{u}\mathbf{A}$ is continuous on D , it is also uniformly continuous on D , hence

$$(19) \quad \begin{aligned} & \forall \varepsilon > 0, \exists \eta > 0, \forall \mathbf{u}, \mathbf{u}' \in D, \\ & \|\mathbf{u} - \mathbf{u}'\| < \eta \Rightarrow \|\mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u}) - \mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u}')\| \leq \varepsilon. \end{aligned}$$

Furthermore, given any $\mathbf{u}, \mathbf{u}' \in D$ and setting

$$\mathbf{f}(t) \equiv \mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u} + t(\mathbf{u}' - \mathbf{u})),$$

as f is continuously differentiable, we get

$$\begin{aligned} \mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u}) - \mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u}') &= \mathbf{f}(1) - \mathbf{f}(0) = \int_0^1 \mathbf{f}'(t) dt = \\ &= \int_0^1 \mathbf{D}_{\mathbf{uu}}\mathbf{A}(\mathbf{u} + t(\mathbf{u}' - \mathbf{u})) dt (\mathbf{u}' - \mathbf{u}). \end{aligned}$$

Then

$$\begin{aligned} & \forall \mathbf{u}, \mathbf{u}' \in D, \quad \|\mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u}) - \mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u}')\| = \\ &= \left\| \int_0^1 \mathbf{D}_{\mathbf{uu}}\mathbf{A}(\mathbf{u} + t(\mathbf{u}' - \mathbf{u})) dt (\mathbf{u}' - \mathbf{u}) \right\| \leq \kappa \|\mathbf{u} - \mathbf{u}'\| \end{aligned}$$

so that we may choose $\eta = \frac{\varepsilon}{\kappa}$.

In addition, for every $\mathbf{u}^0 \in D$, we have

$$\|\mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u})^{-1}\| \leq \frac{\|\mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u}^0)^{-1}\|}{1 - \|\mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u}^0)^{-1}\| \|\mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u}) - \mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u}^0)\|} \leq \frac{\|\mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u}^0)^{-1}\|}{1 - \varepsilon \|\mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u}^0)^{-1}\|}.$$

Consequently, equation (17) holds with $c' = \frac{c}{1-\varepsilon c}$ so that we choose

$$(20) \quad \varepsilon \in \left[0, \frac{1}{2c}\right].$$

This completes the proof. \square

Following [11] (see [16]) with the sequence $(\mathbf{u}_\nu^k)_k$, we associate the quantities $\beta_{k,\nu}$, $\eta_{k,\nu}$, $\gamma_{k,\nu}$, $t_{k,\nu}^\pm$ according to recurrence introduced as follows. Let $\beta_{0,\nu}$, $\eta_{0,\nu}$, $\gamma_{0,\nu}$, $t_{0,\nu}^\pm$ be defined as

$$(21) \quad \|\mathbf{D}_{\mathbf{u}}\mathbf{A}(\mathbf{u}_\nu^0)^{-1}\| \leq \beta_{0,\nu} \equiv \frac{c}{1-\varepsilon c} < +\infty,$$

$$(22) \quad \frac{|1-\alpha_\nu|}{|\alpha_\nu|} \|(\mathbf{D}_{\mathbf{u}}\mathbf{A}(\mathbf{u}_\nu^0))^{-1}\mathbf{r}_\nu\| \leq \eta_{0,\nu} < +\infty,$$

$$(23) \quad \gamma_{0,\nu} \equiv \eta_{0,\nu} \beta_{0,\nu} \kappa,$$

$$(24) \quad t_{0,\nu}^\pm = \frac{1}{\kappa \beta_{0,\nu}} (1 \pm \sqrt{1-2\gamma_{0,\nu}}).$$

For each $\nu \geq 1$, define the sequences $\beta_{k,\nu}$, $\eta_{k,\nu}$, $\gamma_{k,\nu}$, $t_{k,\nu}^\pm$ by

$$\begin{aligned} \gamma_{k,\nu} &= \beta_{k,\nu} \eta_{k,\nu} \kappa, & \beta_{k+1,\nu} &= \frac{\beta_{k,\nu}}{1-\gamma_{k,\nu}}, \\ \eta_{k+1,\nu} &= \frac{\gamma_{k,\nu} \eta_{k,\nu}}{2(1-\gamma_{k,\nu})}, & t_{k,\nu}^\pm &= \frac{1}{\kappa \beta_{k,\nu}} (1 \pm \sqrt{1-2\gamma_{k,\nu}}). \end{aligned}$$

Letting $k \rightarrow \infty$ for the residue Newton-Raphson scheme (12), we have

$$(25) \quad \|\mathbf{u}_\nu^0 - \mathbf{u}(\alpha_\nu)\| \equiv \|\mathbf{u}_\nu^0 - \mathbf{u}_\nu\| \leq t_{0,\nu}^- = t_{p_{\nu-1},\nu-1}^-$$

and, according to the definition of $\mathbf{u}_\nu^0 \equiv \mathbf{u}_{\nu-1}$,

$$(26) \quad \|\mathbf{u}_\nu^0 - \mathbf{u}(\alpha_{\nu-1})\| \equiv \|\mathbf{u}_{\nu-1}^{p_{\nu-1}} - \mathbf{u}_{\nu-1}\| \leq 2\eta_{p_{\nu-1},\nu-1} \leq \frac{2\eta_{0,\nu-1}}{2^{p_{\nu-1}}}.$$

Kantorovich's theorem then reads as follows.

COROLLARY 3.2. *For each ν , the sequence $(\mathbf{u}_\nu^k)_k$ generated by the scheme (12) converges to the unique solution $\mathbf{u}_\nu \equiv \mathbf{u}(\alpha_\nu)$ of the system*

$$\mathbf{A}(\mathbf{u}_\nu) - \alpha_\nu \mathbf{r}_{\nu-1} = 0, \quad \mathbf{r}_{\nu-1} \equiv \mathbf{A}(\mathbf{u}_{\nu-1})$$

in the open ball $B(\mathbf{u}_\nu^0, t_{0,\nu}^+)$ with

$$t_{0,\nu}^+ = \frac{1}{\kappa \beta_{0,\nu}} (1 + \sqrt{1-2\gamma_{0,\nu}}),$$

where κ , $\beta_{0,\nu}$ and $\gamma_{0,\nu}$ are defined in (18), (21) and (23), respectively.

A sufficient condition for the convergence of (12) can now be stated as follows.

PROPOSITION 3.3. *A sufficient condition for the sequence $(\mathbf{u}_\nu^k)_{k \geq 0}$ to converge towards $\mathbf{u}(\alpha_\nu)$ is that α_ν satisfies*

$$(27) \quad 0 < \frac{|1 - \alpha_\nu|}{|\alpha_\nu|} \|\mathbf{r}_\nu\| < \frac{1}{2c} \min \left(\frac{3 - \sqrt{5}}{2\kappa c}, t_{0,\nu}^- \right) \equiv \Lambda_\nu,$$

where the constant c has been defined in (14).

Proof. We look for a condition able to ensure (19) with $\eta = \frac{\varepsilon}{\kappa}$, $\mathbf{u} = \mathbf{u}_\nu^0$ and $\mathbf{u}' = \mathbf{u}(\alpha_\nu)$, that is,

$$(28) \quad \|\mathbf{u}_\nu^0 - \mathbf{u}(\alpha_\nu)\| < \frac{\varepsilon}{\kappa}.$$

Besides, this should be compatible with (25). In order to achieve (28), we first notice that

$$(29) \quad \|\mathbf{u}_\nu^0 - \mathbf{u}(\alpha_\nu)\| \leq \|\mathbf{u}_\nu^0 - \mathbf{u}(\alpha_{\nu-1})\| + \|\mathbf{u}(\alpha_{\nu-1}) - \mathbf{u}(\alpha_\nu)\|.$$

Taking into account (26), we may choose $p_{\nu-1}$ such that

$$\frac{2\eta_{0,\nu-1}}{2^{p_{\nu-1}}} < \frac{\varepsilon}{2\kappa},$$

that is,

$$2^{p_{\nu-1}} > \frac{4\eta_{0,\nu-1}\kappa}{\varepsilon}.$$

which fixes $p_{\nu-1}$ and also $\mathbf{u}_\nu^0 = \mathbf{u}_{\nu-1}^{p_{\nu-1}}$.

Recall that

$$(30) \quad \forall \alpha \in I_\nu : \quad \mathbf{A}(\mathbf{u}(\alpha)) = \frac{(1 - \alpha_\nu)\alpha - (1 - \alpha_{\nu-1})\alpha_\nu}{\alpha_{\nu-1} - \alpha_\nu} \mathbf{r}_{\nu-1},$$

where we set

$$(31) \quad I_\nu \equiv [\min(\alpha_{\nu-1}, \alpha_\nu), \max(\alpha_{\nu-1}, \alpha_\nu)].$$

Then, (15) yields

$$\begin{aligned} \|\mathbf{u}(\alpha_{\nu-1}) - \mathbf{u}(\alpha_\nu)\| &= \left\| \int_{\alpha_{\nu-1}}^{\alpha_\nu} \mathbf{u}'(\alpha) d\alpha \right\| = \\ &= \frac{|1 - \alpha_\nu|}{|\alpha_{\nu-1} - \alpha_\nu|} \left\| \int_{\alpha_{\nu-1}}^{\alpha_\nu} \mathbf{D}_\mathbf{u} \mathbf{A}(\mathbf{u}(\alpha))^{-1} \mathbf{r}_{\nu-1} d\alpha \right\| \leq \\ &\leq \frac{|1 - \alpha_\nu|}{|\alpha_{\nu-1} - \alpha_\nu|} \left| \int_{\alpha_\nu}^{\alpha_{\nu-1}} \|\mathbf{D}_\mathbf{u} \mathbf{A}(\mathbf{u}(\alpha))^{-1} \mathbf{r}_{\nu-1}\| d\alpha \right| \leq \\ &\leq c |1 - \alpha_\nu| \|\mathbf{r}_{\nu-1}\| = c \frac{|1 - \alpha_\nu|}{|\alpha_\nu|} \|\mathbf{r}_\nu\|. \end{aligned}$$

By (27), we may choose ε such that

$$(32) \quad \frac{2\eta_{0,\nu-1}}{2^{p\nu-1}} = c \frac{|1-\alpha_\nu|}{|\alpha_\nu|} \|\mathbf{r}_\nu\| < \frac{\varepsilon}{2\kappa} < \frac{3-\sqrt{5}}{4\kappa c}.$$

After substitution into (29), this expression can be written as

$$(33) \quad \|\mathbf{u}_\nu^0 - \mathbf{u}(\alpha_\nu)\| \leq 2c \frac{|1-\alpha_\nu|}{|\alpha_\nu|} \|\mathbf{r}_\nu\| < \frac{\varepsilon}{\kappa},$$

that is, (28).

As for (25), the same argument with $t_{0,\nu}^-$ instead of $\frac{\varepsilon}{\kappa}$ shows that (27) leads to

$$\|\mathbf{u}_\nu^0 - \mathbf{u}(\alpha_\nu)\| \leq 2c \frac{|1-\alpha_\nu|}{|\alpha_\nu|} \|\mathbf{r}_\nu\| < t_{0,\nu}^-,$$

which yields (25). Moreover, a sufficient condition for convergence is

$$(34) \quad 0 < \eta_{0,\nu} < \frac{1-\varepsilon c}{2\kappa c}.$$

Indeed, from [16], a sufficient condition for Newton's method to converge at the given step ν is that $\gamma_{0,\nu} < \frac{1}{2}$. Arguing as in (17), we get

$$(35) \quad \begin{aligned} \|\mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u}_\nu^0)^{-1}\| &\leq \frac{\|\mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u}(\alpha_\nu))^{-1}\|}{1 - \|\mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u}_\nu^0) - \mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u}(\alpha_\nu))\| \|\mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u}(\alpha_\nu))^{-1}\|} \leq \\ &\leq \frac{c}{1-\varepsilon c} = c' \equiv \beta_{0,\nu}. \end{aligned}$$

Then, the condition $\gamma_{0,\nu} < \frac{1}{2}$ becomes

$$(36) \quad 0 < \eta_{0,\nu} < \frac{1}{2\beta_{0,\nu}\kappa} = \frac{1-\varepsilon c}{2c\kappa},$$

which is (34). Notice that

$$\eta_{0,\nu} \geq \frac{|1-\alpha_\nu|}{|\alpha_\nu|} \|(\mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u}_\nu^0))^{-1}\mathbf{r}_\nu\|$$

with

$$\|(\mathbf{D}_\mathbf{u}\mathbf{A}(\mathbf{u}_\nu^0))^{-1}\mathbf{r}_\nu\| \leq c' \|\mathbf{r}_\nu\|.$$

Then, on account of equation (35), a sufficient condition reads as

$$\frac{|1-\alpha_\nu|}{|\alpha_\nu|} \|\mathbf{r}_\nu\| < \frac{1}{2\kappa} \left(\frac{1-\varepsilon c}{c} \right)^2.$$

Comparing with (27), we get

$$0 < \frac{|1-\alpha_\nu|}{|\alpha_\nu|} \|\mathbf{r}_\nu\| < \frac{1}{2} \min \left(\frac{\varepsilon}{\kappa c}, \frac{t_{0,\nu}^-}{c}, \frac{1}{\kappa} \left(\frac{1-\varepsilon c}{c} \right)^2 \right)$$

and

$$\frac{\varepsilon}{\kappa c} < \frac{1}{\kappa} \left(\frac{1 - \varepsilon c}{c} \right)^2 \iff \varepsilon^2 - \frac{3\varepsilon}{c} + \frac{1}{c^2} > 0.$$

This holds as soon as

$$0 < \varepsilon < \frac{3 - \sqrt{5}}{2c},$$

thus completing the proof. \square

3.2. PSEUDO-ARCLENGTH CONTINUATION

Remember that the pseudo-arclength continuation could also be applied in the case of a non-regular Jacobian $\mathbf{D}_{\mathbf{u}}\mathbf{A}$, so that the Implicit Function Theorem does not apply anymore.

For any given \mathbf{r} and assuming that $\text{rank}(\mathbf{D}_{\mathbf{u}}\mathbf{A}) = n - 1$, let us introduce the operator $\mathcal{F} : \mathbb{R}^{n+1} \mapsto \mathbb{R}^n$ by

$$\mathcal{F}(\mathbf{u}(s), \alpha(s); s) \equiv \begin{pmatrix} \mathcal{H}(\mathbf{u}(s), \alpha(s)) \\ \mathcal{N}(\mathbf{u}(s), \alpha(s); s) \end{pmatrix}.$$

When the Implicit Function Theorem is applied for any given \mathbf{r} , the global homotopy (7) can be written as

$$(37) \quad \mathcal{F}(\mathbf{u}(s), \alpha(s); \mathbf{r}) = 0, \quad k_{\mathbf{u}} \cdot \mathbf{u}(s) + k_{\mu} \mu(\mathbf{u}(s)) = 0.$$

For some fixed $s_{\nu} > 0$, consider Newton's scheme (10) written in the equivalent form below. For $\nu = 1, \dots, N$ and $k = 0, \dots, p_{\nu} - 1$,

$$\begin{aligned} \mathbf{D}_{\mathbf{u}}\mathbf{A}_{\nu}^k(\mathbf{u}_{\nu}^{k+1} - \mathbf{u}_{\nu}^k) - (\alpha_{\nu}^{k+1} - \alpha_{\nu}^k)\mathbf{r}_{\nu-1} &= -\mathbf{A}_{\nu}^k + \alpha_{\nu}^k\mathbf{r}_{\nu-1}, \\ \mathbf{D}_{\mathbf{u}}\mathbf{N}_{\nu}^k(\mathbf{u}_{\nu}^{k+1} - \mathbf{u}_{\nu}^k) + \mathbf{D}_{\alpha}\mathbf{N}_{\nu}^k(\alpha_{\nu}^{k+1} - \alpha_{\nu}^k) &= -\mathcal{N}_{\nu}^k, \\ (\mathbf{u}_{\nu}^k, \alpha_{\nu}^k) &\leftarrow (\mathbf{u}_{\nu}^{k+1}, \alpha_{\nu}^{k+1}). \end{aligned}$$

For any $1 \leq \nu \leq N$, the corresponding value of the control parameter μ_{ν} is such that

$$(38) \quad k_{\mathbf{u}} \cdot (\mathbf{u}_{\nu} - \mathbf{u}_{\nu-1}) + k_{\mu}(\mu_{\nu} - \mu_{\nu-1}) = 0,$$

where the initialization point $(\mathbf{u}_{\nu}^0, \alpha_{\nu}^0) \equiv (\mathbf{u}_{\nu-1}, \alpha_{\nu-1})$ is taken to be solution of (7) with $s \equiv s_{\nu-1}$ and $\mathbf{r} \equiv \mathbf{r}_{\nu-2}$.

In the sequel, we assume that the matrix

$$\mathcal{B}(\mathbf{u}, \alpha) \equiv \begin{pmatrix} \mathbf{D}_{\mathbf{u}}\mathbf{A}(\mathbf{u}) & -\mathbf{r} \\ \mathbf{D}_{\mathbf{u}}\mathbf{N}(\mathbf{u}, \alpha) & \mathbf{D}_{\alpha}\mathbf{N}(\mathbf{u}, \alpha) \end{pmatrix}$$

is nonsingular for every $(\mathbf{u}, \alpha) \in \mathbb{R}^n \times \mathbb{R}$ and that there is a constant $c > 0$ such that

$$\|\mathcal{B}^{-1}(\mathbf{u}, \alpha)\| \leq c, \quad \forall (\mathbf{u}, \alpha) \in \mathbb{R}^n \times \mathbb{R}.$$

Since, by construction, $\mathcal{F}(\cdot, \cdot; s, \mathbf{r})$ is a homeomorphism $\mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n \times \mathbb{R}$, (37) admits a unique solution

$$\begin{pmatrix} \mathbf{u}(s) \\ \alpha(s) \end{pmatrix} = \mathcal{F}(\cdot, \cdot; s)^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for every $s > 0$. As \mathcal{F}^{-1} is continuously differentiable, $s \mapsto \mathcal{F}(\cdot, \cdot; s)^{-1}$ is of class C^1 as well as $s \mapsto (\mathbf{u}(s), \alpha(s))$. In particular, there exists a constant $c > 0$ such that

$$(39) \quad \|\mathcal{B}(\mathbf{u}(s), \alpha(s))^{-1}\| \leq c, \quad \forall s \in \mathbb{R},$$

and we have

$$(40) \quad \begin{pmatrix} \mathbf{u}'(s) \\ \alpha'(s) \end{pmatrix} = \mathcal{B}(\mathbf{u}(s), \alpha(s))^{-1} \begin{pmatrix} 0 \\ -\mathbf{D}_s \mathcal{N}(\mathbf{u}(s), \alpha(s)) \end{pmatrix}.$$

Consider the sequence $(\mathbf{y}_\nu^k)_{\nu, k} \equiv (\mathbf{u}_\nu^k, \alpha_\nu^k)_{\nu, k}$ defined for a division

$$s_0 < s_1 < \dots < s_N, \quad h_\nu = s_\nu - s_{\nu-1} > 0,$$

of $I \subset \mathbb{R}$, by the scheme below. For $\nu = 1, \dots, N$

$$(41) \quad \begin{aligned} \mathbf{y}_\nu^0 &= \mathbf{y}_{\nu-1}, \\ \mathbf{y}_\nu^{k+1} &= \mathbf{y}_\nu^k - \mathcal{B}(\mathbf{y}_\nu^k)^{-1} \begin{pmatrix} \mathbf{A}_\nu^k - \alpha_\nu^k \mathbf{r}_{\nu-1} \\ \mathcal{N}_\nu^k \end{pmatrix}, \quad k = 0, \dots, p_\nu - 1, \\ \mathbf{y}_\nu &= \mathbf{y}_\nu^{p_\nu}, \quad \mathbf{r}_\nu = \mathbf{A}(\mathbf{u}_\nu). \end{aligned}$$

Defining the set

$$\mathcal{C} \equiv \{(\mathbf{u}(s), \alpha(s)) \in \mathbb{R}^n \times \mathbb{R}, s \in [s_0, s_N]\},$$

there exists a compact convex set $D \subset \mathbb{R}^n \times \mathbb{R}$ such that $\mathcal{C} \subset D$. As $\mathbf{D}_y \mathcal{B}$ is continuous, it is also bounded on D and we have

$$\|\mathbf{D}_y \mathcal{B}(\mathbf{y})\| \leq \kappa, \quad \forall \mathbf{y} \equiv (\mathbf{u}, \alpha) \in D$$

for some constant $\kappa > 0$. As \mathcal{B} is continuous on D , it is also uniformly continuous on D , i.e.,

$$(42) \quad \forall \varepsilon > 0, \exists \eta > 0, \forall \mathbf{y}, \mathbf{y}' \in D, \quad \|\mathbf{y} - \mathbf{y}'\| < \eta \Rightarrow \|\mathcal{B}(\mathbf{y}) - \mathcal{B}(\mathbf{y}')\| \leq \varepsilon.$$

Furthermore,

$$\begin{aligned} &\|\mathcal{B}(\mathbf{y}) - \mathcal{B}(\mathbf{y}')\| = \\ &= \left\| \int_0^1 \mathbf{D}_y \mathcal{B}(\mathbf{y} + t(\mathbf{y}' - \mathbf{y})) dt (\mathbf{y}' - \mathbf{y}) \right\| \leq \kappa \|\mathbf{y} - \mathbf{y}'\|, \quad \forall \mathbf{y}, \mathbf{y}' \in D, \end{aligned}$$

so that we may choose $\eta = \frac{\varepsilon}{\kappa}$. Remark that

$$(43) \quad \|\mathcal{B}(\mathbf{y})^{-1}\| \leq \frac{\|\mathcal{B}(\mathbf{y}^0)^{-1}\|}{1 - \|\mathcal{B}(\mathbf{y}^0)^{-1}\| \|\mathcal{B}(\mathbf{y}) - \mathcal{B}(\mathbf{y}^0)\|} \leq \frac{\|\mathcal{B}(\mathbf{y}^0)^{-1}\|}{1 - \varepsilon \|\mathcal{B}(\mathbf{y}^0)^{-1}\|} \leq \frac{c}{1 - \varepsilon c} \equiv c'$$

so that we choose $\varepsilon \in [0, \frac{1}{2c}]$.

With each sequence $(\mathbf{y}_\nu^k)_k$ we may associate quantities $\beta_{k,\nu}, \eta_{k,\nu}, \gamma_{k,\nu}, t_{k,\nu}^-$:

$$(44) \quad \|\mathbf{D}_y \mathcal{B}(\mathbf{y})\| \leq \kappa, \quad \forall \mathbf{y} \in D,$$

$$(45) \quad \|\mathcal{B}(\mathbf{y}_\nu^0)^{-1}\| \leq c' \equiv \beta_{0,\nu} < +\infty,$$

$$(46) \quad \left\| (\mathcal{B}(\mathbf{y}_\nu^0))^{-1} \left(\frac{(1 - \alpha_\nu^0)}{\alpha_\nu^0} \mathbf{r}_\nu, 0 \right)^T \right\| \leq \eta_{0,\nu} < +\infty,$$

$$(47) \quad \gamma_{0,\nu} \equiv \eta_{0,\nu} \beta_{0,\nu} \kappa \leq \frac{1}{2},$$

$$(48) \quad t_{k,\nu}^\pm = \frac{1}{\kappa \beta_{k,\nu}} (1 \pm \sqrt{1 - 2\gamma_{k,\nu}}),$$

$$(49) \quad \gamma_{k,\nu} = \beta_{k,\nu} \eta_{k,\nu} \kappa,$$

$$(50) \quad \beta_{k+1,\nu} = \frac{\beta_{k,\nu}}{1 - \gamma_{k,\nu}},$$

$$(51) \quad \eta_{k+1,\nu} = \frac{\gamma_{k,\nu} \eta_{k,\nu}}{2(1 - \gamma_{k,\nu})},$$

where we took into account that

$$\mathcal{N}_\nu^0 = \mathbf{D}_u \mathbf{A}_\nu^0 \delta \mathbf{u}_\nu^0 + \mathbf{D}_\mu \mathbf{A}_\nu^0 \delta \mu_\nu^0 \equiv \mathbf{D}_u \mathbf{A}_\nu^0 (\mathbf{u}_\nu^0 - \mathbf{u}_{\nu-1}^{p_{\nu-1}}) + \mathbf{D}_\alpha \mathbf{A}_\nu^0 (\alpha_\nu^0 - \alpha_{\nu-1}^{p_{\nu-1}}) = 0,$$

where

$$(52) \quad \mathbf{y}_\nu^{k+1} = \mathbf{y}_\nu^k - \mathcal{B}(\mathbf{y}_\nu^k)^{-1} \mathcal{F}(\mathbf{y}_\nu^k; s_\nu, \mathbf{r}_{\nu-1}).$$

Now, the arguments from the previous section with $\mathbf{D}_u \mathbf{A}, \mathbf{u}, \alpha$ replaced by $\mathcal{B}, \mathbf{y}, s$, respectively, yield the result below.

COROLLARY 3.4. *For each ν , the sequence $(\mathbf{y}_\nu^k)_k$ generated by the scheme (41) converges to the unique solution $\mathbf{y}_\nu \equiv \mathbf{y}(s_\nu) = (\mathbf{u}(s_\nu), \alpha(s_\nu))$ of the system*

$$\mathbf{A}(\mathbf{u}_\nu) - \alpha_\nu \mathbf{r}_{\nu-1} = 0, \quad \mathcal{K}(\mathbf{u}_\nu) = 0, \quad \mathcal{N}(\mathbf{y}_\nu; s_\nu) = 0, \quad \mathbf{r}_{\nu-1} \equiv \mathbf{A}(\mathbf{u}_{\nu-1})$$

in the open ball $B(\mathbf{y}_\nu^0, t_{0,\nu}^+)$, where

$$t_{0,\nu}^+ = \frac{1}{\kappa \beta_{0,\nu}} (1 + \sqrt{1 - 2\gamma_{0,\nu}})$$

and $\kappa, \beta_{0,\nu}$ and $\gamma_{0,\nu}$ are defined in (44), (45) and (47), respectively.

PROPOSITION 3.5. *A sufficient condition for the sequence $(\mathbf{y}_\nu^k)_{k \geq 0}$ to converge to $\mathbf{y}(s_\nu)$ is that*

$$(53) \quad 0 < h_\nu < \frac{1}{2c\|\mathbf{D}_s\mathcal{N}\|} \min\left(\frac{1}{2\kappa}, t_{0,\nu}^-\right)$$

and

$$0 < \frac{|1 - \alpha_\nu^0|}{|\alpha_\nu^0|} \|\mathbf{r}_\nu\| < \frac{1}{2\kappa c^2} \equiv \Lambda_\nu,$$

where the constant c has been defined in (39).

Proof. The proof follows the same lines as that of Proposition 3.3. Condition (42) must hold with $\eta = \frac{\varepsilon}{\kappa}$, $\mathbf{y} = \mathbf{y}_\nu^0$, $\mathbf{y}' = \mathbf{y}(\alpha_\nu)$, that is,

$$\|\mathbf{y}_\nu^0 - \mathbf{y}(\alpha_\nu)\| < \frac{\varepsilon}{\kappa}.$$

In addition, we have the analogue of (19). First, notice that

$$(54) \quad \|\mathbf{y}_\nu^0 - \mathbf{y}(s_\nu)\| \leq \|\mathbf{y}_\nu^0 - \mathbf{y}(s_{\nu-1})\| + \|\mathbf{y}(s_{\nu-1}) - \mathbf{y}(s_\nu)\|.$$

The analogue of (26) holds, namely,

$$\|\mathbf{y}_\nu^0 - \mathbf{y}(s_{\nu-1})\| \equiv \|\mathbf{y}_{\nu-1}^{p_{\nu-1}} - \mathbf{y}_{\nu-1}\| \leq 2\eta_{p_{\nu-1}, \nu-1} \leq \frac{2\eta_{0, \nu-1}}{2^{p_{\nu-1}}}.$$

Therefore, we may choose $p_{\nu-1}$ such that

$$\frac{2\eta_{0, \nu-1}}{2^{p_{\nu-1}}} < \frac{\varepsilon}{2\kappa},$$

that is,

$$2^{p_{\nu-1}} > \frac{4\eta_{0, \nu-1}\kappa}{\varepsilon},$$

which fixes $p_{\nu-1}$ and also $\mathbf{y}_\nu^0 = \mathbf{y}_{\nu-1}^{p_{\nu-1}}$. Moreover, equation (40) yields

$$(55) \quad \begin{aligned} \|\mathbf{y}(s_{\nu-1}) - \mathbf{y}(s_\nu)\| &= \left\| \int_{s_{\nu-1}}^{s_\nu} \mathbf{y}'(s) ds \right\| = \\ &= \left\| \int_{s_{\nu-1}}^{s_\nu} \mathcal{B}(\mathbf{y}(s))^{-1} (0, -\mathbf{D}_s\mathcal{N}(\mathbf{y}(s)))^T ds \right\| \leq \\ &\leq \int_{s_{\nu-1}}^{s_\nu} \|\mathcal{B}(\mathbf{y}(s))^{-1}\| \|\mathbf{D}_s\mathcal{N}\| ds \leq c\|\mathbf{D}_s\mathcal{N}\| h_\nu. \end{aligned}$$

Then (53) implies that we may choose ε such that

$$\frac{2\eta_{0, \nu-1}}{2^{p_{\nu-1}}} = c\|\mathbf{D}_s\mathcal{N}\| h_\nu < \frac{\varepsilon}{2\kappa} < \frac{1}{4\kappa c},$$

in accordance with (20), that is,

$$0 < h_\nu < \frac{\varepsilon}{2\kappa c\|\mathbf{D}_s\mathcal{N}\|}.$$

After substitution in (54), we get

$$(56) \quad \|\mathbf{y}_\nu^0 - \mathbf{y}(s_\nu)\| \leq c \|\mathbf{D}_s \mathcal{N}\| h_\nu < \frac{\varepsilon}{\kappa},$$

which is (42).

The analogue of (25) reads as

$$(57) \quad \|\mathbf{y}_\nu^0 - \mathbf{y}(s_\nu)\| \equiv \|\mathbf{y}_\nu^0 - \mathbf{y}_\nu\| \leq t_{0,\nu}^- = t_{p_{\nu-1}, \nu-1}^-.$$

Then the same argument, with $t_{0,\nu}^-$ instead of $\frac{\varepsilon}{\kappa}$, yields

$$0 < h_\nu < \frac{1}{2c \|\mathbf{D}_s \mathcal{N}\|} \min\left(\frac{\varepsilon}{\kappa}, t_{0,\nu}^-\right).$$

Moreover, arguing as in the previous section we find that the requirement (34) still holds, that is,

$$(58) \quad 0 < \eta_{0,\nu} < \frac{1 - \varepsilon c}{2\kappa c}.$$

Recall that

$$\eta_{0,\nu} \geq \left\| (\mathcal{B}(\mathbf{y}_\nu^0))^{-1} \left(\frac{(1 - \alpha_\nu^0)}{\alpha_\nu^0} \mathbf{r}_\nu, 0 \right)^T \right\|,$$

so that $\eta_{0,\nu}$ may be chosen as

$$\left\| (\mathcal{B}(\mathbf{y}_\nu^0))^{-1} \left(\frac{(1 - \alpha_\nu^0)}{\alpha_\nu^0} \mathbf{r}_\nu, 0 \right)^T \right\| \leq \frac{c}{(1 - \varepsilon c)} \left(\frac{|1 - \alpha_\nu^0|}{|\alpha_\nu^0|} \|\mathbf{r}_\nu\| \right) \equiv \eta_{0,\nu}.$$

This implies

$$\eta_{0,\nu} \leq \frac{1}{8\kappa c(1 - \varepsilon c)}$$

while (20) yields $1 - \varepsilon c > \frac{1}{2}$. Thus,

$$\frac{\eta_{0,\nu}}{1 - \varepsilon c} < \frac{1}{8\kappa c(1 - \varepsilon c)^2} < \frac{1}{2\kappa c^2},$$

which is (58). \square

4. NUMERICAL EXPERIMENTS

In this section, we illustrate the residue continuation method along four examples. The first example addresses the problem of finding a fixed point solution of a scalar equation (i.e. $\mu \in \mathbb{R}$ and $u \in \mathbb{R}$) starting from an arbitrary initial guess. In the second and the third examples a scalar equation is again used and is shown how the method is able to reach an isolated solution; the convergence properties of the method are also shown. These three examples illustrate the key ideas and convergence properties of the residue continuation

method. The last example is an application to a multidimensional set of equations encountered in the analysis of the stability of mechanical structures [22, 9].

4.1. STARTING FROM A REMOTE ESTIMATE

This subsection illustrates how to find a fixed point solution on a branch using the residue continuation method, starting from a disjoint guess. Consider the scalar equation

$$(59) \quad \mathbf{A}(u, \mu) = (u - 1)^2 + \mu + 1$$

with $\mathbf{A} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. The branch of steady states is carried out as follows. Consider $(u^* = 55, \mu^* = -10)$ as the initial guess marked with a square in Figure 1(a). It is a far too coarse guess for the classical continuation methods. Nevertheless, using the residue continuation method with $k_{\mathbf{u}} = 0, k_{\mu} = 1$, a path is found (denoted by pt#1 in Figure 1(a)) from this remote estimate to the corresponding solution on the branch marked with a dot. Starting from this point, the branch of steady states can be determined using the classical pseudo-arclength continuation method (solid and dash-dotted lines). From the initial remote estimate and choosing other directions, other paths can be also taken, and an example where $k_{\mathbf{u}} = 0.1, k_{\mu} = 1$ is shown in Figure 1(b).

The example illustrates that for some given discretized operator, one can start from a remote initial guess and find, using systematically the residue continuation method, a fixed point solution of the operator. One could have alternatively started the Newton-Raphson scheme from the initial estimate, but this is not guaranteed to converge. The method can thus be applied for problems where no trivial or analytical solution is known, and determine a first fixed point solution. Also, in case one does not want to compute solutions from the trivial state because of computational constraints, the method here can provide an efficient initial nontrivial solution.

4.2. LOCATION OF AN ISOLATED BRANCH

Suppose a dynamical system has more than a single branch of fixed point solutions and that only one of these branches has been computed (for example, by starting from a trivial solution). In this example, we illustrate how the residue continuation method can be used to compute the other branches. Again, we take a simple scalar equation, in this case

$$(60) \quad \mathbf{A}(u, \mu) \equiv ((u - 1)^2 + \mu + 1)((u - 10)^2 - \mu - 5)((u - 7)^2 + \mu + 10),$$

where $\mathbf{A} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. As shown in Figure 2(a), there are three branches of steady states labelled #1, #2 and #3 corresponding to the three factors of the right hand side of equation (60). The branches #1 and #3 are connected through a transcritical bifurcation while the branch #2 is isolated. Our target is to compute a point on the branch #2 starting from a point of the branch #1 as a remote estimate.

Using the procedure in Subsection 4.1, we first reach a point on branch #3 through the residue continuation path pt#1 (Figure 2(a)). From this fixed point, the branch #3 is computed using standard pseudo-arclength continuation rounding the saddle-node bifurcation at $u = 7$, $\mu = -10$. In a typical application, one would detect the transcritical bifurcation at $(u = 4.75, \mu^* = -15)$ and then use a branch switching method to calculate the branch #1. Alternatively, one can also take the point $(\mathbf{u} = 5.56, \mu^* = -12)$ on the branch #3 as a remote estimate of a point on the branch #1 and determine the latter using the procedure as in Subsection 4.1. The dotted line pt#2 (Figure 2(a)) represents the corresponding residue continuation method path to the solution on the branch #1 $(\mathbf{u} = 4.32, \mu = -12)$. Therefore, the residue continuation scheme can also be used to switch branches. From the endpoint of pt#2, the branch #1 can again be computed with pseudo-arclength continuation. Subsequently, starting from the point $(\mathbf{u} = 2.74, \mu = -4)$ on the branch #1, the isolated branch #2 is reached through the residue continuation path pt #3 (Figure 2(a)).

Using the residue continuation method, no specific treatment is necessary for the switch and no specific correctors are needed as is the case of predictor methods based on interpolation [1, 19, 18, 2]. Furthermore, only the operator itself and its Jacobian are needed in contrast to the predictor method via the tangent [12], where higher order derivatives are needed.

4.3. CONVERGENCE AND ESTIMATES

In the case of a scalar operator $\mathbf{A} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, we now consider the sufficient convergence conditions for the residue continuation methods (9) and (10) as in Propositions 3.3 and 3.5. The local evolution of the norm of the residue versus the residue parameter is constrained by a curve similar to, say, the function

$$(61) \quad f : \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R}^+, \quad \alpha_\nu \mapsto \left| \frac{\alpha_\nu}{1 - \alpha_\nu} \right|,$$

(and plotted as a solid line in Figure 2(c)). This curve partitions the domain $\{\alpha \in \mathbb{R} \setminus \{1\}, \mathbf{r} \geq 0\}$ in two zones. The Newton-Raphson method quadratically converges only below the curve.

For the example in Section 4.2, consider the residue path pt#3 of Figure 2(a). Along this path, the norm of the residue versus the residue parameter for the three values of the arclength step ds is plotted in Figure 2(b). For $ds = 0.01$, the norm of the residue versus α , together with the zone of quadratic convergence, is plotted in Figure 2(c). The starting point is marked by a square ($\alpha = 1$). As already mentioned in Section 2, the residue first increases ($\alpha > 1$) up to the maximum for which $\alpha = 1$. As a matter of fact, denoting $\mathbf{r}(\alpha) \equiv \mathbf{A}(\mathbf{u}(\alpha))$, it follows from the identity

$$\frac{d}{d\alpha} \|\mathbf{r}\|^2 = 2\|\mathbf{r}\| \frac{d}{d\alpha} \|\mathbf{r}\|$$

that every non zero extremum of the norm of the residue is attained for $\alpha \in I_\nu$ as defined in (31) when

$$\frac{d}{d\alpha} \|\mathbf{r}\| = 0.$$

That is, taking into account equation (30)–(31), when either $\|\mathbf{r}_{\nu-1}\| = 0$ or $\alpha_\nu = 1$. Therefore, as long as \mathbf{r}_ν does not vanish, the extrema of the residue are located along the line $\{\alpha = 1\}$ of the graph of $\|\mathbf{r}\|$ as a function of α . Moreover, if $0 < \alpha_{\nu-1} < \alpha_\nu = 1 < \alpha_{\nu+1}$ then

$$\|\mathbf{r}_\nu\| = \|\mathbf{r}_{\nu-1}\| < \|\mathbf{r}_{\nu+1}\| \quad \text{and} \quad \|\mathbf{r}_{\nu-2}\| > \|\mathbf{r}_\nu\| = \|\mathbf{r}_{\nu-1}\|,$$

that is, if α_ν crosses $\alpha_\nu = 1$ from below, then $\|\mathbf{r}_\nu\| = \|\mathbf{r}_{\nu-1}\|$ is a local minimum. The same arguments show that, conversely, if α_ν crosses $\alpha_\nu = 1$ from above, then $\|\mathbf{r}_\nu\| = \|\mathbf{r}_{\nu-1}\|$ is a local maximum.

In general, the sequence $(\alpha_\nu)_\nu$ is not monotonous, while $\|\mathbf{r}_\nu\|$ is monotonous as long as $|\alpha_\nu|$ remains in either intervals $]0, 1[$ or $]1, +\infty[$. For example, if $\alpha_{\nu_0} > 1$ is, at least locally, a maximum value of the parameter α_ν , then $\|\mathbf{r}_\nu\|$ still increases beyond $\|\mathbf{r}_{\nu_0}\|$ for $\nu > \nu_0$, but at a lower rate than for $\nu < \nu_0$. Indeed, in the example above, the residue $\|\mathbf{r}_\nu\|$ (respectively α_ν) increases from the zero value (from $\alpha_\nu = 1$) until α_ν yields a local maximum $\alpha_{\max} > 1$ from which the residue still increases but at a lower rate. One observes this fold point in both Figures 2(b) and 3(b). Then, α_ν crosses $\alpha_\nu = 1$ from above which causes the residue to decrease until $\alpha_\nu = 0$. This leads to $\mathbf{r}_\nu = 0$. Eventually, the residue decreases from the maximum down to zero. The corresponding end points of the path #3 are marked with a circle for $ds = 10^{-3}$, $ds = 10^{-2}$ and $ds = 5 \cdot 10^{-5}$ from right to left on the α -axis on Figure 2(b). In Figures 3(a) and 3(b) are plotted the corresponding curves of the residue as a function of \mathbf{u} , and \mathbf{u} versus α .

4.4. SHALLOW TRUSS ARCHES

As an illustration of the residue continuation method in a multidimensional dynamical system, we analyze the dynamics of shallow truss arches [9], as depicted in Figure 4(a). In the (x, y) plane, this two rod system is characterized by the Young moduli E_i and the areas A_i of the sections of the rods, $i = 1, 2$. Denote by E and A the nominal values of the perfect system. Let $(x_1 = -1, y_1 = 1)$ and $(x_2 = 1, y_2 = 1)$ be the coordinates of stands 1 and 2. Define the lengths of the rods at the equilibrium before loading and for a given vertical load f as

$$l_i = ((x_3 - x_i)^2 + (y_3 - y_i)^2)^{1/2}$$

$$\hat{l}_i = ((x - x_i)^2 + (y - y_i)^2)^{1/2},$$

where (x_3, y_3) and (x, y) are the coordinates of stand 3 of the two rod system without or with a given vertical load $|f| > 0$.

The equilibrium of the arches is governed by the system of equations

$$\begin{pmatrix} F_x \\ F_y \end{pmatrix} \equiv \begin{pmatrix} \sum_{i=1}^2 \frac{E_i A_i}{EA} \left(\frac{1}{l_i} - \frac{1}{\hat{l}_i} \right) (x - x_i) \\ \sum_{i=1}^2 \frac{E_i A_i}{EA} \left(\frac{1}{l_i} - \frac{1}{\hat{l}_i} \right) (y - y_i) - f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The vector of the unknowns is $\mathbf{u} \equiv (x, y)$ and the vector of the parameters is

$$\left(x_1, y_1, x_2, y_2, x_3, y_3, \frac{E_1 A_1}{EA}, \frac{E_2 A_2}{EA}, f \right).$$

For the perfect case, set

$$x_1 = -1, y_1 = 1, x_2 = 1, y_2 = 1, x_3 = 0, y_3 = 0, \frac{E_1 A_1}{EA} = \frac{E_2 A_2}{EA} = 1.$$

In this case, Figure 4(b) depicts y versus the load f . The solid and the dotted lines denote the stable and unstable parts of the branch, respectively. There are two saddle-node bifurcations which lead to hysteresis behavior as a function of the load f . Starting from the point a on the branch for which $f = 0$, a vertical load ($f > 0$) yields eventually the saddle-node bifurcation at point b . For a slightly larger load, the system jumps down to the stable fixed point c . Then, if the load is decreased from point c , the system eventually reaches the saddle-node bifurcation at point d . For slightly smaller values of f , the system jumps up to the stable fixed point e . This behavior is the well known snap-through phenomenon.

For $x_3 = 0.1$, an imperfect branch #2 (Figure 4(c)) appears. Starting from a point on the perfect branch $x = 0, y = 0$ for $f = 0$ (denoted by #1 in Figure 4(c)), the residue path towards the imperfect branch #2 is again computed using the residue continuation method. Thereafter, the imperfect branch #2 is computed using classical pseudo-arclength continuation method. In the imperfect case ($x_3 = 0.1$) hysteresis occurs both for x and y (Figure 5). In this example, the residue continuation method is an efficient tool to compute the isolated branches, hence to provide insight into the imperfect snap-through of the shallow truss arches system.

5. SUMMARY AND DISCUSSION

Many physical systems exhibit disjointed equilibria. In this paper, the so-called residue (homotopy) continuation method is introduced. The method is based on the Global Homotopy (6) and can start from a remote initial estimate. We have derived explicit conditions ensuring the quadratic convergence of the Newton-Raphson algorithm for the residue continuation method in both natural and pseudo-arclength continuation.

Along a few examples, the capabilities of the residue continuation method are illustrated. Isolated branches are indeed determined in quite general cases. As shown in Subsection 4.2, the method may even be used as a branch switching algorithm. The branch switching near a transcritical bifurcation, which is classically a delicate issue, enters the scope of the method.

As an illustration of the residue continuation method in a more practical (although still low dimensional) system, we considered in Subsection 4.4 the imperfect snap-through in a shallow two truss arch system [9]. In this example, we show that an isolated branch can be reached. The residue continuation method is likely to be a good candidate for the treatment of the imperfect bifurcations.

With respect to earlier methods suggested to determine isolated branches, a complete picture of the fixed point set is attainable with the residue continuation method without referring to any preliminary hierarchy of the singularities. Any singular points on the residue continuation path can be handled by means of a local finite step of the control or residue parameter. Finally, no higher order derivatives (only the Jacobian) are necessary for the residue continuation scheme in contrast to the predictor method via the tangent [12]. The structure of fixed point solutions is a delicate and important issue in the natural systems, in particular those that exhibit imperfections. The residue continuation is reasonably efficient, robust and easy to implement as only the operator and its Jacobian are needed.

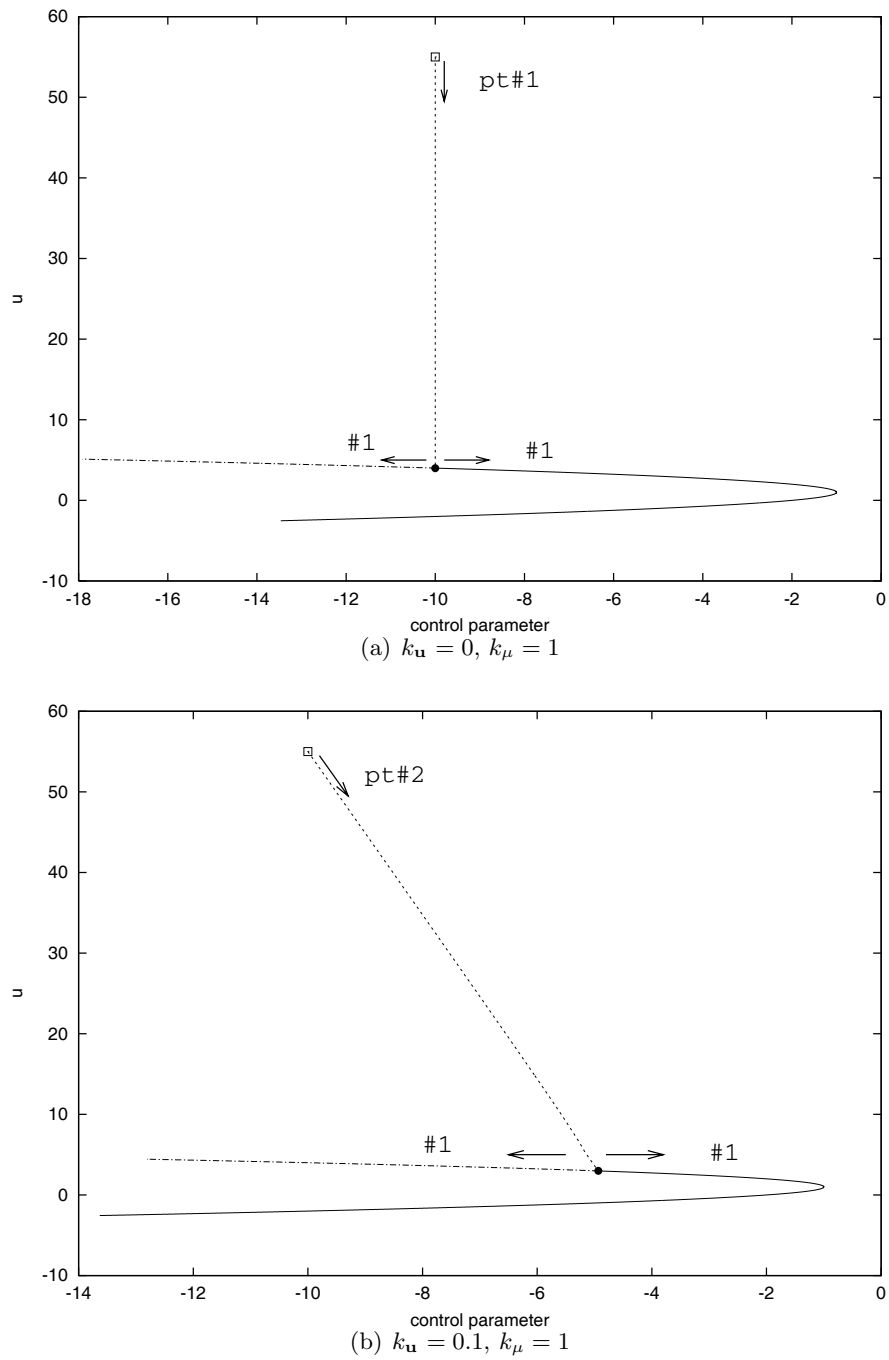
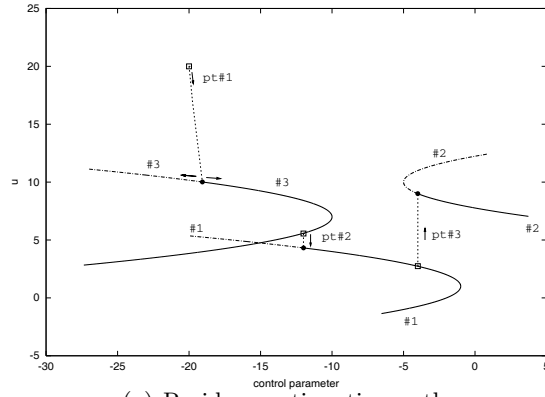
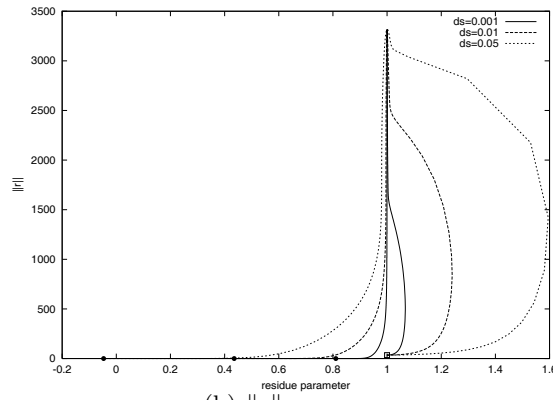
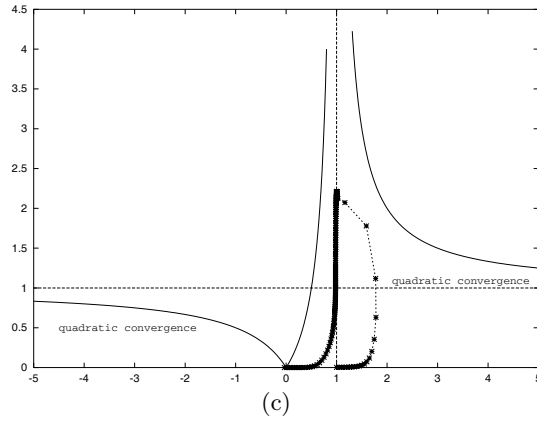


Fig. 1. Plot of the residue continuation path pt#1 (dotted line) from a coarse initial guess (marked with a square) of the operator (59).



(a) Residue continuation paths

(b) $\|r\|$ versus α 

(c)

Fig. 2. For the operator (60), (a) continuation paths, (b) the norm of the residue versus α along pt#3 with $ds = 10^{-2}$, and (c) the sketch of the evolution of the norm of the residue (divided by 1500 for convenience) for pt#3 together with the function (61) as a solid line.

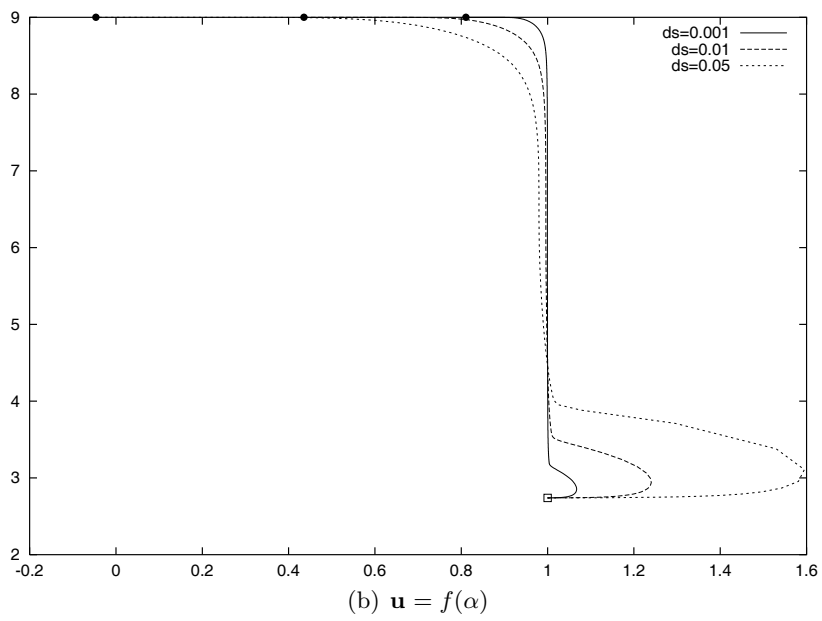
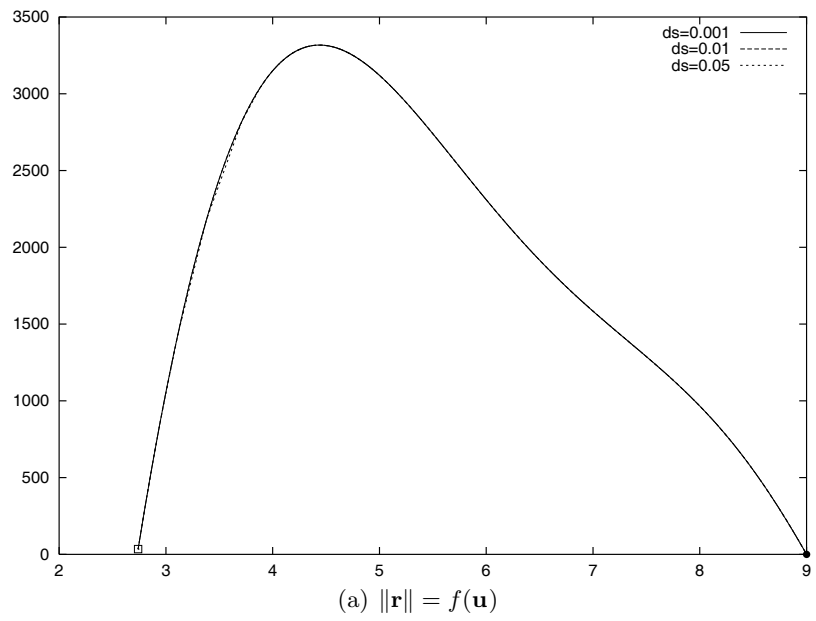
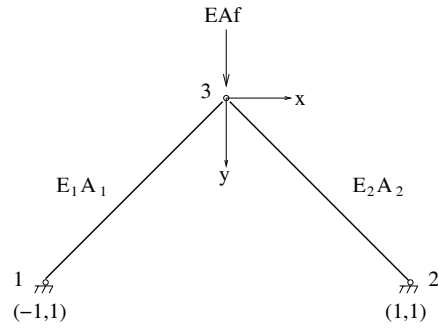


Fig. 3. (a) Norm of the residue versus \mathbf{u} , and (b) \mathbf{u} as a function of the residue parameter corresponding to Figure 2.



(a) Residue continuation paths

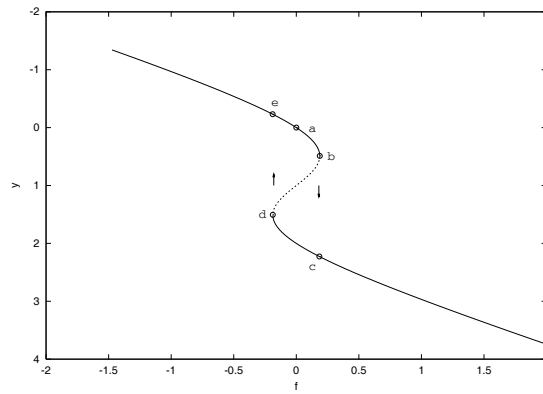
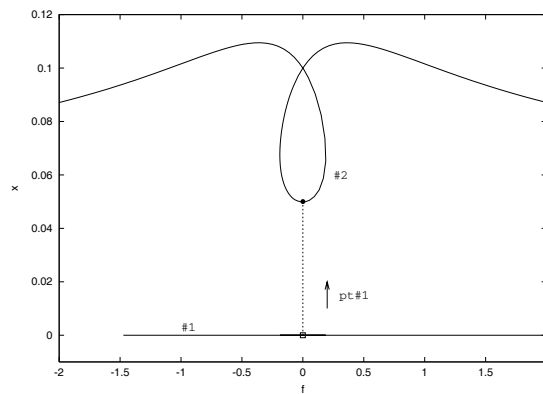
(b) y versus the load f (c) x versus the load f

Fig. 4. (a) A sketch of the shallow truss arches system. (b) Bifurcation diagram for the perfect shallow truss arches system. (c) Bifurcation diagram of the imperfect ($x_3 = 0.1$) shallow truss arches system.

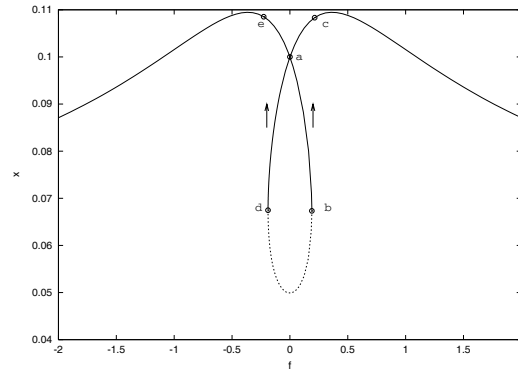
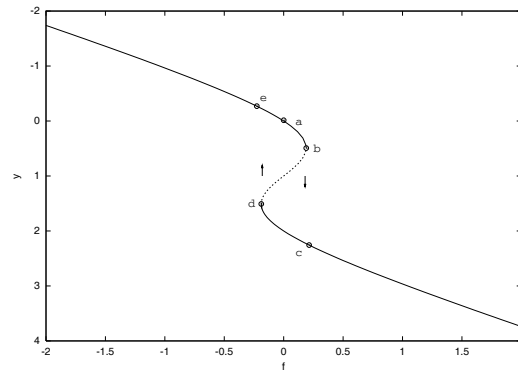
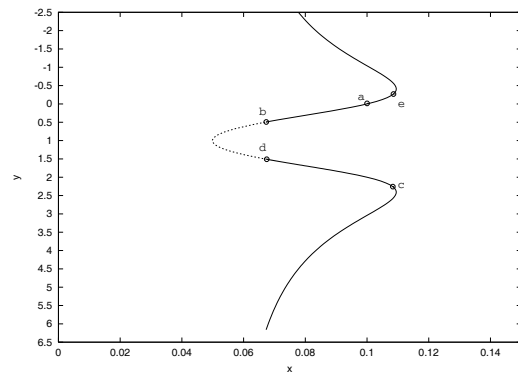
(a) x versus f (b) y versus f (c) $y(f)$ versus $x(f)$

Fig. 5. Bifurcation diagram for the shallow truss arches in the imperfect case. In (a) and (b), y and x are plotted versus the load f , respectively, while in (c) the parametric curve $(y(f), x(f))$ is shown. Solid and dotted linestyles denote the stable and unstable parts of the branches, respectively. In the imperfect case ($x_3 = 0.1$) hysteresis occurs for both y and x as the load f is varied.

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