REVLEX IDEALS IN THE EXTERIOR ALGEBRA

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Let K be a field, V a K-vector space with basis $e_1, \ldots, e_n$ and let E be the exterior algebra of V. We study the Hilbert function of reverse lexicographic ideals in E and their Bass numbers.

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1. INTRODUCTION

Let K be a field, V a K-vector space with basis $e_1, \ldots, e_n$ and let E be the exterior algebra of V.

Let I be a graded ideal in E. The most important theorem in the study of the Hilbert functions of graded algebras of the form E/I is the Kruskal-Katona theorem [1, Theorem 4.1], which is the precise analogue to Macaulay's theorem [3] on the Hilbert functions of homogeneous commutative rings. An important role in this context play the lexsegment ideals (see, for example [1, 2, 5, 6] and the references therein). In fact, if $I \subsetneq E$ is a graded ideal, then there exists a unique lexsegment ideal with the same Hilbert function as that of I [1]. The reverse lexicographic ideals have not in general the same behaviour. Indeed, given an Hilbert function H there is often no reverse lexicographic ideal attaining H.

One of the purposes of this paper is to study the Hilbert function of E/I, when I is a reverse lexicographic ideal and state when a finite sequence of non negative integers determines the Hilbert function of a reverse lexicographic ideal.

It is known that the exterior algebra is self-dual. Hence, some results on Hilbert functions or resolutions have dual counterparts [1].

In [7], the authors showed that the reverse lexicographic ideal has the smallest graded Betti numbers among all strongly stable ideals with the same Hilbert function in an exterior algebra. In this paper, we show that the reverse lexicographic ideals give the lowest graded Bass numbers among all strongly stable ideals with the same Hilbert function.

The paper is organized as follows. Section 2 contains preliminary notions and results. In Section 3, the Hilbert functions of reverse lexicographic ideals are studied. The main result in this section consists in proving that, given a finite sequence of non negative integers $(h_1, \dots h_n)$, then there exists a reverse lexicographic ideal I in E such that $H_{E/I} = 1 + \sum_{i=1}^{n} h_i t^i$ (Theorem 3.1). In Section 4, we compare the Bass numbers of a strongly stable ideal in E and the Bass numbers of a reverse lexicographic ideal with the same Hilbert function as I (Proposition 4.2).

2. PRELIMINARIES AND NOTATIONS

Let K be a field. We denote by $E = K\langle e_1, \dots, e_n\rangle$ the exterior algebra of a K-vector space V with basis $e_1, \dots, e_n$. For any subset $\sigma = \{i_1, \dots, i_d\}$ of $\{1, \dots, n\}$ with $1 \le i_1 < i_2 < \dots < i_d \le n$ we write $e_\sigma = e_{i_1} \wedge \dots \wedge e_{i_d}$ and call e_σ a monomial of degree d. The set of monomials in E forms a K-basis of E of cardinality 2^n.

In order to simplify the notation we put $fg = f \wedge g$ for any two elements f and g in E. An element $f \in E$ is called *homogeneous* of degree j if $f \in E_j$, where $E_j = \bigwedge^j V$. An ideal I is called *graded* if I is generated by homogeneous elements. If I is graded, then $I = \oplus_{j\ge 0} I_j$, where I_j is the K-vector space of all homogeneous elements $f \in I$ of degree j.

Let $e_\sigma = e_{i_1} e_{i_2} \cdots e_{i_d}$ be a monomial of degree d. We define

$$\mathrm{supp}(e_\sigma) = \{i : e_i \text{ divides } e_\sigma\},$$

and we write

$$\mathrm{m}(e_\sigma) = \max\{i : i \in \mathrm{supp}(e_\sigma)\}.$$

Definition 2.1. Let $I \subsetneq E$ be a monomial ideal. I is called *stable* if for each monomial $e_\sigma \in I$ and each $j < \mathrm{m}(e_\sigma)$ one has $e_j e_{\sigma \setminus \{\mathrm{m}(e_\sigma)\}} \in I$. I is called *strongly stable* if for each monomial $e_\sigma \in I$ and each $j \in \sigma$ one has that $e_i e_{\sigma\setminus\{j\}} \in I$, for all $i < j$.

Let I be a graded ideal of E. The function $H_{E/I}(j) = \dim_K (E/I)_j$, $j = 0, 1, \dots,$ is called the *Hilbert function* of E/I and the polynomial $H_{E/I} = \sum_{j\ge 0} H_{E/I}(j) t^i$ is called the *Hilbert series* of E/I.

It is known that if I is a graded ideal in E, then E/I has the unique minimal graded free resolution over E:

$$F : \dots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to E/I \to 0.$$

where $F_i = \oplus_j E(-j)^{\beta_{i,j}(E/I)}$. The integers $\beta_{i,j}(E/I) = \dim_K \mathrm{Tor}_i^E(E/I, K)_j$ are called the *graded Betti numbers*.

Now, let M_d denote the set of all monomials of degree $d \geq 1$ in E. For any subset S of E, we denote by $M(S)$ the set of all monomials in S and we denote by $|S|$ its cardinality.

We write $>_{\text{revlex}}$ for the *reverse lexicographic order* (revlex for short) on the finite set M_d, *i.e.*, if $u = e_{i_1}e_{i_2}\cdots e_{i_d}$ and $v = e_{j_1}e_{j_2}\cdots e_{j_d}$ are monomials belonging to M_d with $1 \leq i_1 < i_2 < \ldots < i_d \leq n$ and $1 \leq j_1 < j_2 < \ldots < j_d \leq n$, then

$$u >_{\text{revlex}} v \text{ if } i_d = j_d, i_{d-1} = j_{d-1}, \ldots, i_{s+1} = j_{s+1} \text{ and } i_s < j_s$$

for some $1 \leq s \leq d$.

From now on, in order to simply the notations, we will write $>$ instead of $>_{\text{revlex}}$.

Definition 2.2. A nonempty set $M \subseteq M_d$ is called a *reverse lexicographic segment of degree* d (revlex segment of degree d, for short) if for all $v \in M$ and all $u \in M_d$ such that $u > v$, we have that $u \in M$.

If M is a revlex segment of degree d and $|M|=\ell$, ℓ is called the *length* of M.

Definition 2.3. Let $I = \oplus_{j\geq 0} I_j$ be a monomial ideal of E. We say that I is a *reverse lexicographic ideal* of E if, for every j, I_j is spanned by a revlex segment (as K-vector space).

From now on, for the sake of simplicity, given a monomial ideal $I = \oplus_{j\geq 0} I_j$ of E we will say that I_j is *a reverse lexicographic segment of degree* j if I_j is spanned as K-vector space by a reverse lexicographic segment of degree j.

Definition 2.4. Let M be a subset of monomials of E. Set $\mathbf{e}_i = \{e_1, \ldots, e_i\}$. We define the set

$$\mathbf{e}_i M = \{ue_j : u \in M, \quad j \notin \text{supp}(u), \quad j = 1, \ldots, i\}.$$

Note that $\mathbf{e}_i M = \emptyset$ if, for every monomial $u \in M$ and for every $j = 1, \ldots, i$, one has $j \in \text{supp}(u)$.

If M is a set of monomial of degree $d < n$, $\mathbf{e}_n M$ is called the *shadow* of M and is denoted by $\text{Shad}(M)$ [4]:

$$\text{Shad}(M) = \{ue_j : u \in M, \ j \notin \text{supp}(u), \quad j = 1, \ldots, n\}.$$

We define the i-th shadow recursively by $\text{Shad}^i(M) = \text{Shad}(\text{Shad}^{i-1}(M))$.

If M is a revlex segment of degree d, then $\text{Shad}(M)$ needs not be a revlex segment of degree $d+1$ [7].

For a subset M of monomials of degree d of E we denote by $E_1 M$ the K-vector space generated by $\text{Shad}(M)$. Moreover, if $I = \oplus_{j\geq 0} I_j$ is a monomial ideal of E, we denote by $E_1 I_d$ the K-vector space generated by $\text{Shad}(M(I_d))$ and by $E_{d_2-d_1} I_{d_1}$ the K-vector space generated by $\text{Shad}^{d_2-d_1}(M(I_{d_1}))$, $d_2 > d_1$.

If $I \subsetneq E$ is a monomial ideal, we denote by $G(I)$ the unique minimal set of monomial generators of I and we define the following sets:

$$G(I)_d = \{u \in G(I) : \ \deg(u) = d\}, \ G(I;i) = \{u \in G(I) : \ \mathrm{m}(u) = i\},$$
$$\mathrm{m}_i(I) = |G(I;i)|,$$

for $d > 0$ and $1 \leq i \leq n$.

Proposition 2.1. *Let M be a revlex segment of degree $d < n-2$ of E. Then the following conditions are equivalent:*

(a) $\mathrm{Shad}(M)$ *is a revlex segment of degree $d+1$;*
(b) $|M| \geq \binom{n-2}{d}$*;*
(c) $e_{n-(d+1)} \cdots e_{n-2} \in M$*;*
(d) *all iterated shadows of M are revlex segments.*

If one of the above conditions is fulfilled, then $\mathrm{Shad}^2(M) = M_{d+2}$.

Proof. (a) $\Leftrightarrow$ (b). See [7, Proposition 4.1].
(a) $\Leftrightarrow$ (c). See [7, Corollary 3.8].
(c) $\Rightarrow$ (d). If $e_{n-(d+1)} \cdots e_{n-2} \in M$, then $e_{n-(d+1)} \cdots e_{n-2}e_n \in \mathrm{Shad}(M)$.
Hence, $\mathrm{Shad}^2(M) = M_{d+2}$, and, consequently, $\mathrm{Shad}^i(M) = M_{d+i}$, for all $i \geq 3$.
(d) $\Rightarrow$ (c). Obvious. $\square$

As a consequence of Proposition 2.1, we obtain the following result.

Corollary 2.1. *Let $I \subsetneq E$ be a revlex ideal generated in degree $d < n-2$. Then* $\mathrm{Shad}(M(I_d))$ *is a revlex segment of degree $d+1$ if and only if*

$$H_{E/I}(d) \leq \binom{n}{d} - \binom{n-2}{d}.$$

3. HILBERT FUNCTION

In this section, we state under which conditions a finite sequence of non negative integers determines the Hilbert function of a revlex ideal.

For a graded ideal $I = \oplus_{j \geq 0} I_j$ of E, we denote by $\mathrm{indeg}(I)$, the *initial degree* of I, that is, the minimum s such that $I_s \neq 0$.

Proposition 3.1. *A revlex ideal $I \subsetneq E$ is minimally generated in at most two consecutive degrees.*

Proof. Let $d = \mathrm{indeg}(I)$. Then $e_1e_2 \cdots e_d \in I_d$, and $e_1e_2 \cdots e_de_n \in I_{d+1}$. As I is a revlex ideal, it follows that $e_{n-(d+2)} \cdots e_{n-2} \in I_{d+1}$, and consequently $\mathrm{Shad}(M(I_{d+1})) = M_{d+2}$. Therefore, the minimal monomial generators of I are at most of degree d and $d+1$. $\square$

PROPOSITION 3.2. *Let $I \subsetneq E$ be a revlex ideal generated in degree $d < n-2$. If $\dim_K I_d = \binom{n-2}{d} + m_{n-1}(I)$, then*

$$\dim_K I_{d+1} = \binom{n-2}{d} + \binom{n-1}{d+1} + m_{n-1}(I).$$

Proof. As $\binom{n-2}{d-1} - m_{n-1}(I)$ is the number of monomials $v \in M_d \setminus G(I)$ such that $\mathrm{m}(v) = n-1$, it follows that

$$\begin{aligned}\dim_K I_{d+1} &= \dim_K E_{d+1} - \left[\binom{n-2}{d-1} - m_{n-1}(I)\right]\\ &= \binom{n}{d+1} - \binom{n-2}{d-1} + m_{n-1}(I)\\ &= \binom{n}{d+1} - \binom{n-1}{d} + \binom{n-2}{d} + m_{n-1}(I)\\ &= \binom{n-1}{d+1} + \binom{n-2}{d} + m_{n-1}(I). \quad \square\end{aligned}$$

The next result will be crucial in the sequel.

COROLLARY 3.1. *Let $I \subsetneq E$ be a revlex ideal generated in degree $d < n-2$ and b a positive integer such that $b \le \binom{n}{d} - \binom{n-2}{d}$.*
Let $\dim_K E_d/I_d = b$, then

(i) $\dim_K E_{d+1}/E_1 I_d = b - \binom{n-1}{d-1}$, *if* $b > \binom{n-1}{d-1}$;

(ii) $\dim_K E_{d+1}/E_1 I_d = 0$, *if* $b \le \binom{n-1}{d-1}$.

Proof. First of all, observe that $E_1 I_d = I_{d+1}$ is a revlex segment set of degree $d+1$ (Corollary 2.1) and that $\binom{n-1}{d-1}$ is the number of all monomials $z \in M_d$ such that $\mathrm{m}(z) = n$.

(i). Let $b > \binom{n-1}{d-1}$.

Claim. $b - \binom{n-1}{d-1} = \binom{n-2}{d-1} - m_{n-1}(I)$.

Since $\binom{n-2}{d-1}$ is the number of all monomials $w \in M_d$ such that $\mathrm{m}(w) = n-1$, then $\binom{n-2}{d-1} - m_{n-1}(I)$ is the number of all monomials $u \in M_d \setminus G(I)$ such that $\mathrm{m}(u) = n-1$. Hence,

$$\binom{n-2}{d-1} - m_{n-1}(I) = \binom{n}{d} - \binom{n-1}{d-1} - \dim_K I_d.$$

Since $\dim_K I_d = \binom{n}{d} - b$, then

$$\binom{n-2}{d-1} - m_{n-1}(I) = b - \binom{n-1}{d-1}, \tag{1}$$

and the claim is proved.

Note that under our hypothesis, if u is the smallest monomial of $G(I)$, then $\mathrm{m}(u) \in \{n-2, n-1\}$.

(Case 1). Suppose $\mathrm{m}(u) = n-2$. It follows that $u = e_{n-(d+1)} \cdots e_{n-2}$ is the smallest monomial of $G(I)$ and consequently $m_{n-1}(I) = 0$.

Therefore, by Proposition 3.2, we have:

$$\begin{aligned}\dim_K E_{d+1}/E_1 I_d &= \binom{n}{d+1} - \binom{n-2}{d} - \binom{n-1}{d+1} \\ &= \binom{n-1}{d} - \binom{n-2}{d} = \binom{n-2}{d-1}\end{aligned}$$

and from the claim, we get the assert.

(Case 2). Suppose $\mathrm{m}(u) = n-1$, then $\dim_K I_d = \binom{n-2}{d} + m_{n-1}(I)$. Since, by Proposition 3.2, $\dim_K I_{d+1} = \binom{n-2}{d} + \binom{n-1}{d+1} + m_{n-1}(I)$, it follows that

$$\begin{aligned}\dim_K E_{d+1}/E_1 I_d &= \binom{n}{d+1} - \binom{n-2}{d} - \binom{n-1}{d+1} - m_{n-1}(I) \\ &= \binom{n-2}{d-1} - m_{n-1}(I).\end{aligned}$$

From the claim we get the desired equality.

(ii). Let $b \leq \binom{n-1}{d-1}$. In this case, if u is the smallest monomial of $G(I)$, then $\mathrm{m}(u) = n$.

Hence, $\mathrm{Shad}(M(I_d)) = M_{d+1}$ and $\dim_K E_{d+1}/E_1 I_d = 0$. $\square$

Theorem 3.1. *Let $(h_1, h_2, \ldots, h_n)$ be a sequence of non negative integers and d' a positive integer such that $d' < n-2$.*

Suppose that:

(1) $h_d = \binom{n}{d}$, *for* $d = 0, \ldots, d'-1$;

(2) $h_{d'} \leq \binom{n}{d'} - \binom{n-2}{d'}$;

(3) $\begin{cases} h_{d'+1} \leq h_{d'} - \binom{n-1}{d'-1}, & \text{if } h_{d'} > \binom{n-1}{d'-1}, \\ h_{d'+1} = 0, & \text{if } h_{d'} \leq \binom{n-1}{d'-1}; \end{cases}$

(4) $h_d = 0$, *for* $d > d'+1$.

Then there exists a unique revlex ideal $J \subsetneq E = K\langle e_1, \ldots, e_n\rangle$ of initial degree d' and such that $H_{E/J}(t) = 1 + \sum_{i=1}^n h_i t^i$.

Proof. Let J_d be the K-vector space generated by the revlex segment of degree d and length $\binom{n}{d} - h_d$ in the exterior algebra $E = K\langle e_1, \ldots, e_n\rangle$, for $d = 0, \ldots, n$.

(Case 1). Let $h_{d'} > \binom{n-1}{d'-1}$. First of all, note that $J_d = 0$, for $d = 0, \dots, d'-1$, and $J_d = E_d$, for $d > d'+1$.

By definition $J_{d'}$ is the K-vector space spanned by the revlex segment of degree d' and length $\binom{n}{d'} - h_{d'}$. By (2), $h_{d'} \le \binom{n}{d'} - \binom{n-2}{d'}$, then $\dim_K J_{d'} \ge \binom{n-2}{d'}$ and from Proposition 2.1, $\mathrm{Shad}(M(J_{d'}))$ is a revlex segment of degree $d'+1$, *i.e.*, $E_1 J_{d'}$ is a K-vector subspace of $E_{d'+1}$ generated by a revlex segment of degree $d'+1$. Hence, applying the same reasoning of Corollary 3.1, proof of (i), we can state that $\dim_K E_{d'+1}/E_1 J_{d'} = h_{d'} - \binom{n-1}{d'-1}$. Thus, from (3), it follows that $h_{d'+1} \le \dim_K E_{d'+1}/E_1 J_{d'}$ and so $\dim_K J_{d'+1} \ge \dim_K E_1 J_{d'}$. Hence, $E_1 J_{d'} \subseteq J_{d'+1}$.

On the other hand, $E_1 J_{d'+1} = E_{d'+2} = J_{d'+2}$ by Proposition 2.1. Therefore $J = \oplus_{d=d'}^{n} J_d$ is indeed an ideal of $E = K\langle e_1, \dots, e_n \rangle$ of initial degree d' and E/J has the desired Hilbert function. Note that J has generators in degrees d' and $d'+1$.

(Case 2). Let $h_{d'} \le \binom{n-1}{d'-1}$. In this case, $J_d = 0$, for $d = 0, \dots, d'-1$, and $J_d = E_d$, for $d \ge d'+1$.

Moreover, the smallest monomial u belonging to $M(J_{d'})$ is such that $m(u) = n$. Hence, $E_1 J_{d'} = \langle M_{d'+1} \rangle = J_{d'+1}$. It follows that $J = \oplus_{d=d'}^{n} J_d$ is an ideal of $E = K\langle e_1, \dots, e_n \rangle$ of initial degree d' and $H_{E/J} = 1 + \sum_{i=1}^{n} h_i t^i$.

Note that J is an ideal generated in degree d'.

The uniqueness part of the theorem is obvious. $\square$

Remark 3.1. Let $(h_1, h_2, \dots, h_n)$ be a sequence of non negative integers satisfying the conditions stated in the Kruskal-Katona theorem [1, Theorem 4.1]. There exists always a (unique) lexicographic ideal in E such that $H_{E/I} = 1 + \sum_{i=1}^{n} h_i t^i$ [1]. While there is often no revlex ideal J in E attaining $H_{E/I}$.

For example, consider the sequence of integers $(1, 5, 8, 5, 0)$; there exists the lexicographic ideal $I = (e_1e_2, e_1e_3) \subsetneq E = K\langle e_1, e_2, e_3, e_4, e_5 \rangle$ such that $H_{E/I} = 1 + 5t + 8t^2 + 5t^3$, but no revlex ideal such that $H_{E/J} = H_{E/I}$ can be found.

Note that the sequence examined does not satisfy all the conditions in Theorem 3.1.

4. BASS NUMBERS

In this section, we analyze the Bass numbers of revlex ideals in the exterior algebra $E = K\langle e_1, \dots, e_n \rangle$.

Let M be a finite left (right) E-module. We denote by M' the right (left) E-module $\mathrm{Hom}_E(M, E)$. The graded Bass numbers of M' are the integers defined as follows [3]:

$$\mu_{i,j}(M') = \dim_K \mathrm{Ext}^i_E(K, M')_j.$$

The next result due to Aramova, Herzog and Hibi [1, Proposition 5.2] relates the graded Betti numbers of the left E-module M to the graded Bass numbers of the dual right E-module M'.

PROPOSITION 4.1. *Let M be a graded right E-module. Then*

$$\beta_{i,j}(M) = \mu_{i,n-j}(M'), \qquad \text{for all } i, j.$$

If $I \subsetneq E$ is an ideal, then $\mathrm{Hom}_E(E/I, E) \simeq 0 : I$, where $0 : I$ is the annihilator of I, that is, the set of all elements $b \in E$ such that $ba = 0$, for all $a \in I$.

Thus, if $I \subsetneq E$ is a graded ideal, it follows that [1, Corollary 5.3]:

$$\dim_K(E/I)_i = \dim_K(0 : I)_{n-i}, \quad \text{for all } i. \tag{2}$$

For more details on this subject see [1].

LEMMA 4.1. *Let $I \subsetneq E$ be a graded ideal.*

(1) *If I is a strongly stable ideal, then $0 : I$ is a strongly stable ideal in E.*

(2) *If I is a revlex ideal, then $0 : I$ is a revlex ideal in E.*

Proof. The ideal $0 : I$ is spanned as K-vector space by all monomials $e_{\bar{\sigma}}$ such that $e_\sigma \notin I$, where $\bar{\sigma}$ is the complement of σ in the set $\{1, \ldots, n\}$ (see [1, Proposition 5.7], proof).

(1). If T is a strongly stable set in E, *i.e.*, a set of monomials of degree $d \geq 1$ in E such that for each monomial $e_\sigma \in T$ and each $j \in \sigma$ one has that $(-1)^{\alpha(\sigma,i)} e_j e_{\sigma \setminus \{j\}} \in T$, where $\alpha(\sigma, i) = |\{r \in \sigma : r < i\}|$, for all $i < j$, then the set $\{e_{\bar{\sigma}} : e_\sigma \notin T\}$ is a strongly stable set in E, too. Hence, if I is a strongly stable ideal, then $0 : I$ is a strongly stable ideal in E.

(2). If T is a revlex segment in E, then the set $\{e_{\bar{\sigma}} : e_\sigma \notin T\}$ is a revlex segment in E, too. Hence, if I is a revlex ideal, then $0 : I$ is a revlex ideal in E. $\square$

Remark 3.1 has pointed out that given a strongly stable ideal $I \subsetneq E$ there is not always a revlex ideal with the same Hilbert function as I (see [8] for the polynomial case). This fact justifies our assumption in the next results.

LEMMA 4.2. *Let $I \subsetneq E$ be a strongly stable ideal and $J \subsetneq E$ a revlex ideal such that $H_{E/J}(d) = H_{E/I}(d)$, for all d. Then*

$$H_{E/0:J}(d) = H_{E/0:I}(d), \quad \text{for all } d.$$

Proof. From (2) and by our assumptions, it follows that

$$\dim_K(0 : I)_d = \dim_K(E/I)_{n-d} = \dim_K(E/J)_{n-d} = \dim_K(0 : J)_d,$$

and we get the assert. $\square$

As a consequence, we obtain a bound similar to [7, Theorem 5.6] for the Bass numbers of E/I, for I strongly stable ideal.

PROPOSITION 4.2. *Let $I \subsetneq E$ be a strongly stable ideal and $J \subsetneq E$ a revlex ideal such that $H_{E/J}(d) = H_{E/I}(d)$, for all d. Then*

$$\mu_{i,j}(E/J) \leq \mu_{i,j}(E/I), \qquad \text{for all } i, j.$$

Proof. First of all observe that, from Lemmas 4.1 and 4.2, $0 : I$ is a strongly stable ideal with the same Hilbert function as the revlex ideal $0 : J$. Hence, from Proposition 4.1 and [7, Theorem 5.6], it follows that:

$$\mu_{i,j}(E/I) = \beta_{i,n-j}(0 : I) \geq \beta_{i,n-j}(0 : J) = \mu_{i,j}(E/J). \quad \square$$

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