ON (-1)-WEAK AMENABILITY OF BANACH ALGEBRAS

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For a Banach algebra A, its second dual A'' is (-1)-weakly amenable if A' is a Banach A''-bimodule and the first cohomology group of A'' with coefficients in A' is zero, *i.e.* $H^1(A'', A') = \{0\}$. In this paper, we study the (-1)-weak amenability of the second dual of James algebras. Moreover, we show that $Lip_\alpha\mathbb{T}$ is not (-1)-weakly amenable if $\frac{1}{2} < \alpha < 1$ and $\mathbb{T}$ is the unit circle.

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1. INTRODUCTION

Let A be a Banach algebra and E be a Banach A-bimodule, then a bounded derivation from A into E is a bounded linear map $D : A \longrightarrow E$ such that for each $a, b \in A$, $D(a \cdot b) = Da \cdot b + a \cdot Db$. For example, let $x \in X$ and define $\delta_x : A \longrightarrow E$ by $\delta_x a = a \cdot x - x \cdot a$, then δ_x is a bounded derivation which is called an inner derivation. Then $Z^1(A, E)$ is the space of all bounded derivations from A into E, $N^1(A, E)$ is the space of all inner derivations from A into E, and the first cohomology group of A with coefficients in E is the quotient space $H^1(A, E) = \frac{Z^1(A,E)}{N^1(A,E)}$.

A Banach algebra A is amenable if $H^1(A, E') = \{0\}$ for each Banach A-bimodule E. This concept was introduced by B.E. Johnson in [5].

The notion of weak amenability was introduced by W.G. Bade, P.C. Curtis and H.G. Dales in [1] for commutative Banach algebras. Later, Johnson defined weak amenability for arbitrary Banach algebras in [6], in fact a Banach algebra A is weakly amenable if $H^1(A, A') = \{0\}$.

Let A be a Banach algebra and A'' be its second dual, for each $a, b \in A$, $f \in A'$ and $F, G \in A''$ we define $f \cdot a$, $a \cdot f$ and $F \cdot f, f \cdot F \in A'$ by

$$f \cdot a(b) = f(a \cdot b), \quad a \cdot f(b) = f(b \cdot a)$$
$$F \cdot f(a) = F(f \cdot a), \; f \cdot F(a) = F(a \cdot f).$$

Now, we define $F \cdot G, F \times G \in A''$ as follows

$$F \cdot G(f) = F(G \cdot f), \qquad F \times G(f) = G(f \cdot F).$$

Then A'' is a Banach algebra with respect to either of the products $\cdot$ and $\times$. These products are called respectively, the first and the second Arens products on A''. Then A is called Arens regular if $F \cdot G = F \times G$, for all $F, G \in A''$. In this paper, A'' is considered with the first Arens product.

In [7], A. Medghalchi and T. Yazdanpanah, introduced the notion of (-1)-weak amenability. A Banach algebra A is (-1)-weakly amenable if A' is a Banach A''-bimodule and $H^1(A'', A') = \{0\}$.

Here, we give some examples which are and some others which are not (-1)-weakly amenable Banach algebras. For example, in Theorem 2.2 we show that $Lip_\alpha \mathbb{T}$ is not (-1)-weakly amenable for $\frac{1}{2} < \alpha < 1$, where $\mathbb{T}$ is the unit circle.

Also, for James algebra $\mathcal{J}, \mathcal{J}''$ is (-1)-weakly amenable, see Example 2.2.

For (-1)-weak amenability of a Banach algebra we need A' to be a Banach A''-bimodule, in [4] we give some conditions, for a Banach A-bimodule X, which makes X' a Banach A''-bimodule.

Let X be a Banach space, then $\iota : X \longrightarrow X''$ is the natural embedding and ι_x is denoted by $\hat{x}$ $(x \in X)$, and $\hat{X}$ is the natural embedding of X in X''.

Let A be a Banach algebra and let E be a Banach A-bimodule, then the iterated conjugates of E, denoted by $E', E'', E''', \ldots$ are Banach A-bimodules, and the map $\rho : E''' \longrightarrow E'$ with $\rho(\Gamma) = \Gamma \mid_{\hat{E}}$ is an A-bimodule homomorphism which is called natural projection.

All concepts and definitions which are not defined in this paper may be found in [2] and [3].

2. MAIN RESULTS

Let A be a Banach algebra with a closed ideal I. Then, we have the identifications

$$(A/I)' \simeq I^0 , \qquad (A/I)'' \simeq A''/I''$$

where $I^0 = \{\lambda \in A' : \; \lambda|_I = 0\}$.

Let E be a Banach A-bimodule and I be a closed ideal in A, then E is a Banach I-bimodule. In the case $IE = EI$, we have E is a Banach A/I-bimodule.

Let I be an ideal in A. Then I has the trace extension property in A, whenever for each $\lambda \in I'$ with $a \cdot \lambda = \lambda \cdot a$ $(a \in A)$, there exists $\tau \in A'$ such that $\tau|_I = \lambda$ and $a \cdot \tau = \tau \cdot a, (a \in A)$.

For a Banach algebra A, we define the center of $(A'', \cdot)$ as follows,

$$Z(A'', \cdot) = \{F \in A'' : F \cdot G = F \times G, (G \in A'')\}.$$

THEOREM 2.1. *Let A be a Banach algebra and, I be a closed ideal in A and its second dual I'', be a closed ideal in A''. Then*

(1) *If A''/I'' is (-1)-weakly amenable, then I has trace extension property.*

(2) *If A'' is (-1)-weakly amenable and $I'' \subseteq Z((A'', \cdot))$ and I'' has the trace extension property, then A''/I'' is (-1)-weakly amenable.*

(3) *If I'' and A''/I'' are (-1)-weakly amenable and $\overline{I''^2} = I''$ and A' is a Banach A''-bimodule, then A'' is (-1)-weakly amenable.*

Proof. (1) Let $\lambda \in I'$ with $a \cdot \lambda = \lambda \cdot a$, for all $a \in A$. By Hahn-Banach theorem there exists $f \in A'$ such that $f|_I = \lambda$. We define

$$D : A''/I'' \longrightarrow I^0 = (A/I)'$$
$$D(F + I'') = F \cdot f - f \cdot F \qquad (F \in A'')$$

then for each $x \in I$ and $F \in A''$ with $F = w^* - \lim_{\alpha} \hat{a}_\alpha$, we have

$$\begin{aligned} D(F + I'')(x) &= (F \cdot f - f \cdot F)(x) = F(f \cdot x - x \cdot f) \\ &= \lim_{\alpha} f(x \cdot a_\alpha - a_\alpha \cdot x) = \lim_{\alpha} \lambda(x \cdot a_\alpha - a_\alpha \cdot x) \\ &= \lim_{\alpha} \hat{a}_\alpha(\lambda \cdot x - x \cdot \lambda) = 0. \end{aligned}$$

It follows that D is a bounded derivation in $Z^1(A''/I'', I^0)$. By (-1)-weak amenability of A''/I'', there exists $\lambda_0 \in I^0$ such that $D(F+I) = F \cdot \lambda_0 - \lambda_0 \cdot F$. Now, put $\tau = f - \lambda_0 \in A'$, then for each $a \in A$

$$a \cdot \tau - \tau \cdot a = (a \cdot f - f \cdot a) - (a \cdot \lambda_0 - \lambda_0 \cdot a) = D(\hat{a} + I'') - D(\hat{a} + I'') = 0$$

so, $a\tau = \tau a$. Moreover, since $\lambda_0 \in I^0$ and $f|_I = \lambda|_I$, then $\tau(x) = f(x) - \lambda_0(x) = \lambda(x)$, and $\tau|_I = \lambda$. It follows that $a\tau = \tau a$, for all $a \in A$.

(2) First, we show that I^0 is a Banach A''/I''-bimodule. Since A'' is (-1)-weakly amenable, then A' is an A''-bimodule and for each $\lambda \in I^0$ and $F \in A''$, we have $\lambda \cdot F, F \cdot \lambda \in I^0$ so I^0 is an A''-bimodule. On the other hand, since I'' is a closed ideal in A'', then I^0 is an I''-bimodule and since $I^0 \cdot I'' = I'' \cdot I^0 = 0$ then I^0 is a Banach A''/I''-bimodule. Let $D \in Z^1(A''/I'', (A/I)')$ and let $\pi : A \longrightarrow A/I$, $a \mapsto a + I$ be the quotient map. We define $\tilde{D} = \pi' \circ D \circ \pi''$ *i.e.*,

$$\tilde{D} : A'' \longrightarrow A'$$
$$\tilde{D}(F)(a) = D(F + I'')(a + I) \qquad (a \in A, F \in A'')$$

then $\tilde{D} \in Z^1(A'', A')$, since for each $F, G \in A''$ with $F = w^* - \lim_{\alpha} \hat{a}_\alpha$ and $G = w^* - \lim_{\beta} \hat{b}_\beta$ we have

$$\begin{aligned} \tilde{D}(F \cdot G)(a) &= D(F \cdot G + I'')(a + I) = D\left((F + I'') \cdot (G + I'')\right)(a + I) \\ &= (D(F + I'') \cdot (G + I'') + (F + I'') \cdot D(G + I''))(a + I) \end{aligned}$$

$$
\begin{aligned}
&= G + I''((a+I)\cdot D(F+I'')) + F + I''(D(G+I'')\cdot(a+I))\\
&= \lim_{\beta}(\hat{b}_\beta + I'')((a+I)\cdot D(F+I''))\\
&\qquad + \lim_{\alpha}(\hat{a}_\alpha + I'')(D(G+I'')\cdot(a+I))\\
&= \lim_{\beta} D(F+I'')(b_\beta\cdot a + I) + \lim_{\alpha} D(G+I'')(a\cdot a_\alpha + I)\\
&= \lim_{\beta}\tilde{D}F(b_\beta\cdot a) + \lim_{\alpha}\tilde{D}G(a\cdot a_\alpha) = \lim_{\beta} a\cdot\tilde{D}F(b_\beta)\\
&\qquad + \lim_{\alpha}\tilde{D}G\cdot a(a_\alpha)\\
&= \lim_{\beta}\hat{b}_\beta(a\cdot\tilde{D}F) + \lim_{\alpha}\hat{a}_\alpha(\tilde{D}G\cdot a) = G(a\cdot\tilde{D}F) + F(\tilde{D}G\cdot a)\\
&= (\tilde{D}F\cdot G + F\cdot\tilde{D}G)(a).
\end{aligned}
$$

On the other hand, if A'' is (-1)-weakly amenable then there exists $\lambda_0 \in A'$ such that $\tilde{D}F = F\cdot\lambda_0 - \lambda_0\cdot F$, for all $F \in A''$. For $G \in I''$ we have $G\cdot\lambda_0 - \lambda_0\cdot G = \tilde{D}G = D(G+I'') = 0$ so, $G\cdot\lambda_0 = \lambda_0\cdot G$. Let $\hat{\lambda}_0 \in A'''$ defined by $\hat{\lambda}_0(F) = F(\lambda_0)$, for all $F \in A''$. We have $I'' \subseteq Z((A'',\cdot))$, then for each $F \in A''$ and $G \in I''$, we have

$$
\begin{aligned}
F\cdot\hat{\lambda}_0(G) &= \hat{\lambda}_0(G\cdot F) = F(\lambda_0\cdot G) = F(G\cdot\lambda_0)\\
&= F\cdot G(\lambda_0) = \hat{\lambda}_0(F\cdot G) = \hat{\lambda}_0\cdot F(G).
\end{aligned}
$$

If we take $\theta_0 = \hat{\lambda}_0|_{I''} \in I'''$, then we have $F\cdot\theta_0 = \theta_0\cdot F$. Since I'' has the trace extension property, then there exists $\Lambda_0 \in A'''$ such that $\Lambda_0|_{I''} = \theta_0$ and $F\cdot\Lambda_0 = \Lambda_0\cdot F$, for all $F \in A''$. For each $a \in A$, we define $\tau \in A'$ by $\tau(a) = \lambda_0(a) - \Lambda_0(\hat{a})$, (more precisely $\tau = \rho(\hat{\lambda}_0 - \Lambda_0)$, where $\rho : A''' \longrightarrow A'$ is the natural projection). Then

$$
\tau(x) = \lambda_0(x) - \Lambda_0(\hat{x}) = \hat{\lambda}_0(\hat{x}) - \Lambda_0(\hat{x}) = \theta_0(\hat{x}) - \Lambda_0(\hat{x}) = 0
$$

for all $x \in I$, we have $\tau \in I^0$. (Note, that since $I^0 = (A/I)'$ then we can assume τ as an element in $(A/I)'$). On the other hand, we have $\tilde{D} = \delta_{\lambda_0}$ and $\tau|_I = 0$ then for each $a \in A$ and $F \in A''$ with $F = w^* - \lim_{\alpha}\hat{a}_\alpha$, we have

$$
\begin{aligned}
D(F+I'')(a+I) &= \tilde{D}(F)(a) = (F\cdot\lambda_0 - \lambda_0\cdot F)(a)\\
&= F(\lambda_0\cdot a - a\cdot\lambda_0) = \lim_{\alpha}\hat{a}_\alpha(\lambda_0\cdot a - a\cdot\lambda_0)\\
&= \lim_{\alpha}\lambda_0(a\cdot a_\alpha - a_\alpha\cdot a) - \lim_{\alpha}(\Lambda_0\cdot\hat{a} - \hat{a}\cdot\Lambda_0)(\hat{a}_\alpha)\\
&= \lim_{\alpha}\Big(\lambda_0(a\cdot a_\alpha - a_\alpha\cdot a) - \Lambda_0(\widehat{a\cdot a_\alpha - a_\alpha\cdot a})\Big)\\
&= \lim_{\alpha}\tau(a\cdot a_\alpha - a_\alpha\cdot a) = \lim_{\alpha}\tau(a\cdot a_\alpha - a_\alpha\cdot a + I)\\
&= \lim_{\alpha}\tau\left((a+I)(a_\alpha+I) - (a_\alpha+I)(a+I)\right)
\end{aligned}
$$

$$\begin{aligned} &= \lim_\alpha (\tau\cdot(a+I)-(a+I)\cdot\tau)(a_\alpha+I) \\ &= \lim_\alpha \hat{a}_\alpha + I''(\tau\cdot(a+I)-(a+I)\cdot\tau) \\ &= (F+I'')(\tau\cdot(a+I)-(a+I)\cdot\tau) \\ &= \big((F+I'')\cdot\tau-\tau\cdot(F+I'')\big)(a+I). \end{aligned}$$

Therefore, $D(F+I'')=\delta_\tau(F+I'')$ and $H^1(A''/I'',(A/I)')=0$.

(3) Let $D\in Z^1(A'',A')$ and $i: I\longrightarrow A$ be the inclusion mapping. For $F\in I''$, $x\in I$ we define $D_1=i'\circ D\circ i''$, *i.e.*

$$D_1(F)=D(i''(F))|_I \quad (F\in A'') .$$

We see immediately that $D_1\in Z^1(I'',I')$. Since I'' is (-1)-weakly amenable, then there exists $\lambda_1\in I'$ with $D_1F=F\cdot\lambda_1-\lambda_1\cdot F$, for all $F\in A''$. Now, extend λ_1 to an element in A', say λ_0 and put $D_2=D-\delta_{\lambda_0}$. Then $D_2\in Z^1(A'',A')$ and $D_2|_{I''}=0$. Let $a\in A$ and $F,G\in I''$ with $F=w^*-\lim_\alpha \hat{a}_\alpha$ and $G=w^*-\lim_\beta \hat{b}_\beta$ where $(a_\alpha)_\alpha$, $(b_\beta)_\beta$ are in I, then we have

$$\begin{aligned} D_2(F\cdot G)(a) &= D_2(F)\cdot G(a)+F\cdot D_2(G)(a) \\ &= D_2(i''F)\cdot G(a)+F\cdot D_2(i''G)(a) \\ &= G(a\cdot D_2(i''F))+F(D_2(i''G)\cdot a) \\ &= \lim_\beta \hat{b}_\beta(a\cdot D_2(i''F))+\lim_\alpha \hat{a}_\alpha(D_2(i''G)\cdot a) \\ &= \lim_\beta D_2(i''F)(b_\beta\cdot a)+\lim_\alpha D_2(i''G)(a\cdot a_\alpha) \\ &= \lim_\beta D_1F(b_\beta\cdot a)+\lim_\alpha D_1G(a\cdot a_\alpha) \\ &\quad -\lim_\beta \delta_{\lambda_0}F(b_\beta\cdot a)-\lim_\alpha \delta_{\lambda_0}G(a\cdot a_\alpha)=0. \end{aligned}$$

This means that $D_2|_{I''^2}=0$ and it follows from $\overline{I''^2}=I''$ that $D_2|_{I''}=0$. Now, we define

$$\begin{aligned} &\tilde{D}: A''/I''\longrightarrow I^0 \\ &\tilde{D}(F+I'')(a)=D_2(F)(a) \quad (F\in A'', a\in A). \end{aligned}$$

Note that $\tilde{D}$ is well-defined since $D_2|_{I''}=0$ and $\tilde{D}\in Z^1(A''/I'',I^0)$. On the other hand, A''/I'' is (-1)-weakly amenable then there exists $f_0\in I^0$ such that $\tilde{D}(F+I'')=\delta_{f_0}(F+I'')$. Let $x\in A$ and $G\in I''$ with $G=w^*-\lim_\alpha \hat{b}_\alpha$. Then we have

$$f_0\cdot G(x)=G(x\cdot f_0)=\lim_\alpha (x\cdot f_0)(b_\alpha)=\lim_\alpha f_0(b_\alpha\cdot x)=0.$$

It follows that $f_0\cdot I''=0$ and similarly $I''\cdot f_0=0$, so

$$D_2(F) = \tilde{D}(F+I'')=\delta_{f_0}(F+I'')$$

$$\begin{aligned} &= (F + I'') \cdot f_0 - f_0 \cdot (F + I'') \\ &= F \cdot f_0 - f_0 \cdot F = \delta_{f_0}(F). \end{aligned}$$

This means that $D_2 = D - \delta_{\lambda_0}$, then $D = \delta_{\lambda_0 + f_0}$ and A'' is (-1)-weakly amenable. $\square$

Example 2.1. Let E be a Banach space without approximation property and $A = E\hat{\otimes}E'$ be the nuclear algebra. Then $\mathcal{F}(E) \simeq E\hat{\otimes}E'$, as linear spaces, where $\mathcal{F}(E)$ is the space of continuous finite-rank operators on E. Let $\mathcal{N}(E)$ be the space of nuclear operators with the nuclear norm $\|\cdot\|_v$, then by using 2.5.3 (iii) of [2], the identification of $E\hat{\otimes}E'$ with $\mathcal{F}(E)$ extends to an epimorphism

$$R : E\hat{\otimes}E' \longrightarrow \mathcal{N}(E)$$

with $I = \ker R$, and $I = \{0\}$ if and only if E has approximation property. Moreover, if $\dim I \geq 2$, then I has not trace extension property and by previous theorem A''/I'' is not (-1)-weakly amenable. For more details, definitions and some theorems which are used in this example, see 2.5.3 and 2.5.4 of [2].

Example 2.2. Let $\mathbb{N}^{<w} = \bigcup_{k\in\mathbb{N}} \mathbb{N}^k$ and P be the set of elements $p = (p_1, \ldots, p_k)$ of $\mathbb{N}^{<w}$ such that $k \geq 2$ and $p_1 < p_2 < \cdots < p_k$. For a sequence $\alpha \in \mathbb{C}^{\mathbb{N}}$, define $N(\alpha, p)$ for $p \in P$ by

$$2N(\alpha, p)^2 = \left(\sum_{j=1}^{k-1} |\alpha_{p_{j+1}} - \alpha_{p_j}|^2 \right) + |\alpha_{p_k} - \alpha_{p_1}|^2$$

set $N(\alpha) = \sup_{p\in P} N(\alpha, p)$ so, $N(\alpha) \in [0, +\infty]$. We define

$$\mathcal{J} = \{\alpha \in C_0 : N(\alpha) < \infty\}$$

where $C_0 = \{\alpha \in \mathbb{C}^{\mathbb{N}} : \lim_{n\to\infty} \alpha_n = 0\}$. $\mathcal{J}$ is a commutative closed subalgebra of l^∞, and is called the James algebra. By 4.1.45 of [2], we have some properties for $\mathcal{J}$ such as:

1) $\mathcal{J}$ is an ideal in $\mathcal{J}''$,
2) $\mathcal{J}$ is Arens regular,
3) $\mathcal{J}'' = \mathcal{M}(\mathcal{J})$ (isometrically isomorphic)
   where $\mathcal{M}(\mathcal{J})$ is the set of all multiplicative functionals on $\mathcal{J}$,
4) $\mathcal{J}$ has a bounded approximate identity,
5) $\mathcal{J}$ is weakly amenable,
6) $\mathcal{J}$ is not amenable.

By using Lemma 3.1, $\mathcal{J}'$ is a Banach $\mathcal{J}''$-bimodule. Since $\mathcal{J}$ has a bounded approximate identity, then $\mathcal{J}$ is essential and by using 2.9.54 of [2], we have

$$H^1(\mathcal{J}, \mathcal{J}') = H^1(\mathcal{M}(\mathcal{J}), \mathcal{J}').$$

Now (3), (5) imply that $\mathcal{J}''$ is (-1)-weakly amenable.

THEOREM 2.2. *Let $\frac{1}{2} < \alpha < 1$ and let $\mathbb{T}$ be the unit circle. If $A = lip_\alpha \mathbb{T}$, then $A'' = Lip_\alpha \mathbb{T}$ is not (-1)-weakly amenable.*

Proof. Define $D : Lip_\alpha \mathbb{T} \longrightarrow (lip_\alpha \mathbb{T})'$ by

$$D(F)(h) = \sum_{k=-\infty}^{+\infty} k\hat{h}(k)\hat{F}(-k)$$

for all $F \in Lip_\alpha \mathbb{T}$ and all $h \in lip_\alpha \mathbb{T}$, where $\hat{h}(k)$ and $\hat{F}(k)$ are the Fourier coefficients of h and F at k, that is

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-ik\theta} \mathrm{d}\theta.$$

First, we check that D is well-defined. It follows from 4.5.14 of [2] that there is constant $C_\alpha > 0$ such that for each $F \in Lip_\alpha \mathbb{T}$

$$\left(\sum_{k=-\infty}^{+\infty} |k| \; |\hat{F}(k)|^2 \right)^{\frac{1}{2}} \leq C_\alpha \|F\|_\alpha$$

so, we have

$$\sum_{k=-\infty}^{+\infty} |k\hat{h}(k)\hat{F}(-k)| \leq \left(\sum_{k=-\infty}^{+\infty} |k| \; |\hat{h}(k)|^2 \right)^{\frac{1}{2}} \left(\sum_{k=-\infty}^{+\infty} |k| \; |\hat{F}(-k)|^2 \right)^{\frac{1}{2}} \leq C_\alpha^2 \|h\|_\alpha \|F\|_\alpha.$$

Then, $|DF(h)| \leq C_\alpha^2 \|h\|_\alpha \|F\|_\alpha$ and so, $DF \in (lip_\alpha \mathbb{T})'$ and D is a bounded linear operator. Let $F, G \in Lip_\alpha \mathbb{T}$ and $h \in lip_\alpha \mathbb{T}$, by 4.4.26 (i), (ii) of [2], $lin \{\varepsilon_x : x \in \mathbb{T}\}$ is dense in $(lip_\alpha \mathbb{T})'$, where $\varepsilon_x f = fx$, for each f in $lip_\alpha \mathbb{T}$. So, there are sequences $(x_n) \subseteq \mathbb{T}$ and $(t_n) \subseteq \mathbb{C}$ such that $D(F) = \sum_{n=1}^{\infty} t_n \varepsilon_{x_n}$. Note that $DF(h) = \sum_{n=1}^{\infty} t_n f(x_n)$, for $f \in lip_\alpha \mathbb{T}$. We can extend DF to $Lip_\alpha \mathbb{T}$, say $\tilde{D}F$. (This is valid since the Bernstein inequality in 4.5.14 of [2] is valid for all f in $Lip_\alpha \mathbb{T}$). For each $l \in lip_\alpha \mathbb{T}$, we have

$$h \cdot DF(l) = DF(l \cdot h) = \sum_{n=1}^{\infty} t_n h(x_n) = \sum_{n=1}^{\infty} t_n h(x_n) \varepsilon_{x_n}(l) = \left(\sum_{n=1}^{\infty} t_n h(x_n) \varepsilon_{x_n} \right) (l).$$

The last equation is valid since $\|x_n\| = 1 (n \in \mathbb{N})$ and

$$\sum_{n=1}^{\infty} \|t_n h(x_n)\varepsilon_{x_n}\| \leq \sum_{n=1}^{\infty} \|t_n \varepsilon_{x_n}\| \|h\| \|x_n\| = \|h\| \sum_{n=1}^{\infty} \|t_n \varepsilon_{x_n}\| < \infty.$$

Hence, we have $h \cdot DF = \sum_{n=1}^{\infty} t_n h(x_n)\varepsilon_{x_n}$. On the other hand, $(lip_\alpha \mathbb{T})''$ is isometrically isomorphic to $Lip_\alpha \mathbb{T}$, by $\tau(F)(x) = F(\varepsilon_x)$ for $x \in \mathbb{T}$ and $F \in Lip_\alpha \mathbb{T}$. So, for $G \in Lip_\alpha \mathbb{T}$ there exists a $\Phi_G \in (lip_\alpha \mathbb{T})''$ such that $\Phi_G(\varepsilon_x) = G(x)$, then by the continuity of Φ_G, we have

$$\begin{aligned} DF \cdot G(h) &= \Phi_G\left(\sum_{n=1}^{\infty} t_n h(x_n)\varepsilon_{x_n}\right) \\ &= \sum_{n=1}^{\infty} t_n h(x_n)\Phi_G(\varepsilon_{x_n}) = \sum_{n=1}^{\infty} t_n h(x_n) G(x_n) \\ &= \sum_{n=1}^{\infty} t_n \varepsilon_{x_n}(h \cdot G) = \tilde{D}F(h \cdot G). \end{aligned}$$

Similarly, $F \cdot DG(h) = \tilde{D}G(h \cdot F)$. Let $F, G \in Lip_\alpha \mathbb{T}$ then $\hat{F}, \hat{G} \in L^2(\mathbb{T})$ and $\widehat{F \cdot G} = \hat{F} * \hat{G}$ where $*$ is convolution product with

$$\hat{F} * \hat{G}(k) = \sum_{j=-\infty}^{+\infty} \hat{F}(j)\hat{G}(k-j).$$

Then, we have

$$\begin{aligned} D(F \cdot G)(h) &= \sum_{k=-\infty}^{+\infty} k\hat{h}(k)\widehat{F \cdot G}(-k) = \sum_{k=-\infty}^{+\infty} k\hat{h}(k)\left(\sum_{j=-\infty}^{+\infty} \hat{F}(j)\hat{G}(-k-j)\right) \\ &= \sum_{k=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} k\hat{h}(k)\hat{F}(j)\hat{G}(-k-j) \\ &= \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} k\hat{h}(k)\hat{F}(j)\hat{G}(-k-j). \end{aligned}$$

Moreover,

$$\begin{aligned} \tilde{D}F(h \cdot G) &= \sum_{k=-\infty}^{+\infty} k\hat{F}(-k)\widehat{h \cdot G}(k) = \sum_{k=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} k\hat{F}(-k)\hat{h}(j)\hat{G}(k-j) \\ &= \sum_{k=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} k\hat{F}(-k)\hat{h}(k-j)\hat{G}(j) = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} k\hat{F}(-k)\hat{h}(k-j)\hat{G}(j) \end{aligned}$$

$$= \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} (k+j)\hat{F}(-k-j)\hat{h}(k)\hat{G}(j)$$

and we have

$$\begin{aligned}\tilde{D}(G)(h\cdot F) &= \sum_{j=-\infty}^{+\infty} j\widehat{G}(-j)\widehat{h\cdot F}(j) = \sum_{j=-\infty}^{+\infty} (-j)\hat{G}(j)\widehat{h\cdot F}(-j) \\ &= \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} (-j)\hat{G}(j)\hat{h}(k)\hat{F}(-j-k).\end{aligned}$$

Now, by the definition of $\tilde{D}F$, we have

$$DF\cdot G(h) + F\cdot DG(h) = \tilde{D}F(G\cdot h) + \tilde{D}G(h\cdot F) = D(F\cdot G)(h)$$

it follows that D is a derivation. We have $|DF(h)| \leq C_\alpha^2 \|h\|_\alpha \|F\|_\alpha$, then D is a nonzero bounded derivation in $Z^1(Lip_\alpha\mathbb{T}, (lip_\alpha\mathbb{T})')$. On the other hand, $Lip_\alpha\mathbb{T}$ is commutative then we have $H^1\,(Lip_\alpha\mathbb{T}, (lip_\alpha\mathbb{T})') \neq \{0\}$. It follows that $Lip_\alpha\mathbb{T}$ is not (-1)-weakly amenable. □

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