SOLUTION OF HALLEN'S INTEGRAL EQUATION BY USING RADIAL BASIS FUNCTIONS

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In this paper, we present a numerical method for solving Hallen's integral equation based on radial basis functions (RBFs). This method will represent the solution of Hallen's integral equation by interpolating the radial basis functions based on Legendre-Gauss-Lobatto(LGL) nodes and weights. The numerical results show that the proposed method for Hallen's integral equation is very accurate and efficient.

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1. INTRODUCTION

Erik Hallen [1] wrote his famous integral equation to give an exact treatment of antenna current wave reflection at the end of the tube shaped cylindrical antenna in 1956, but his first work on this subject [2] probably goes back to 1938. This equation enabled him to show that on thin wire the current distribution is approximately sinusoidal and propagates with nearly the speed of the light.

The Hallen's integral equation is a Fredholm integral equation of the first kind. This equation for the thin-wire cylindrical antenna of length l and radius a with $a \ll l$ is given by

$$\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} K(x',y')I(y')dy' = \frac{\jmath}{2\zeta_0} V sin(\beta|x'|) + A cos(\beta x'), \quad \frac{-l}{2} < x' < \frac{-l}{2} \tag{1.1}$$

where

$$K(x',y') = \frac{1}{4\pi}\frac{e^{-j\beta\sqrt{(x'-y')^2+a^2}}}{\sqrt{(x'-y')^2+a^2}}. \tag{1.2}$$

In Eqs. (1.1) and (1.2), $\zeta_0 = 120\pi$ and $\beta = 2\pi/\lambda$ is the free wavenumber where λ is wavelenght, $I(y')$ is the current, V is the driven voltage and A is a constant. The boundary conditions for the equation (1) are $I(-\ell/2) = I(\ell/2) = 0$.

The kernel in Eq. (1.2) is sharply peaked when $x' = y'$, particularly for small value of a that may lead to an ill conditioned numerical formulation, thus making difficult to formulate a robust numerical algorithm. There are two choices of $K(x', y')$. The two kernels are usually referred to as the exact and the approximate or reduced kernel [3]. Many different methods have been used to estimate the solution of Hallen's integral equation such as, finite element methods [4], a general method for solving dual integral equation is presented and applied to Hallen's integral equation in [5], multi-wavelets [6] and sinc collocation method [7]. In [3, 4] the difficulties associated with solving Hallen's equation numerically were considered in detail.

Radial basis functions(RBFs), introduced by R. Hardy [9, 10], form a primary tool for multivariate interpolation, and they are also receiving increased attention for solving PDE in irregular domains [8]. Hardy [9] showed that multiquadrics RBFs are related to a consistent solution of the biharmonic potential problem and thus, has a physical foundation. Buhmann and Micchelli [11] and Chiu *et al.* [8, 12] have shown that RBFs are related to pre-wavelets (wavelets that do not have orthogonality properties). RBFs have been popular in high-dimensional approximation [8] and are increasingly being used in the numerical solution of partial differential equations. Also, Alipanah and Dehghan [14], using RBFs for solution of nonlinear integral equation in one dimensional.

In this paper, we use the idea of the interpolation by RBFs to approximate the solution of the Hallen's integral equation. Also, the collocation points for the interpolation and quadrature formula are based on Legendre-Gauss-Lobbato (LGL) nodes.

This paper organized as follows: in Section 2 we introduce positive definite or radial basis functions and their properties. In Section 3, we introduce Legendre-Gauss-Lobatto nodes and weights. In Section 4, the proposed method is used to approximate the Hallen's integral equation. Finally, in Section 5 we report our numerical finding and demonstrate the accuracy of the proposed numerical scheme by considering numerical examples.

2. STRICTLY POSITIVE DEFINITE FUNCTIONS

Let the values of function $f \in L^2[a, b]$ on nodes $x_i,\ i = 0, 1, \cdots, N$ be given, if

$$F(x_i) = f(x_i),\ i = 0, 1, \cdots, N,$$

then we say that $F(x)$ interpolates the given data $\{(x_i, f(x_i))\}_{i=0}^{N}$.

The linear interpolation problem is a special case that arises when $F(x)$ is to be chosen from a prescribed N dimensional vector space of functions on $[a,b]$.

Then the function $F(x)$ can be represented as

$$F(x)=\sum_{j=0}^{N} c_j u_j(x).$$

When the interpolation conditions are imposed on $F(x)$, we obtain

$$f(x_i)=F(x_i)=\sum_{j=0}^{N} c_j u_j(x_i), \quad i=0,1,\cdots,N.$$

This is a system of $(N+1)$ linear equations in $(N+1)$ unknowns $Ac=y$, *i.e.*

$$\begin{bmatrix} u_0(x_0) & u_1(x_0) & \cdots & u_N(x_0) \\ u_0(x_1) & u_1(x_1) & \cdots & u_N(x_1) \\ \vdots & \vdots & \vdots & \vdots \\ u_0(x_N) & u_1(x_N) & \cdots & u_N(x_N) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix}.$$

The $(N+1)\times(N+1)$ matrix A is called the interpolation matrix. In order that this system is solvable for any choice of $f(x_i)$, it is necessary and sufficient that the interpolation matrix be nonsingular.

Definition. Let the values of function $f(x)$ at nodes $x_i, i=0,1,\cdots,N$ be given. We say that $f(x)$ is interpolated by translating a single function $\phi(x)$, if a linear combination of $(N+1)$ transition of $\phi(x)$ exists such that $F(x)$ interpolates $f(x)$ at nodes [17], *i.e.*,

$$F(x)=\sum_{j=0}^{N} c_j \phi(x-v_j),$$

and

$$f(x_i)=F(x_i)=\sum_{j=0}^{N} c_j \phi(x_i-v_j),\ i=0,1,\cdots,N.$$

There are a lot of functions $\phi(x)$ that can be used as a linear combination for interpolating a function. Generally, it isn't necessary that $v_j=x_i, i,j=0,1,\cdots,N$, but in this paper for simplicity, we use a class of functions which it's transition can be used in the case $v_i=x_i, i=0,1,\cdots,N$ in interpolation, *i.e.*

$$f(x_i)=F(x_i)=\sum_{j=0}^{N} c_j \phi(x_i-x_j), \quad i=0,1,\cdots,N.$$

Definition. A function ϕ on $[a, b]$ is said to be positive definite if for any set of points $x_0, x_1, \cdots, x_N$ in $[a, b]$ the $(N+1)\times(N+1)$ matrix $A_{ij} = \phi(x_i - x_j)$ is nonnegative definite, *i.e.*

$$U^T AU = \sum_{i=0}^{N}\sum_{j=0}^{N} u_i u_j A_{ij} \geq 0,$$

for all $U \in \mathbb{R}^{(N+1)}$. If $U^T AU > 0$ whenever the points x_i are distinct and $U \neq 0$, then we say that $\phi(x)$ is strictly a positive definite function [12–17].

If $\phi(x)$ is a strictly positive definite function on a linear space, then the eigenvalues of A are positive and its determinant is positive. Therefore, we can use a linear combination translation of $\phi(x)$ to interpolate [17].

Definition. A function $f(x)$ is said to be completely monotone on $[0, \infty)$, if for $x > 0$ we have

$$1. \quad f \in C^\infty[0, \infty), \qquad 2. \quad (-1)^k f^{(k)}(x) \geq 0.$$

A real-valued function F on an inner-product space is said [17] to be radial if $F(x) = F(y)$ whenever $\|x\| = \|y\|$. Now, we present a theorem that introduces a large number of strictly positive definite or radial basis functions.

Theorem (Bochner's Theorem [17]). *Let f be a nonnegative Borel function on $\mathbb{R}$, if $0 < \int_{\mathbb{R}} f < \infty$, then $\hat{f}$ is strictly positive definite, where $\hat{f}$ is the Fourier transform of function f, which*

$$\hat{f}(x) = \int_{-\infty}^{+\infty} f(y) e^{ixy} \mathrm{d}y.$$

We can find many strictly positive definite functions by using this theorem. In Table 1, we give some positive definite functions (RBFs) by using Bochner's Theorem.

Table 1
Strictly positive definite functions that satisfy Bochner's Theorem

$f_1(x) = \frac{1+x^2}{\pi}$	$\hat{f}_1(x) = e^{-\lvert x\rvert}$
$f_2(x) = \frac{e^{-\lvert x\rvert}}{2}$	$\hat{f}_2(x) = \frac{1}{1+x^2}$
$f_3(x) = \pi^{-\frac{1}{2}} e^{-x^2}$	$\hat{f}_3(x) = e^{-\frac{x^2}{4}}$
$f_4(x) = \frac{1+x^{-2}}{2\pi}$	$\hat{f}_4(x) = \lvert x\rvert^{-1}(1 - e^{-\lvert x\rvert})$
$f_5(x) = sech(\pi x)$	$\hat{f}_5(x) = sech(\frac{x}{2})$
$f_6(x) = \frac{1-xcsch(x)}{2x^2}$	$\hat{f}_6(x) = log(1 + e^{-\frac{\pi}{\lvert x\rvert}})$

Thus, for any set of distinct points $x_0, x_1, \cdots, x_N$ on X, the matrix $A_{ij} = \hat{f}(\|x_i - x_j\|^2)$ is strictly positive definite.

3. LEGENDRE-GAUSS-LOBATTO NODES AND WEIGHTS

Let $L_N(x)$ be the shifted Legendre polynomial of order N on $[-1, 1]$. Then the Legendre-Gauss-Lobatto nodes are

$$x_0 = -1 < x_1 < \cdots < x_{N-1} < x_N = 1, \tag{3.1}$$

and $x_m, \quad 1 \leq m \leq N-1$ are the zeros of $\dot{L}(x)$, where $\dot{L}(x)$ is the derivative of $L_N(x)$ with respect to x. No explicit formulas are known for the points x_m, and so, they are computed numerically using subroutines [20].

Also, we approximate the integral of f on $[-1, 1]$ as

$$\int_{-1}^{1} f(x)dx = \sum_{i=0}^{N} w_i f(x_i), \tag{3.2}$$

where x_i are Legendre-Guass-Lobatto nodes in equation (3.1) and the weights w_i given in [20, p. 76].

$$w_i = \frac{2}{N(N+1)[L_N(x_i)]^2}, \quad i = 0, 1 \cdots, N.$$

It is well known that the integration in equation (3.2) is exact whenever $f(x)$ is a polynomial of degree $\leq 2N+1$.

4. ILLUSTRATE THE SOLUTION OF HALLEN'S INTEGRAL EQUATION

Equation (1.1) can be normalized by changing the independent variables to $x' = \frac{\ell}{2}x$ and $y' = \frac{\ell}{2}y$. Therefore, the Hallen's integral equation and the condition $I(-\frac{\ell}{2}) = I(\frac{\ell}{2}) = 0$ may be written as follows

$$\frac{\ell}{2}\int_{-1}^{1} K\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right) I(y)dy = f(x), \ -1 < x < 1 \tag{4.1}$$

and

$$I(-1) = I(1) = 0, \tag{4.2}$$

where

$$f(x) = \frac{\jmath}{2\zeta_0} V sin(\beta|\frac{\ell}{2}x|) + Acos(\beta\frac{\ell}{2}x), \quad -1 < x < 1.$$

When $x = y$, the kernel in equation (4.1) is sharply peaked, particulary for small value of a. Therefore, from the computational point of view, it would be advantageous to isolate and extract the singularity from kernel. This may be accomplish by writing $K\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right)$ as

$$K\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right) = K_n\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right) + K_s\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right), \tag{4.3}$$

where $K_n\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right)$ and $K_s\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right)$ denote the nonsingular and singular parts of kernel K respectively, and are given in [3] as

$$K_n\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right) = \frac{1}{4\pi}\frac{e^{-j\beta\sqrt{(\frac{\ell}{2}x-\frac{\ell}{2}y)^2+a^2}} - 1}{\sqrt{(\frac{\ell}{2}x-\frac{\ell}{2}y)^2+a^2}}, \tag{4.4}$$

$$K_s\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right) = \frac{1}{4\pi}\frac{1}{\sqrt{(\frac{\ell}{2}x-\frac{\ell}{2}y)^2+a^2}}, \tag{4.5}$$

By using equation (4.3) we can express equation (4.1) as

$$\frac{\ell}{2}\int_{-1}^{1} K_n\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right) I(y)\mathrm{d}y + \frac{\ell}{2}\int_{-1}^{1} K_s\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right) I(y)\mathrm{d}y = f(x),\ -1<x<1. \tag{4.6}$$

The integrand of the first integral in equation (4.6) is well behaved and as a consequence may be evaluated numerically, but the second integral contains a singularity and will be evaluated as follows. Let

$$\int_{-1}^{1} K_s\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right) I(y)\mathrm{d}y = S_1(x) + S_2(x), \tag{4.7}$$

where

$$S_1(x) = \int_{-1}^{1} K_s\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right)\left(I(y) - I(x)\right)\mathrm{d}y, \tag{4.8}$$

and

$$S_2(x) = I(x)\int_{-1}^{1} K_s\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right)\mathrm{d}y, \tag{4.9}$$

The integrand of the integral in equation (4.8) is well behaved, and the integral in equation (4.9) can be evaluated as

$$H(x) = \int_{-1}^{1} K_s\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right)\mathrm{d}y = \frac{1}{4\pi l} ln\left[\frac{\sqrt{(lx-l)^2+4a^2} + lx - l}{\sqrt{(lx-l)^2+4a^2} - lx - l}\right]. \tag{4.10}$$

In view of equations (4.6)–(4.10), equation (4.1) is expressed by

$$\frac{\ell}{2}\int_{-1}^{1} K_n\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right) I(y)dy + \frac{\ell}{2}\int_{-1}^{1} K_s\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right)(I(y) - I(x))dy + \frac{\ell}{2}I(x)H(x) = f(x). \tag{4.11}$$

We approximate $I(x)$, in equation (4.11) with interpolation by EBFs $\phi(x)$ as follows,

$$I_N(x) = \sum_{j=0}^{N} c_j \phi(x - x_j) = C^T \Psi(x), \tag{4.12}$$

where $x_j, j = 0, 1, \cdots, N$ are LGL nodes given in equation (3.1) and $\Psi(x) = \big[\phi(x - x_0), \phi(x - x_1), \cdots, \phi(x - x_N)\big]^T$ and $C = \big[c_0, c_1, \cdots, c_N\big]^T$.

By substituting equation (4.12) in equation (4.1) we have that

$$C^T\Bigg(\int_{-1}^{1} K_n\Big(\frac{\ell}{2}x, \frac{\ell}{2}y\Big)\Psi(y)\mathrm{d}y + \int_{-1}^{1} K_s\Big(\frac{\ell}{2}x, \frac{\ell}{2}y\Big)(\Psi(y) - \Psi(x))\mathrm{d}y + \Psi(x)H(x)\Bigg) = \frac{2f(x)}{l}, \tag{4.13}$$

For obtaining $c_j, j = 0, 1, \cdots, N$ in the above equation, by collocating at the points $x = x_i$ for $i = 0, 1, 2, \cdots, N$ we have that

$$C^T\Bigg(\int_{-1}^{1}\Big[K_n\Big(\frac{\ell}{2}x_i, \frac{\ell}{2}y\Big)\Psi(y) + K_s\Big(\frac{\ell}{2}x_i, \frac{\ell}{2}y\Big)\big(\Psi(y) - \Psi(x_i)\big)\Big]\mathrm{d}y + \Psi(x_i)H(x_i)\Bigg) = \frac{2f(x_i)}{l}, \tag{4.14}$$

By applying numerical integration methods in equation (3.2), we can approximate the integrals in equation (4.14) hence, this equation can be written as follow

$$C^T\Big(Q(x_i) + \Psi(x_i)H(x_i)\Big)\frac{l}{2} = f(x_i), \quad i = 0, 1, \cdots, N. \tag{4.15}$$

where $Q(x_i)$ is a vector as follows

$$Q(x_i) = \sum_{j=0}^{N} w_j \left[K_n\left(\frac{\ell}{2}x_i, \frac{\ell}{2}x_j\right)\Psi(x_j) + K_s\left(\frac{\ell}{2}x_i, \frac{\ell}{2}x_j\right)(\Psi(x_j) - \Psi(x_i))\right].$$

Equation (4.15) is a linear system of equations that we can solve by iteration methods to obtain unknown vector C^T.

5. NUMERICAL EXAMPLES

In this section, two numerical examples are presented to illustrate the validity and the merits of this technique. Using two RBFs $\phi_1(x) = e^{-r^2x^2}$ and $\phi_2(x) = \frac{1}{1+x^2}$ to approximate the Hallen's integral equation. In both examples data are given for two selected wire lengths so that they include special cases of practical interest, e.g., $l = \frac{\lambda}{2}$ and $l = \lambda$ and $a = 0.0005\lambda(\lambda = 2)$ and $V = 1$. In Fig. 1, $|I_N(x)|$ for $l = \lambda$, $N = 20$ and $r = 6$ has been shown. Also, the magnitude of $I_N(x)$ for $l = \frac{\lambda}{2}$ and $N = 20$ has been shown in Fig. 2. All computational efforts in this work have been done by Maple software in 20 decimal digits.

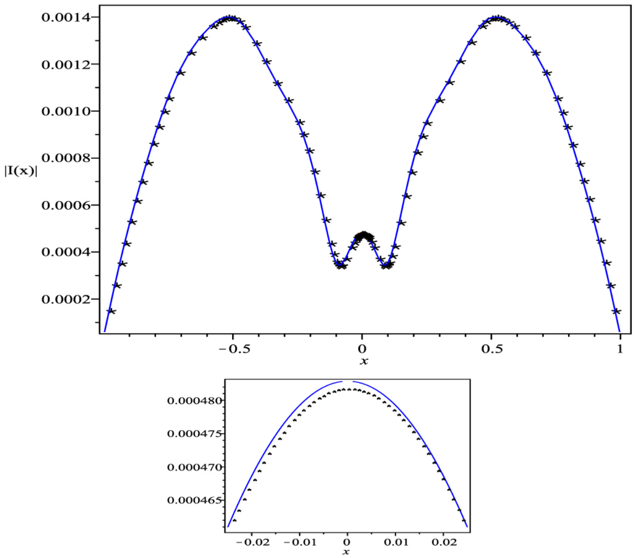

Fig. 1 – The magnitude of $I_{20}(x)$ for $l = \lambda$.

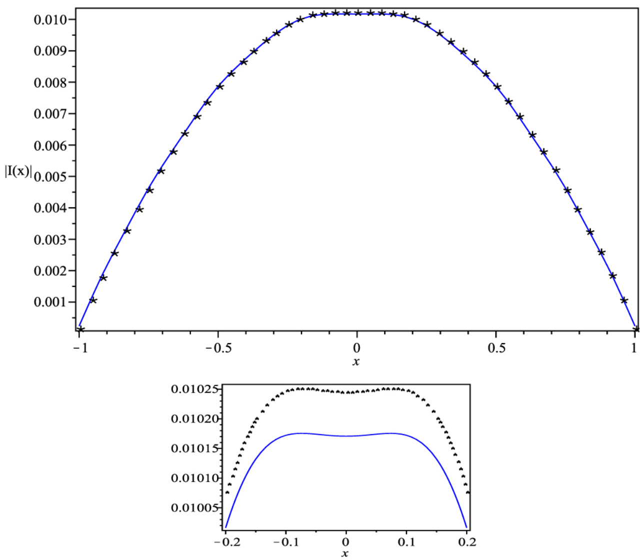

Fig. 2 – The magnitude of $I_{20}(x)$ for $l = \frac{\lambda}{2}$.

6. CONCLUSION

In this article, we have investigated the application of interpolation via radial basis functions for solving the Hallen's integral equation. The basis is very simple and simplicity can be used rather than the other basis [3–8] to approximate the Hallen's integral equation. Also, this method has high accuracy.

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