ON THE GROWTH OF SOLUTIONS OF COMPLEX DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS OF FINITE LOGARITHMIC ORDER

TING-BIN CAO, KAI LIU and JUN WANG

Communicated by the former editorial board

The concept of logarithmic order is used to investigate the growth of solutions of the linear differential equations

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_1(z)f' + A_0(z)f = 0,$$

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_1(z)f' + A_0(z)f = F(z),$$

where $A_0 \neq 0$, $A_1, \ldots, A_{k-1}$ and $F \neq 0$ are transcendental entire functions with orders zero.

AMS 2010 Subject Classification: 34M10, 30D35.

Key words: entire function, differential equation, Nevanlinna theory, finite logarithmic order.

1. DEFINITIONS AND INTRODUCTIONS

We shall assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions (see *e.g.* [10, 25]). Let us define inductively, for $r \in [0, +\infty)$, $\exp_1 r := e^r$ and $\exp_{n+1} r := \exp(\exp_n r)$, $n \in \mathbb{N}$. For all r sufficiently large, we define $\log_1 r := \log^+ r = \max\{\log r, 0\}$ and $\log_{n+1} r := \log(\log_n r)$, $n \in \mathbb{N}$. We also denote $\exp_0 r := r =: \log_0 r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$. For the unity of notations, we here introduce the concepts of (p,q)-order and (p,q)-type (see *e.g.* [16, 17]) as follows. Note that we here assume that p and q are all integers.

Definition 1.1. The (p, q)-order of a meromorphic function f in the plane is defined by

$$\sigma_{(p,q)}(f) = \limsup_{r\to+\infty} \frac{\log_p T(r, f)}{\log_q r},$$

where $T(r, f)$ denotes the Nevanlinna character of the function f.

Remark 1.1. (i) $\sigma_{(p,1)}(f) := \sigma_p(f)$ is just the iterated p-order of f (see *e.g.* [18, 22, 23]. In particular, $\sigma_{(1,1)}(f) := \sigma(f)$ and $\sigma_{(2,1)}(f) := \sigma_2(f)$ are just the order and hyper-order of f, respectively (see *e.g.* [26]).

(ii) If f is an entire function, then

$$\sigma_{(p,q)}(f) = \limsup_{r\to+\infty} \frac{\log_{p+1} M(r,f)}{\log_q r},$$

where $M(r,f)$ denotes the maximum modulus of f in the circle $|z| = r$.

(iii) $\sigma_{(1,2)}(f) := \sigma_{\log}(f)$ is just the logarithmic order of f (see *e.g.* [5]).

(iv) Obviously, the logarithmic order of any non-constant rational function f is one, and thus, any transcendental meromorphic function in the plane has logarithmic order no less than one. Moreover, any meromorphic function with finite logarithmic order in the plane is of order zero.

Definition 1.2. The (p,q)-type of a meromorphic function f with (p,q)-order $0 < \sigma_{(p,q)}(f) < +\infty$ in the plane is defined by

$$\tau_{(p,q)}(f) = \limsup_{r\to+\infty} \frac{\log_{p-1} T(r,f)}{(\log_{q-1} r)^{\sigma_{(p,q)}(f)}}.$$

Remark 1.2. (i) $\tau_{(1,1)}(f) := \tau(f)$ is just the type of f.

(ii) If f is an entire function, then

$$\tau_{(p,q)}(f) = \limsup_{r\to+\infty} \frac{\log_{p} M(r,f)}{(\log_{q-1} r)^{\sigma_{(p,q)}(f)}}.$$

(iii) $\tau_{(1,2)}(f) := \tau_{\log}(f)$ is just the logarithmic type of f.

(iv) It is obvious that the logarithmic type of any non-constant polynomial P equals its degree $deg(P)$, that any non-constant rational function is of finite logarithmic type, and that any transcendental meromorphic function whose logarithmic order equals one in the plane must be of infinite logarithmic type.

Definition 1.3. The (p,q)-exponent of convergence of zeros or distinct zeros of a meromorphic function f in the plane are defined by

$$\lambda_{(p,q)}(f) = \limsup_{r\to+\infty} \frac{\log_p n(r,\frac{1}{f})}{\log_q r}$$

or

$$\overline{\lambda}_{(p,q)}(f) = \limsup_{r\to+\infty} \frac{\log_p \overline{n}(r,\frac{1}{f})}{\log_q r}$$

respectively, where $n(r,\frac{1}{f})$ (or $\overline{n}(r,\frac{1}{f})$) denotes the number of zeros (or distinct zeros) of f in the disc $|z| \le r$.

Remark 1.3. (i) $\lambda_{(1,2)}(f) := \lambda_{\log}(f)$ is just the logarithmic exponent of convergence of zeros of f (see *e.g.* [5]).

(ii) It is trivial that for the case $p \geq q \geq 1$, $\lambda_{(p,q)}(f)$ (or $\overline{\lambda}_{(p,q)}(f)$) can also be given by making use of the notation of the counting function of zeros (or distinct zeros) of f, $N(r, \frac{1}{f})$ (or $\overline{N}(r, \frac{1}{f})$), to replace the notations $n(r, \frac{1}{f})$ (or $\overline{n}(r, \frac{1}{f})$), respectively (see *e.g.* [21], Definitions 10 and 11). However, it does not hold for the case that $p < q$. For example, the logarithmic order of $N(r, \frac{1}{f})$ equals $\lambda_{\log}(f) + 1$, (see [5], Theorem 4.1).

It is well-known that all solutions of the $k(\geq 2)$ order linear differential equations

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_1(z)f' + A_0(z)f = 0 \tag{1}$$

and

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_1(z)f' + A_0(z)f = F(z) \tag{2}$$

are entire functions when the coefficients $A_0 \not\equiv 0, A_1, \ldots, A_{k-1}$ and $F \not\equiv 0$ are entire functions. Moreover, Wittich [24] proved that the coefficients $A_0, \ldots, A_{k-1}$ in (1) are all polynomials if and only if all solutions of the equation (1) are of finite order of growth. For the case that coefficients in (1) are all polynomials, there are a lot of classical and relevant results to estimate the growth of solutions of finite rational order (see for example [9], or for more details see [19, 20]. In this paper, we consider the case where there is at least one transcendental coefficient in (1).

By Wittich's result, there exists at least a solution with infinite order when any coefficient in (1) is transcendental. Frei [7] showed that if A_j is the last transcendental function in the coefficients, sequence $A_0, \ldots, A_{k-1}$ in (1), then the equation (1) possesses at most j linearly independent solutions f of finite order. Also, it follows from the lemma of logarithmic derivative that if A_0 is the unique transcendental entire function, while other coefficients are all polynomials, then all solutions of (1) are of infinite order. Thus, it is an important subject on how to express explicitly the growth of solutions of infinite order.

As far as we known, Bernal [2] firstly introduced the idea of iterated order to express the fast growth of solutions of complex linear differential equations. Since then, many authors obtained further results on iterated order of solutions of (1) and (2), see *e.g.* [1, 3, 4, 18, 19, 20]. We here state some of them. The *finiteness degree of growth* $i(f)$ of a meromorphic function f in the plane is $i(f) = 0$, for rational functions, $i(f) := \min\{j \in \mathbb{N} : \sigma_{(j,1)}(f) < +\infty\}$, and $i(f) = +\infty$ otherwise (see *e.g.* [22, 23], also [18, 19]).

Theorem 1.1 ([18], Theorem 2.3, [3], Corollary 2.1, [1], Corollary 1.7). *Let* $A_0, A_1, \ldots, A_{k-1}$ *be entire functions, and let* $i(A_0) = p$ $(0 < p < +\infty)$. *Assume that either*

$$\max\{i(A_j) : j = 1, 2, \ldots, k-1\} < p$$

or

$$\max\{\sigma_{(p,1)}(A_j) : j = 1, 2, \ldots, k-1\} \le \sigma_{(p,1)}(A_0) := \sigma \ (0 < \sigma < +\infty),$$
$$\max\{\tau_{(p,1)}(A_j) : \sigma_{(p,1)}(A_j) = \sigma_{(p,1)}(A_0)\} < \tau_{(p,1)}(A_0) := \tau \ (0 < \tau < +\infty).$$

Then, every solution $f \not\equiv 0$ *of the equation* (1) *satisfies* $i(f) = p + 1$ *and* $\sigma_{(p+1,1)}(f) = \sigma_{(p,1)}(A_0)$.

Note that for the special case that $\max\{i(A_j) : j = 1, 2, \ldots, k-1\} < i(A_0) = p$ and $\sigma_{(p,1)}(A_0) = 0$ in the above theorem, every solution $f \not\equiv 0$ of (1) satisfies $i(f) = p + 1$ and $\sigma_{(p+1,1)}(f) = \sigma_{(p,1)}(A_0) = 0$. For so many solutions of iterated $p+1$-order zero, how to express better the growth of solutions than now?

Recently, Liu, Tu and Shi [21] firstly introduced the concepts of (p, q)-order and (p, q)-type for the case $p \ge q \ge 1$ to investigate the entire solutions of (1) and (2), and obtained some results which improve and generalize some previous results, see for example:

Theorem 1.2 ([21], Theorems 2.2–2.3). *Let* $p \ge q \ge 1$, *and let* $A_0, A_1, \ldots, A_{k-1}$ *be entire functions such that either*

$$\max\{\sigma_{(p,q)}(A_j) : j \ne 0\} < \sigma_{(p,q)}(A_0) < +\infty,$$

or

$$\max\{\sigma_{(p,q)}(A_j) : j \ne 0\} \le \sigma_{(p,q)}(A_0) < +\infty,$$
$$\max\{\tau_{(p,q)}(A_j) : \sigma_{(p,q)}(A_j) = \sigma_{(p,q)}(A_0) > 0\} < \tau_{(p,q)}(A_0),$$

then every nontrivial solution f *of* (1) *satisfies* $\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_0)$.

In essence, for a meromorphic function h in the plane satisfying $\sigma_{(p,1)}(h) = 0$ $(p \ge 2)$, the fast growth of h can be carefully expressed by taking suitable $q (p \ge q \ge 2)$ such that its (p,q)-order is a positive and finite value. However, this case for $p = 1$ does not hold. Hence, Theorem 1.2 expresses the fast growth of solutions more precise than Theorem 1.1 for the special case $\sigma_{(p,1)}(A_0) = 0$ $(p \ge 2)$. Note that if A_0 is a transcendental function with order zero in Theorem 1.1, every solution $f \not\equiv 0$ of (1) is of infinite order and hyper-order zero, and that the authors did not consider this case in Theorem 1.2. Thus, it arises a natural problem: *How to express the growth of solutions of* (1) *when now the dominant coefficient* A_0 *is transcendental and of order zero?*

In this paper, for so many solutions of hyper-order zero, we want to solve this problem by making use of the idea of logarithmic order due to Chern [5]. One of the key tools is an extension of the well-known logarithmic derivative lemma due to Heittokangas, Korhonen and Rättyä (see Lemma 3.1 in Section 3). The remainder of the paper is organized as follows. In Section 2, we shall show our main results which supplement the research for the case of the dominant transcendental coefficient with order zero. Section 3 is for some lemmas and basic theorems, in which we prove some important results that are very interesting by themselves. The other sections are for the proofs of our main results.

2. MAIN RESULTS

Firstly, we consider the case that an arbitrary coefficient A_s ($s \in \{0, 1, \dots, k-1\}$) dominates the growth of solutions, and obtain the following theorem.

THEOREM 2.1. *Let $A_0, A_1, \dots, A_{k-1}$ be entire functions such that there exists one transcendental function $A_s (0 \le s \le k-1)$ satisfying*

$$\max\{\sigma_{(1,2)}(A_j) : j \neq s \, and \, j = 0, 1, \dots, k-1\} < \sigma_{(1,2)}(A_s) < +\infty.$$

Then every transcendental entire solution f of equation (1) satisfies $\sigma_{(2,1)}(f) = 0$ and

$$\sigma_{(2,2)}(f) \le \sigma_{(1,2)}(A_s) + 1 \le \sigma_{(1,2)}(f) + 1.$$

If f is not transcendental, then it must be a polynomial with degree no greater than $s-1$. Furthermore, there is at least one entire solution, say f_1, which satisfies

$$1 \le \sigma_{(1,2)}(A_s) \le \sigma_{(2,2)}(f_1) \le \sigma_{(1,2)}(A_s) + 1.$$

For the special case that the dominant coefficient is A_0, we obtain the next result.

THEOREM 2.2. *Let $A_0, A_1, \dots, A_{k-1}$ be entire functions. If A_0 is transcendental and satisfies*

$$\max\{\sigma_{(1,2)}(A_j) : j = 1, 2, \dots, k-1\} < \sigma_{(1,2)}(A_0) < +\infty,$$

then every nonzero solution f of (1) satisfies $\sigma_{(2,1)}(f) = 0$ and

$$1 \le \sigma_{(1,2)}(A_0) \le \sigma_{(2,2)}(f) \le \sigma_{(1,2)}(A_0) + 1.$$

By Theorems 1.1 and 2.2, we obtain immediately the following corollary for second order differential equations.

Corollary 2.1. *Let A be a transcendental entire function with finite logarithmic order, then every nonzero solution f of equation $f'' + A(z)f = 0$ satisfies $\sigma_{(1,1)}(f) = +\infty$, $\sigma_{(2,1)}(f) = 0$ and $1 \leq \sigma_{(1,2)}(A_0) \leq \sigma_{(2,2)}(f) \leq \sigma_{(1,2)}(A_0) + 1$.*

If there exist some coefficients whose finite logarithmic orders are the same as the logarithmic order of the last coefficient A_0, then we obtain the following theorem.

Theorem 2.3. *Let $A_0, A_1, \ldots, A_{k-1}$ be entire functions. If A_0 is transcendental and satisfies*

$$\max\{\sigma_{(1,2)}(A_j) : j = 1, 2, \ldots, k-1\} = \sigma_{(1,2)}(A_0) < +\infty,$$

and

$$\max\{\tau_{(1,2)}(A_j) : \sigma_{(1,2)}(A_j) = \sigma_{(1,2)}(A_0)\} < \tau_{(1,2)}(A_0) \leq +\infty,$$

then any transcendental entire solution f of (1) satisfies $\sigma_{(2,1)}(f) = 0$ and

$$1 \leq \sigma_{(1,2)}(A_0) \leq \sigma_{(2,2)}(f) \leq \sigma_{(1,2)}(A_0) + 1.$$

Furthermore, if $\sigma_{(1,2)}(A_0) > 1$, then the degree of any nonzero polynomial solution of (1) is not less than $\tau_{(1,2)}(A_0)$; if $\sigma_{(1,2)}(A_0) = 1$, then any nonzero solution of (1) can not be a polynomial.

The following theorem is for the special case that A_0 is transcendental and other coefficients are all polynomials.

Theorem 2.4. *Let A_0 be a transcendental entire function with finite logarithmic order, and let $A_1, \ldots, A_{k-1}$ be polynomials. Then any nonzero entire solution f of (1) satisfies $\sigma_{(1,1)}(f) = +\infty$, $\sigma_{(2,1)}(f) = 0$ and*

$$1 \leq \sigma_{(1,2)}(A_0) \leq \sigma_{(2,2)}(f) \leq \sigma_{(1,2)}(A_0) + 1.$$

Considering nonhomogeneous linear differential equations (2), we obtain the following three results corresponding to the above theorems.

Theorem 2.5. *Assume that $A_0, A_1, \ldots, A_{k-1}$ satisfy the hypotheses of Theorem 2.1. Let F be an entire function. Set f_0 is a solution of the equation (2), and $g_1, g_2, \ldots, g_k$ are a solution base of the corresponding homogeneous equation (1) of equation (2). Then*

(i) *if either $\sigma_{(2,2)}(F) > \sigma_{(1,2)}(A_s) + 1$, then all solutions of (2) satisfy $\sigma(f) = \sigma(F)$;*

(ii) *if $\sigma_{(2,2)}(F) < \sigma_{(1,2)}(A_s) + 1$, then all solutions f of (2) satisfy that $\sigma_{(1,2)}(A_s) + 1 \geq \sigma_{(2,2)}(f)$, and that $\sigma_{(2,2)}(f) = \lambda_{(2,2)}(f) = \overline{\lambda}_{(2,2)}(f)$ holds for any solution which satisfies $\sigma_{(1,2)}(A_s) + 1 = \sigma_{(2,2)}(f)$.*

Theorem 2.6. *Assume that $A_0, A_1, \ldots, A_{k-1}$ satisfy the hypotheses of Theorem 2.2 or Theorem 2.4. Let F be an entire function. Then*

(i) *if $\sigma_{(2,2)}(F) > \sigma_{(1,2)}(A_0)+1$, then all solutions f of* (2) *satisfy $\sigma_{(2,2)}(f) = \sigma_{(2,2)}(F)$;*

(ii) *if $\sigma_{(2,2)}(F) < \sigma_{(1,2)}(A_0) + 1$, then all solutions f of* (2) *satisfy that $\sigma_{(1,2)}(A_0)+1 \geq \sigma_{(2,2)}(f)$, that $\sigma_{(2,2)}(f) \geq \sigma_{(1,2)}(A_0)$ possibly outside one exceptional solution, and that $\sigma_{(2,2)}(f) = \lambda_{(2,2)}(f) = \overline{\lambda}_{(2,2)}(f)$ holds for any solution which satisfies $\sigma_{(1,2)}(A_0) + 1 = \sigma_{(2,2)}(f)$.*

Theorem 2.7. *Assume that $A_0, A_1, \ldots, A_{k-1}$ satisfy the hypotheses of Theorem 2.3. Let F be an entire function. Then*

(i) *if $\sigma_{(2,2)}(F) > \sigma_{(1,2)}(A_0)+1$, then all solutions f of* (2) *satisfy $\sigma_{(2,2)}(f) = \sigma_{(2,2)}(F)$;*

(ii) *if $\sigma_{(2,2)}(F) < \sigma_{(1,2)}(A_0) + 1$, then all solutions f of* (2) *satisfy that $\sigma_{(1,2)}(A_0) + 1 \geq \sigma_{(2,2)}(f)$, and that $\sigma_{(2,2)}(f) = \lambda_{(2,2)}(f) = \overline{\lambda}_{(2,2)}(f)$ holds for any solution which satisfies $\sigma_{(1,2)}(A_0) + 1 = \sigma_{(2,2)}(f)$.*

3. SOME BASIC THEOREMS AND LEMMAS

The first lemma is an extension of the well-known logarithmic derivative lemma due to Heittokangas, Korhonen and Rättyä [13].

Lemma 3.1 ([13], Theorem 4.1 and Remark 4.2). *Let k and j be integers such that $k > j \geq 0$. Let f be a meromorphic function in the plane $\mathbb{C}$ such that $f^{(j)}$ does not vanish identically. Then, there exists an $r_0 > 1$ such that*

$$m(r, \frac{f^{(k)}}{f^{(j)}}) \leq (k-j) \log^+ \frac{\rho(T(\rho, f))}{r(\rho - r)} + \log \frac{k!}{j!} + (k-j)5.3078$$

for all $r_0 < r < \rho < +\infty$. If f is of finite order s, then

$$\limsup_{r \to +\infty} \frac{m(r, \frac{f^{(k)}}{f^{(j)}})}{\log r} \leq \max\{0, (k-j)(s-1)\}.$$

Lemma 3.2 ([19], Lemma 1.1.2). *Let $g : (0, +\infty) \to \mathbb{R}$ and $h : (0, +\infty) \to \mathbb{R}$ be monotone nondecreasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E_2 of finite logarithmic measure. Then for any $\beta > 1$, there exists r_0 such that $g(r) \leq h(r^\beta)$ for all $r > r_0$.*

Lemma 3.3 ([6]). *Let $f_1, f_2, \ldots, f_k$ be linearly independent meromorphic solutions of the differential equation* (1) *with meromorphic functions A_0, A_1, $\ldots$, A_{k-1} in the plane as the coefficients, then*

$$m(r, A_j) = O\left\{\log\left(\max_{1 \leq n \leq k} T(r, f_n)\right)\right\} \quad (j = 0, 1, \ldots, k-1).$$

LEMMA 3.4 ([5], Theorem 6.1). *If f is a transcendental meromorphic function in the plane $\mathbb{C}$ with finite logarithmic order, then f and f' have the same logarithmic order.*

LEMMA 3.5. *Suppose that h is a transcendental entire function with finite logarithmic order and $f = e^h$. Then $\sigma_{(2,2)}(f) = \sigma_{(1,2)}(h)$.*

Proof. By the proof of Theorem 1.45 in [26] we have

$$\log T(r, f) \le 3T(2r, h)$$

and

$$T(r, h) \le \log\{6T(4r, f)\} + 3|h(0)|.$$

This implies $\sigma_{(2,2)}(f) = \sigma_{(1,2)}(h)$. □

LEMMA 3.6. *Let $\Phi(r)$ be a continuous and positive increasing function, defined for r on $(0, +\infty)$, with logarithmic order $\sigma_{(1,2)}(\Phi)$. Then for any subset E_1 of $[0, +\infty)$ that has finite linear measure, there exists a sequence $\{r_n\}, r_n \notin E_1$ such that*

$$\sigma_{(1,2)}(\Phi) = \lim_{r_n \to +\infty} \frac{\log \Phi(r_n)}{\log \log r_n}.$$

Proof. Since $\sigma_{(1,2)}(\Phi) = \limsup_{r\to+\infty} \frac{\log \Phi(r)}{\log\log r}$, there exists a sequence $\{r_n'\}$ $(r_n' \to +\infty)$ such that

$$\lim_{r_n' \to +\infty} \frac{\log \Phi(r_n')}{\log \log r_n'} := \sigma.$$

Set $mE_1 = \delta < +\infty$. Then for $r_n \in [r_n', r_n' + \delta + 1] \setminus E_1$, we have

$$\frac{\log \Phi(r_n)}{\log \log r_n} \ge \frac{\log \Phi(r_n')}{\log \log(r_n' + \delta + 1)} = \frac{\log \Phi(r_n')}{\log(\log r_n' + \log(1 + \frac{\delta+1}{r_n'}))}.$$

Hence,

$$\lim_{r_n \to +\infty} \frac{\log \Phi(r_n)}{\log \log r_n} \ge \lim_{r_n' \to +\infty} \frac{\log \Phi(r_n')}{\log(\log r_n' + \log(1 + \frac{\delta+1}{r_n'}))} = \sigma_{(1,2)}(\Phi).$$

This gives

$$\lim_{r_n \to +\infty} \frac{\log \Phi(r_n)}{\log \log r_n} = \sigma_{(1,2)}(\Phi). \quad \square$$

LEMMA 3.7. *Let f be a transcendental entire function with finite logarithmic order $0 < \sigma_{(1,2)}(f) < +\infty$ and finite logarithmic type $0 < \tau_{(1,2)}(f) < +\infty$, then for any given $\beta < \tau_{(1,2)}(f)$, there exists a subset I of $[1, +\infty)$ that has infinite logarithmic measure such that $\log M(r, f) > \beta(\log r)^{\sigma_{(1,2)}(f)}$ holds for all $r \in I$.*

Proof. By definitions of logarithmic order and logarithmic type, there exists an increasing sequence $\{r_m\}$ ($r_m \to +\infty$) satisfying $(1+\frac{1}{m})r_m < r_{m+1}$ and

$$\lim_{m\to+\infty} \frac{\log M(r_m,f)}{(\log r_m)^{\sigma_{(1,2)}(f)}} = \tau_{(1,2)}(f).$$

Then there exists a positive integer m_0 such that for all $m > m_0$ and for any given $0 < \varepsilon < \tau_{(1,2)}(f) - \beta$, we have

$$\log M(r_m,f) > (\tau_{(1,2)}(f) - \varepsilon)(\log r_m)^{\sigma_{(1,2)}(f)}. \tag{3}$$

For any given $\beta < \tau_{(1,2)}(f)$, there exists a positive integer m_1 such that for all $m > m_1$ we have

$$\left(\frac{m}{m+1}\right)^{\sigma_{(1,2)}(f)} > \frac{\beta}{\tau_{(1,2)}(f) - \varepsilon}. \tag{4}$$

Take $m \geq m_2 = \max\{m_1, m_0\}$. By (3) and (4), for any $r \in [r_m, (1+\frac{1}{m})r_m]$ we have

$$\begin{aligned}\log M(r,f) &\geq \log M(r_m,f) > (\tau_{(1,2)}(f) - \varepsilon)(\log r_m)^{\sigma_{(1,2)}(f)} \\ &\geq (\tau_{(1,2)}(f) - \varepsilon)\left(\frac{m\log r}{1+m}\right)^{\sigma_{(1,2)}(f)} > \beta(\log r)^{\sigma_{(1,2)}(f)}.\end{aligned}$$

Set $I = \bigcup_{m=m_2}^{+\infty}[r_m, (1+\frac{1}{m})r_m]$, then there holds

$$m_l I = \sum_{m=m_2}^{+\infty} \int_{r_m}^{(1+\frac{1}{m})r_m} \frac{\mathrm{d}t}{t} = \sum_{m=m_2}^{+\infty} \log(1+\frac{1}{m}) = +\infty. \quad \square$$

LEMMA 3.8 ([8]). *Let f be a transcendental meromorphic function in the plane, and let $\alpha > 1$ be a given constant. Then there exist a set $E_2 \subset (1,+\infty)$ that has a finite logarithmic measure, and a constant $B > 0$ depending only on α and (m,n) $(m,n \in \{0,1,\ldots,k\})$ $m < n$ such that for all z with $|z| = r \notin [0,1]\bigcup E_2$, we have*

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right| \leq \left(\frac{T(\alpha r,f)}{r}(\log^{\alpha} r)\log T(\alpha r,f)\right)^{n-m}.$$

LEMMA 3.9 (Wiman-Valiron theory [11, 12]). *Let f be a transcendental entire function, δ be a constant such that $0 < \delta < \frac{1}{8}$, and let z be a point with $|z| = r$ at which $|f(z)| > M(r,f)\cdot\nu(r,f)^{-\frac{1}{8}+\delta}$, where $\nu(r,f)$ denote the central index of f, then the estimation*

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu(r,f)}{z}\right)^n(1+\eta_k(z)) \quad (n \in \mathbb{N})$$

holds for all $|z| = r$ outside a subset E_2 of finite logarithmic measure, where

$$\eta_k(z) = O\Big((\nu(r,f))^{-\frac{1}{8}+\delta}\Big).$$

LEMMA 3.10 ([12], Theorems 1.9–1.10, [15], Satz 4.3–4.4). *Let* $g(z) = \sum_{n=0}^{+\infty} a_n z^n$ *be an entire function,* $\mu(r,g)$ *be the maximum term, and* $\nu(r,g)$ *be the central-index. Then*

(i) *if* $|a_0| \neq 0$, *then* $\log \mu(r,g) = \log |a_0| + \int_0^r \frac{\nu(t,g)}{t} dt$,

(ii) *if* $r < R$, *then* $M(r,g) < \mu(r)\{\nu(R,g) + \frac{R}{R-r}\}$.

By Wiman-Valiron theory we obtain the following result which is an upper bound of the growth of solutions of (1) depending on the finite logarithmic order of entire coefficients. Note that this result can also be obtained by making use of growth estimates for solutions of (1) in the plane due to Heittokangas, Korhonen and Rättyä [14].

THEOREM 3.1. *Let* A_0, A_1, $\ldots$, A_{k-1} *be entire functions such that* $\max\{\sigma_{(1,2)}(A_j) : j = 0, 1, \ldots, k-1\} \leq \alpha < +\infty$. *Then any solution* f *of* (1) *satisfies* $\sigma_{(2,1)}(f) = 0$ *and* $\sigma_{(2,2)}(f) \leq \alpha + 1$.

Proof. Obviously, if f is a polynomial then $\sigma_{(2,1)}(f) = 0$ and $\sigma_{(2,2)}(f) = 0 < \alpha + 1$. Thus, we may assume that f is a transcendental solution of (1). By Lemma 3.9, let $E_2 \subset \mathbb{R}_+$ be a set of finite logarithmic measure such that

$$\frac{f^{(j)}(z)}{f(z)} = \Big(\frac{\nu(r,f)}{z}\Big)^j \big(1 + \eta_k(z)\big) \tag{5}$$

holds for $j = 0, \ldots, k$ and for $r = |z| \notin E_2$, z was chosen as in Lemma 3.9. For instance, we may assume that $|f(z)| = M(r,f)$. Substituting (5) into (1), we get that

$$(1 + o(1))\nu(r,f) \leq r^k \sum_{j=0}^{k-1} |A_j(z)|$$

outside of a possible exceptional set E_2 of finite logarithmic measure. Since A_0, A_1, $\ldots$, A_{k-1} are of finite logarithmic order, it is obvious that

$$M(r, A_j) \leq \exp\big\{(\log r)^{\alpha+\varepsilon}\big\} \quad (j = 0, 1, \ldots, k-1)$$

for any $\varepsilon > 0$ and sufficiently large r. Given $\beta > 1$, by Lemma 3.2 we have

$$\nu(r,f) \leq r^{\beta k} \exp\{(\log r)^{\alpha+2\varepsilon}\}$$

for all r sufficiently large. By Lemma 3.10 we obtain for any $\varepsilon > 0$,

$$\begin{aligned}
\log M(r,f) &\leq \log \mu(r,f) + \log(\nu(2r,f) + 2) \\
&\leq \nu(r,f) \log r + \log(\nu(2r,f)) + O(1) \\
&\leq r^{\beta k} \exp\{(\log r)^{\alpha+3\varepsilon}\} \log r + (\beta k) \log r + (\log r)^{\alpha+3\varepsilon} + O(1).
\end{aligned}$$

This results in

$$\log^+ \log^+ \log^+ M(r,f) \leq (\alpha + 1 + 4\varepsilon) \log\log r + \log\log\log r + O(1),$$

and therefore, $\sigma_{(2,1)}(f) = 0$ and $\sigma_{(2,2)}(f) \leq \alpha + 1 < +\infty$. $\square$

In the following result, we obtain a lower bound of (2,2)-order of solutions depending on the growth of logarithmic order of the dominant coefficient A_0 in [1].

THEOREM 3.2. *Let $A_0, A_1, \ldots, A_{k-1}$ be entire functions such that*

$$\max\{\sigma_{(1,2)}(A_j) : j = 1, 2, \ldots, k-1\} < \sigma_{(1,2)}(A_0) < +\infty.$$

Then any nonzero entire solution f of (1) satisfies

$$\sigma_{(2,2)}(f) \geq \sigma_{(1,2)}(A_0).$$

Proof. Suppose that f is a nonzero solution of the equation (1). By (1), we get

$$-A_0(z) = \frac{f^{(k)}}{f} + A_{k-1}(z)\frac{f^{(k-1)}}{f} + \cdots + A_1(z)\frac{f'}{f}. \tag{6}$$

By Lemma 3.1 and (6), we get that

$$m(r, A_0) \leq \sum_{j=1}^{k-1} m(r, A_j) + \sum_{j=1}^{k} m(r, \frac{f^{(j)}}{f}) = \sum_{j=1}^{k-1} m(r, A_j) + k \log^+ T(2r, f) + O(1). \tag{7}$$

Set

$$b =: \max\{\sigma_{(1,2)}(A_j), j = 1, 2, \ldots, k-1\} < \sigma_{(1,2)}(A_0).$$

Then we have

$$m(r, A_j) \leq T(r, A_j) \leq (\log r)^{b+\varepsilon} \quad (j = 1, 2, \ldots, k-1). \tag{8}$$

Since $\sigma_{(1,2)}(A_0) := \sigma > 0$, by Lemma 3.6 there exists a sequence $\{r_n\}$ such that for all $r_n \notin E_1$,

$$m(r_n, A_0) = T(r_n, A_0) \geq (\log r_n)^{\sigma-\varepsilon} \tag{9}$$

holds for any given ε $(0 < 3\varepsilon < \sigma - b)$. Therefore, substituting (8) and (9) into (7),

$$(\log r_n)^{\sigma-\varepsilon} \leq (\log r_n)^{b+2\varepsilon} + k \log^+ T(2r_n, f) + O(1),$$

namely,

$$(1 - o(1))(\log r_n)^{\sigma-\varepsilon} \leq k \log^+ T(2r_n, f)$$

for all $r_n \notin E_1$. This results in $\sigma_{(2,2)}(f) \geq \sigma_{(1,2)}(A_0)$. $\square$

The following result is a lower bound for transcendental solutions of the equation (1) when an arbitrary A_s $(s \in \{0, 1, \ldots, k-1\})$ is the dominant coefficient.

THEOREM 3.3. *Let A_0, A_1, ..., A_{k-1} be entire functions such that*

$$\max\{\sigma_{(1,2)}(A_j) : j \neq s \text{ and } j = 0, 1, \ldots, k-1\} < \sigma_{(1,2)}(A_s) < +\infty.$$

Then any transcendental entire solution f of (1) satisfies

$$\sigma_{(1,2)}(f) \geq \sigma_{(1,2)}(A_s).$$

Proof. We assume that f is a transcendental meromorphic solution of (1). By (1) we get

$$-A_s = (\frac{f^{(k)}}{f^{(s)}} + A_{k-1}\frac{f^{(k-1)}}{f^{(s)}} + \ldots + A_{s+1}\frac{f^{(s+1)}}{f^{(s)}} + A_{s-1}\frac{f^{(s-1)}}{f^{(s)}} + \ldots + A_0\frac{f}{f^{(s)}})$$

$$= [\frac{f^{(k)}}{f^{(s)}} + A_{k-1}\frac{f^{(k-1)}}{f^{(s)}} + \ldots + A_{s+1}\frac{f^{(s+1)}}{f^{(s)}}] + \frac{f}{f^{(s)}}[A_{s-1}\frac{f^{(s-1)}}{f} + \ldots + A_1\frac{f'}{f} + A_0].$$

Obviously,

$$\begin{aligned} m(r, \frac{f}{f^{(s)}}) &\leq T(r,f) + T(r, \frac{1}{f^{(s)}}) \\ &\leq T(r,f) + m(r, f^{(s)}) + N(r, f^{(s)}) + O(1) \\ &\leq (s+2)T(r,f) + m(r, \frac{f^{(s)}}{f}) + O(1). \end{aligned}$$

Together with Lemma 3.1, we get that

$$\begin{aligned} T(r, A_s) &= m(r, A_s) \leq \sum_{j\neq s} m(r, A_j) + \max\{(k-s), s\} \log T(r,f) \\ &\quad +(s+2)T(r,f) + O(1), \end{aligned}$$

and thus,

$$T(r, A_s) \leq \sum_{j\neq s} T(r, A_j) + (s+2)(1-o(1))T(r,f). \tag{10}$$

Set

$$b =: \max\{\sigma_{(1,2)}(A_j), j \in \{0, 1, \ldots, k-1\} \setminus \{s\}\} < \sigma_{(1,2)}(A_s).$$

Then we have

$$T(r, A_j) \leq (\log r)^{b+\varepsilon} \quad (j \in \{0, 1, \ldots, k-1\} \setminus \{s\}). \tag{11}$$

Since $\sigma_{(1,2)}(A_s) := \sigma > 0$, by Lemma 3.6 there exists a sequence $\{r_n\}$ such that for all $r_n \notin E_1$,

$$T(r_n, A_s) \geq (\log r_n)^{\sigma-\varepsilon} \tag{12}$$

holds for any given ε $(0 < 3\varepsilon < \sigma - b)$. Therefore, substituting (11) and (12) into (10),

$$(\log r_n)^{\sigma-\varepsilon} \leq (\log r_n)^{b+2\varepsilon} + (s+2)(1-o(1))T(r_n, f),$$

namely,

$$(1-o(1))(\log r_n)^{\sigma-\varepsilon} \leq (s+2)(1-o(1))T(r_n,f)$$

for all $r_n \notin E_1$. This results in $\sigma_{(1,2)}(f) \geq \sigma_{(1,2)}(A_s)$. $\square$

The last result in this section plays the key role to consider the growth of solutions of the equation (2) and the fixed points of solutions of the equations (1) and (2).

THEOREM 3.4. *Let $A_0 \neq 0, A_1, \ldots, A_{k-1}$ and $F \neq 0$ be meromorphic functions in the complex plane and let f be a meromorphic solution of equation* (2) *such that*

$$\max\{\sigma_{(2,2)}(F), \sigma_{(2,2)}(A_j) : j=0,1,\ldots,k-1\} < \sigma_{(2,2)}(f) < +\infty,$$

then we have $\overline{\lambda}_{(2,2)}(f) = \lambda_{(2,2)}(f) = \sigma_{(2,2)}(f)$.

Proof. By equation (2), we have

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \ldots + A_0\right). \tag{13}$$

If f has a zero at z_0 of order $\gamma(>k)$ and if A_0, A_1, $\ldots$, A_{k-1} are all analytic at z_0, then F has a zero at z_0 of order at least $\gamma - k$. Hence, we have

$$n(r,\frac{1}{f}) \leq k\cdot\overline{n}(r,\frac{1}{f}) + n(r,\frac{1}{F}) + \sum_{j=0}^{k-1} n(r,A_j),$$

$$N(r,\frac{1}{f}) \leq k\cdot\overline{N}(r,\frac{1}{f}) + N(r,\frac{1}{F}) + \sum_{j=0}^{k-1} N(r,A_j). \tag{14}$$

By Lemma 3.1 and (13), we get that

$$m(r,\frac{1}{f}) \leq m(r,\frac{1}{F}) + \sum_{j=0}^{k-1} m(r,A_j) + k\log^+ T(r,f) + O(1). \tag{15}$$

Therefore, by (14), (15) and the first main theorem, there holds

$$T(r,f) = T(r,\frac{1}{f}) + O(1) \leq k\overline{N}(r,\frac{1}{f}) + T(r,F) + \sum_{j=0}^{k-1} T(r,A_j) + k\log^+ T(r,f) + O(1).$$

This results in

$$(1-o(1))T(r,f) \leq k\overline{N}(r,\frac{1}{f}) + T(r,F) + \sum_{j=0}^{k-1} T(r,A_j). \tag{16}$$

The assumption $\max\{\sigma_{(2,2)}(F), \sigma_{(2,2)}(A_j)(j = 0, 1, \dots, k-1)\} := b < \sigma_{(2,2)}(f) < +\infty$ implies that

$$\max\{T(r, F), T(r, A_j) : j = 0, 1, \dots, k-1\} \leq \exp((\log r)^{b+\varepsilon}). \tag{17}$$

By Lemma 3.6, for the set E_1 there exists a sequence $\{r_n\}, r_n \notin E_1$ such that

$$\lim_{r_n \to +\infty} \frac{\log \log T(r_n, f)}{\log \log r_n} = \sigma_{(2,2)}(f) := \sigma.$$

Hence, we get that for all sufficiently large $r_n \notin E_1$, there holds

$$T(r_n, f) \geq \exp((\log r_n)^{\sigma-\varepsilon}), \tag{18}$$

for any given ε $(0 < 2\varepsilon < \sigma - b)$. By (17) and (18) we get that for all sufficiently large $r_n \notin E_1$ there holds

$$\max\left\{\frac{T(r_n, F)}{T(r_n, f)}, \frac{T(r_n, A_j)}{T(r_n, f)} : j = 0, 1, \dots, k-1\right\} \to 0, \quad (r_n \to +\infty). \tag{19}$$

Hence, by (16) and (19) we get that for sufficiently large $r_n \notin E_1$ there holds

$$(1 - o(1))T(r_n, f) \leq k\overline{N}(r_n, \frac{1}{f}).$$

This implies that $\overline{\lambda}_{(2,2)}(f) = \lambda_{(2,2)}(f) = \sigma_{(2,2)}(f)$. □

4. PROOF OF THEOREM 2.1

We first assume that f is a transcendental entire solution of equation (1). By Theorem 3.3, we have $\sigma_{(1,2)}(f) \geq \sigma_{(1,2)}(A_s) \geq 1$. On the other hand, by Theorem 3.1 we have $\sigma_{(2,1)}(f) = 0$ and $\sigma_{(2,2)}(f) \leq \sigma_{(1,2)}(A_s) + 1$. Hence, $\sigma_{(2,1)}(f) = 0$ and $\sigma_{(2,2)}(f) \leq \sigma_{(1,2)}(A_s) + 1 \leq \sigma_{(1,2)}(f) + 1$.

Suppose that $\{f_1, \dots, f_k\}$ is a solution base of equation (1). By Lemma 3.3, we get

$$T(r, A_s) = m(r, A_s) = O\{\log(\max_{1 \leq j \leq k} T(r, f_j)\}.$$

This implies that there exists one of $\{f_1, f_2, \dots, f_k\}$, say f_1, satisfying $T(r, A_s) = O\{\log T(r, f_1)\}$. Thus, $\sigma_{(2,2)}(f_1) \geq \sigma_{(1,2)}(A_s) \geq 1$. Therefore, $\sigma_{(1,2)}(A_s) + 1 \geq \sigma_{(2,2)}(f_1) \geq \sigma_{(1,2)}(A_s) \geq 1$.

Suppose that f is a polynomial with degree $deg(f) \geq s$, then $f^{(s)}(z) \not\equiv 0$. By a similar discussion as in the proof of Theorem 3.3, we obtain

$$T(r, A_s) = m(r, A_s) \leq \sum_{j \neq s} m(r, A_j) + 2d(\log r) + m(r, \frac{f^{(k)}}{f}) + O(1).$$

Together with Lemma 3.1

$$T(r, A_s) = m(r, A_s) \leq \sum_{j \neq s} m(r, A_j) + 2d(\log r) + O(1). \tag{20}$$

We first assume that

$b := \max\{\sigma_{(1,2)}(A_j) : j \neq s \,\text{and}\, j = 0, 1, \ldots, k-1\} < \sigma_{(1,2)}(A_s) = \sigma.$

Then we have

$$m(r, A_j) = T(r, A_j) \leq (\log r)^{b+\varepsilon} \quad (j \in \{0, 1, \ldots, k-1\} \setminus \{s\}). \tag{21}$$

Since $\sigma_{(1,2)}(A_0) = \sigma > 0$, by Lemma 3.6 there exists a sequence $\{r_n\}$ such that for all $r_n \notin E_1$,

$$T(r_n, A_s) \geq (\log r_n)^{\sigma-\varepsilon} \tag{22}$$

holds for any given ε $(0 < 4\varepsilon < \sigma - b)$. Therefore, substituting (21) and (22) into (20),

$$(\log r_n)^{\sigma-\varepsilon} \leq (\log r_n)^{b+2\varepsilon} + 2d\,(\log r_n)$$

for all $r_n \notin E_1$. Since $b \geq 1$,

$$(\log r_n)^{\sigma-\varepsilon} \leq (\log r_n)^{b+3\varepsilon}$$

for all $r_n \notin E_1$. This is a contradiction. Therefore, if f is not a transcendental solution, then it must be a polynomial with degree $deg(f) \leq s - 1$.

5. PROOFS OF THEOREM 2.2

For the case that

$$\max\{\sigma_{(1,2)}(A_j) : j = 1, 2, \ldots, k-1\} < \sigma_{(1,2)}(A_0) < +\infty,$$

we get by Theorem 3.2 that every nonzero entire solution f of (1) satisfies $\sigma_{(2,2)}(f) \geq \sigma_{(1,2)}(A_0) \geq 1$. On the other hand, by Theorem 3.1 we get that every solution f of (1) satisfies $\sigma_{(2,1)}(f) = 0$ and $\sigma_{(2,2)}(f) \leq \sigma_{(1,2)}(A_0) + 1 < +\infty$.

6. PROOFS OF THEOREM 2.3

Assume that f is transcendental. By (6) we get that

$$|A_0(z)| = \left|\frac{f^{(k)}}{f}\right| + |A_{k-1}(z)| \left|\frac{f^{(k-1)}}{f}\right| + \ldots + |A_1(z)| \left|\frac{f'}{f}\right|. \tag{23}$$

Then by Lemma 3.8, there exists a set $E_2 \subset (1, +\infty)$ that has a finite logarithmic measure, and a constant $B > 0$ depending only on $\alpha(> 1)$ and $j \in \{1, \ldots, k\})$ such that for all z with $|z| = r \notin [0,1] \bigcup E_2$, we have

$$\left|\frac{f^{(j)}(z)}{f}(z)\right| \leq \left(\frac{T(\alpha r, f)}{r}(\log^{\alpha} r) \log T(\alpha r, f)\right)^{j}. \tag{24}$$

By the assumption that

$$\max\{\sigma_{(1,2)}(A_j) : j = 1, 2, \ldots, k-1\} = \sigma_{(1,2)}(A_0) < +\infty,$$

and

$$\max\{\tau_{(1,2)}(A_j) : \sigma_{(1,2)}(A_j) = \sigma_{(1,2)}(A_0)\} < \tau_{(1,2)}(A_0) \leq +\infty.$$

there exists a nonempty set $J_1 \subseteq \{1, 2, \ldots, k-1\}$, such that for $j \in J_1$ we have $\sigma_{(1,2)}(A_j) = \sigma_{(1,2)}(A_0) := \sigma$ and $\tau_{(1,2)}(A_j) < \tau_{(1,2)}(A_0) := \tau$, and for $i \in \{1, 2, \ldots, k-1\} \setminus J_1$ we have $\alpha_1 := \max\{\sigma_{(1,2)}(A_i)\} < \sigma_{(1,2)}(A_0) = \sigma$. Hence, there exist constants β_1 and β with $\max\{\tau_{(1,2)}(A_j) : j \in J_1\} < \beta_1 < \beta < \tau$ such that

$$M(r, A_i) \leq \exp(\log r)^{\alpha_1 + \varepsilon} \leq \exp(\log r)^{\sigma - \varepsilon} \leq \exp\{\beta_1 (\log r)^{\sigma}\}, \quad i \in \{1, \ldots, k-1\} \setminus J_1 \tag{25}$$

and

$$M(r, A_j) \leq \exp\{\beta_1 (\log r)^{\sigma}\}, \ j \in J_1. \tag{26}$$

By Lemma 3.7 there exists a set I_0 having infinite logarithmic measure such that for all $r \in I_0$ we have

$$M(r, A_0) > \exp\{\beta (\log r)^{\sigma}\}. \tag{27}$$

Hence, by substituting (24)–(27) into (23) we get that for all z satisfies $|A_0(z)| = M(r, A_0)$ and $|z| = r \in I_0 \setminus E_2$,

$$\exp\{\beta (\log r)^{\sigma}\} \leq k \exp\{\beta_1 (\log r)^{\sigma}\} \left(\frac{T(\alpha r, f)}{r}(\log^{\alpha} r) \log T(\alpha r, f)\right)^{k},$$

$$(1 - o(1)) \exp\{\beta (\log r)^{\sigma}\} \leq \{(1 - o(1)) T(\alpha r, f)(\log^{\alpha} r)\}^{k},$$

This implies $\sigma_{(2,2)}(f) \geq \sigma_{(1,2)}(A_0) \geq 1$. On the other hand, by Theorem 3.1 we get that every solution f of (1) satisfies $\sigma_{(2,1)}(f) = 0$ and $\sigma_{(2,2)}(f) \leq \sigma_{(1,2)}(A_0) + 1 < +\infty$.

Furthermore, assume that $\sigma_{(1,2)}(A_0) > 1$, then A_0 and some other coefficients of $A_1, \ldots, A_{k-1}$ are transcendental. If f is a nonzero polynomial solution with degree less than $\tau_{(1,2)}(A_0)$, then $\tau_{(1,2)}(H) = \tau_{(1,2)}(A_0) > 0$, where $H(z) \equiv f^{(k)}(z) + A_{k-1}(z) f(z) + \ldots + A_0(z) f(z)$. This contradicts $H(z) \equiv 0$.

Assume that $\sigma_{(1,2)}(A_0) = 1$, then $\tau_{(1,2)}(A_0) = +\infty$. If f is a nonzero polynomial solution, then $\tau_{(1,2)}(f) = deg(f) < +\infty$, and thus, $\tau_{(1,2)}(H) = \tau_{(1,2)}(A_0) = +\infty$, where $H(z) \equiv f^{(k)}(z) + A_{k-1}(z)f(z) + \ldots + A_0(z)f(z)$. This contradicts $H(z) \equiv 0$.

7. PROOF OF THEOREM 2.4

Suppose that A_0 is a transcendental entire function with order zero and other coefficients A_j $(j = 1, 2, \ldots, k-1)$ are polynomials, then by Theorem 2.3 in [18] (or see Theorem 1.1) every nonzero solution f of (1) satisfies $\sigma_{(1,1)}(f) = +\infty$, $\sigma_{(2,1)}(f) = \sigma_{(1,1)}(A_0) = 0$.

Now assume that $\sigma_{(1,2)}(A_0) = 1$, then $\tau_{(1,2)}(A_0) = +\infty$. Note that A_j $(j = 1, 2, \ldots, k-1)$ are polynomials, and thus, their logarithmic type are all finite. Hence, for $j = 1, 2, \ldots, k-1$,

$$\sigma_{(1,2)}(A_j) = \sigma_{(1,2)}(A_0) = 1 \quad \text{and} \quad \tau_{(1,2)}(A_j) = deg(A_j) < \tau_{(1,2)}(A_0) = +\infty.$$

This satisfies the conditions of Theorem 2.3.

Assume that $\sigma_{(1,2)}(A_0) > 1$, then we have $1 = \sigma_{(1,2)}(A_j) < \sigma_{(1,2)}(A_0)$ holds for $j = 1, 2, \ldots, k-1$. This satisfies the conditions of Theorem 2.2.

Therefore, any nonzero entire solution f of (1) satisfies $\sigma_{(1,1)}(f) = +\infty$, $\sigma_{(2,1)}(f) = 0$ and

$$1 \leq \sigma_{(1,2)}(A_0) \leq \sigma_{(2,2)}(f) \leq \sigma_{(1,2)}(A_0) + 1.$$

8. PROOF OF THEOREM 2.5

We assume that $\{f_1, \ldots, f_k\}$ is an entire solution base of (1) corresponding to (2). By the elementary theory of differential equations (see, *e.g.* [19]), any solution of (2) can be represented in the form

$$f = (B_1 + C_1)f_1 + (B_2 + C_2)f_2 + \ldots + (B_k + C_k)f_k, \tag{28}$$

where $C_1, \ldots, C_k \in \mathbb{C}$ and $B_1, \ldots, B_k$ are entire functions given by the system of equations

$$\begin{cases} B_1' f_1 + B_2' f_2 + \ldots + B_k' f_k = 0 \\ B_1' f_1' + B_2' f_2' + \ldots + B_k' f_k' = 0 \\ \ldots \\ B_1' f_1^{(k-2)} + B_2' f_2^{(k-2)} + \ldots + B_k' f_k^{(k-2)} = 0 \\ B_1' f_1^{(k-1)} + B_2' f_2^{(k-1)} + \ldots + B_k' f_k^{(k-1)} = F. \end{cases} \tag{29}$$

Since the Wronskian of $f_1, \ldots, f_k$ satisfies $W(f_1, \ldots, f_k) = \exp(-\int A_{k-1} \mathrm{d}z)$, we obtain

$$B_j' = F \cdot G_j(f_1, \ldots, f_k) \cdot \exp(\int A_{k-1} \mathrm{d}z), \ (j = 1, \ldots, k), \tag{30}$$

where $G_j(f_1, \ldots, f_k)$ is a differential polynomial of $f_1, \ldots, f_k$ and of their derivatives, with constant coefficients.

By Theorem 2.1, if

$$\max\{\sigma_{(1,2)}(A_j) : j \in \{0, 1, \ldots, k-1\} \setminus \{s\}\} < \sigma_{(1,2)}(A_s) < +\infty,$$

then $\sigma_{(2,2)}(f_j) \leq \sigma_{(1,2)}(A_s) + 1$.

By Lemma 3.4 and Lemma 3.5 we have

$$\sigma_{(2,2)}(\exp(\int A_{k-1} \mathrm{d}z)) = \sigma_{(1,2)}(A_{k-1}) < \sigma_{(1,2)}(A_s).$$

Again by Lemma 3.4 and together with (28)–(30), we obtain

$$\sigma_{(2,2)}(f) \leq \max\{\sigma_{(1,2)}(A_s) + 1, \sigma_{(2,2)}(F)\}.$$

(1) if $\sigma_{(2,2)}(F) > \sigma_{(1,2)}(A_s) + 1$, then it follows from the equation (2) that $\sigma_{(2,2)}(f) = \sigma_{(2,2)}(F)$.

(2) if $\sigma_{(2,2)}(F) < \sigma_{(1,2)}(A_s) + 1$, then $\sigma_{(2,2)}(f) \leq \sigma_{(1,2)}(A_s) + 1$. Further, assume that a solution f of (2) satisfies $\sigma_{(2,2)}(f) = \sigma_{(1,2)}(A_s) + 1$, then there holds

$$\max\{\sigma_{(2,2)}(F), \sigma_{(2,2)}(A_j) : j = 0, 1, \ldots, k-1\} < \sigma_{(2,2)}(f).$$

By Theorem 3.4, we obtain that the solution f of (2) satisfies $\sigma_{(2,2)}(f) = \overline{\lambda}_{(2,2)}(f) = \lambda_{(2,2)}(f)$.

9. PROOF OF THEOREM 2.6

By a similar discussion as in the proof of Theorem 2.5, we also have (28)-(30). By Theorem 2.2 or Theorem 2.4, f_j $(j = 1, 2, \ldots, k)$ satisfies

$$1 \leq \sigma_{(1,2)}(A_0) \leq \sigma_{(2,2)}(f_j) \leq \sigma_{(1,2)}(A_0) + 1.$$

By Lemma 3.4 and Lemma 3.5 we have

$$\sigma_{(2,2)}(\exp(\int A_{k-1} \mathrm{d}z)) = \sigma_{(1,2)}(A_{k-1}) < \sigma_{(1,2)}(A_0).$$

Again by Lemma 3.4 and together with (28)–(30), we obtain

$$\sigma_{(2,2)}(f) \leq \max\{\sigma_{(1,2)}(A_0) + 1, \sigma_{(2,2)}(F)\}.$$

(i) If $\sigma_{(2,2)}(F) > \sigma_{(1,2)}(A_0) + 1$, then it follows from the equation (2) that $\sigma_{(2,2)}(f) = \sigma_{(2,2)}(F)$.

(ii) If $\sigma_{(2,2)}(F) < \sigma_{(1,2)}(A_0) + 1$, then $\sigma_{(2,2)}(f) \le \sigma_{(1,2)}(A_0) + 1$.

Now, we assert that all solutions f of the equation (2) satisfy $\sigma_{(2,2)}(f) \ge \sigma_{(1,2)}(A_0)$, with at most one exception. In fact, if there exist two distinct meromorphic solutions g_1 and g_2 of (2) satisfying $\sigma_{(2,2)}(g_i) < \sigma_{(1,2)}(A_0), (j = 1, 2)$, then $g = g_1 - g_2$ is a nonzero entire solution of (1) and satisfies $\sigma_{(2,2)}(g) = \sigma_{(2,2)}(g_1 - g_2) < \sigma_{(1,2)}(A_0)$. But by Theorem 2.2 or Theorem 2.4 we have $\sigma_{(2,2)}(g) = \sigma_{(2,2)}(g_1 - g_2) \ge \sigma_{(1,2)}(A_0)$. This is a contradiction.

Assume that a solution f of (2) satisfies $\sigma_{(2,2)}(f) = \sigma_{(2,2)}(A_0) + 1$, then there holds

$$\max\{\sigma_{(2,2)}(F), \sigma_{(2,2)}(A_j) : j = 0, 1, \ldots, k-1\} < \sigma_{(2,2)}(f).$$

By Theorem 3.4, we obtain that the solution f of (2) satisfies $\sigma_{(2,2)}(f) = \overline{\lambda}_{(2,2)}(f) = \lambda_{(2,2)}(f)$.

10. PROOF OF THEOREM 2.7

By a similar discussion as in the proof of Theorem 2.5, we also have (28)–(30). By Theorem 2.3, If

$$\max\{\sigma_{(1,2)}(A_j) : j = 1, 2, \ldots, k-1\} = \sigma_{(1,2)}(A_0) < +\infty,$$

and

$$\max\{\tau_{(1,2)}(A_j) : \sigma_{(1,2)}(A_j) = \sigma_{(1,2)}(A_0)\} < \tau_{(1,2)}(A_0) < +\infty,$$

then f_j $(j = 1, 2, \ldots, k)$ of (1) is either a polynomial with degree one or a transcendental entire function satisfying

$$1 \le \sigma_{(1,2)}(A_0) \le \sigma_{(2,2)}(f_j) \le \sigma_{(1,2)}(A_0) + 1.$$

By Lemma 3.4 and Lemma 3.5 we have

$$\sigma_{(2,2)}(\exp(\int A_{k-1}\mathrm{d}z)) = \sigma_{(1,2)}(A_{k-1}) = \sigma_{(1,2)}(A_0).$$

Again by Lemma 3.4 and together with (28)–(30), we obtain

$$\sigma_{(2,2)}(f) \le \max\{\sigma_{(1,2)}(A_0) + 1, \sigma_{(2,2)}(F)\}.$$

(i) If $\sigma_{(2,2)}(F) > \sigma_{(1,2)}(A_0) + 1$, then it follows from the equation (2) that $\sigma_{(2,2)}(f) = \sigma_{(2,2)}(F)$.

(ii) If $\sigma_{(2,2)}(F) < \sigma_{(1,2)}(A_0) + 1$, then $\sigma_{(2,2)}(f) \le \sigma_{(1,2)}(A_0) + 1$. Further, assume that a solution f of (2) satisfies $\sigma_{(2,2)}(f) = \sigma_{(2,2)}(A_0) + 1$, then there holds

$$\max\{\sigma_{(2,2)}(F), \sigma_{(2,2)}(A_j) : j = 0, 1, \ldots, k-1\} < \sigma_{(2,2)}(f).$$

By Theorem 3.4, we obtain that the solution f of (2) satisfies $\sigma_{(2,2)}(f) = \overline{\lambda}_{(2,2)}(f) = \lambda_{(2,2)}(f)$.

Acknowledgment. This work was partially supported by the NSFC (No. 11101201, No. 11001057, No. 11026110), the NSF of Jiangxi (no. 20122BAB211001, No. 20132BAB211003) and NSF of education department of Jiangxi (no. GJJ13077, GJJ13078) of China.

REFERENCES

[1] B. Belaïdi, *Growth and oscillation of solutions to linear differential equations with entire coefficients having the same order.* Electron. J. of Differential Equations **2009** (2009), *70*, 1–10.

[2] L.G. Bernal, *On growth k-order of solutions of a complex homogeneous linear differential equation.* Proc. Amer. Math. Soc. **101** (1987), *2*, 317–322.

[3] T.B. Cao, J.F. Xu and Z.X. Chen, *On the meromorphic solutions of linear differential equations on the complex plane.* J. Math. Anal. Appl. **364** (2010), 130–142.

[4] Z.X. Chen and C.C. Yang, *Quantitative estimations on the zeros and growths of entire solutions of linear differential equations.* Complex Var. Elliptic Equ. **42** (2000), 119–133.

[5] T.Y. Peter Chern, *On meromorphic functions with finite logarithmic order,* Tran. Amer. Math. Soc. **358** (2005), *2*, 473–489.

[6] G. Frank and S. Hellerstein, *On the meromorphic solutions of non-homogeneous linear differential equations with polynomial coefficients.* Proc. London Math. Soc. **53** (1986), *3*, 407–428.

[7] M. Frei, *Uber die losungen linearer differentialgleichungen mit ganzen funktionen als koeffizienten.* Comment. Math. Helv. **35** (1961), 210–222.

[8] G.G. Gundersen, *Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates.* J. London Math. Soc. **37** (1988), *2*, 88–104.

[9] G.G. Gundersen, E. Steinbart and S. Wang, *The possible orders of solutions of linear differential equations with polynomial coefficients.* Trans. Amer. Math. Soc. **350** (1998), *3*, 1225–1247.

[10] W.K. Hayman, *Meromorphic Functions.* Oxford Mathematical Monographs Clarendon Press, Oxford, 1964.

[11] W. Hayman, *The local growth of power series: A survey of Wiman-Valiron method.* Canad. Math. Bull. **17**(1974), *3*, 317–358.

[12] Y.Z. He and X.Z. Xiao, *Algebroid Functions and Ordinary Differential Equations.* Beijing, Science Press, 1988 (in Chinese).

[13] J. Heittokangas, R. Korhonen and J. Rättyä, *Generalized logarithmic derivative estimates of Gol'dberg-Grinshtein type.* Bull. London Math. Soc. **36** (2004), 105–114.

[14] J. Heittokangas, R. Korhonen and J. Rättyä, *Growth estimates for analytic solutions of complex linear differential equation.* Ann. Acad. Sci. Fenn. Math. **29** (2004), 233–246.

[15] G. Jank and L. Volkman, *Meromorphe Funktionen Und Differentialgeichungen.* Birkäuser, Basel-Boston, 1985.

[16] O.P. Juneja, G.P. Kapoor and S.K. Bajpai, *On the (p, q)-order and lower (p,q)-order of an entire function.* Reine Angew. Math. **282** (1976), 53–67.

[17] O.P. Juneja, G.P. Kapoor and S.K. Bajpai, *On the (p, q)-type and lower (p,q)-type of an entire function.* J. Reine Angew. Math. **290** (1977), 180–190.

[18] L. Kinnunen, *Linear differential equations with solutions of finite iterated order.* Southeast Asian Bull. Math. **22** (1998), *4*, 385–405.

[19] I. Laine, *Nevanlinna Theory and Complex Differential Equations.* W. de Gruyter, Berlin, 1993.

[20] I. Laine, *Complex differential equations.* In: *Handbook of Differential Equations: Ordinary Differential Equations* **4,** North-Holland, 2008, 269–363.

[21] J. Liu, J. Tu and L.Z. Shi, *Linear differential equations with entire coefficients of [p, q]-order in the complex plane.* J. Math. Anal. Appl. **372** (2010), 55–67.

[22] D. Sato, *On the rate of growth of entire functions of fast growth.* Bull. Amer. Math. Soc. **69** (1963), 411–414.

[23] A. Schönhage, *Über das Wachstum zusammengesetzter Funktionen.* Math. Z. **73** (1960), 22–44.

[24] H. Wittich, *Neuere Untersuchungen über eindeutige analytiche Funktionen.* Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955.

[25] L. Yang, *Value Distribution Theory.* Berlin: Springer-Verlag, 1993/ Beijing: Science Press, 1982.

[26] H.X. Yi and C.C. Yang, *Uniqueness Theory of Meromorphic Functions.* Science Press 1995/Kluwer 2003.

Received 16 September 2011

Nanchang University,
Department of Mathematics,
Nanchang, Jiangxi 330031,
P. R. China
tbcao@ncu.edu.cn, tingbincao@hotmail.com

Nanchang University,
Department of Mathematics,
Nanchang, Jiangxi 330031,
P. R. China
liukai418@126.com

Fudan University,
Department of Mathematics,
Shanghai 200433,
P. R. China
majwang@fudan.edu.cn