ON THE FATAL SHOCK MODEL

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In a fatal shock model, the joint distribution of the times-to-failure of the components is not absolutely continuous with respect to the Lebesgue measure. In this paper, we define a specific measure that makes this distribution absolutely continuous. We derive the corresponding density, we show how to use it to evaluate expected values, and we present several examples and particular cases.

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1. INTRODUCTION: THE FATAL SHOCK MODEL

The main objective of system reliability is the construction of a model (life distribution) that describes the times-to-failure of an entire system based on the life distributions of the components from which it is composed. Such a model is an useful tool when choosing the components in order to improve or optimize the overall system reliability, maintainability and/or availability. To accomplish this, the relationship between these components must be studied.

In this paper, we consider an n-component parallel system and let $Y_1, ..., Y_n$ represent the times-to-failure of the components labeled $1, ..., n$ respectively. For calculation purposes, in system reliability, these components are often assumed to be independent, with independent failure times. A more realistic situation is when all the components are exposed to a common stress (shock), and therefore, dependency between these components must be taken into account. In this sense, in the following we consider a special version of the fatal shock model described in [8]. We assume that there exist events (shocks) that are selectively fatal to only one of the components, and events that are simultaneously fatal to all components. Examples of common-cause shocks include extreme environmental conditions, computer viruses, sabotages, power outages, design weaknesses, human errors, etc. For example, in insurance, such shocks might be natural catastrophes; in credit risk modelling, they might be a variety of economic events (*e.g.*, global recession); in operational risk modelling, they might be the failure of various IT systems; in medicine or biology, they could be due to different kinds of trauma, etc. For other types of shock models studied in the literature, see, *e.g.*, [4–6, 10, 11], etc. To mathematically model the above described system, we denote by X_i the moment of the first selective event that affects component $i, i = 1, ..., n$, and by Z the moment of the first event simultaneously affecting all components. Then we obviously have $Y_i = \min(Z, X_i), i = 1, ..., n$. The usual assumption is that $Z, X_1, ..., X_n$ are independent absolutely continuous random variables, often identically distributed.

The purpose of this work is to describe the distribution of the random vector $\mathbf{Y} = (Y_1, ..., Y_n)$ in order to be able to compute, e.g., expected values of the type $\mathbb{E}\varphi(\mathbf{Y})$, for $\varphi : \mathbb{R}^n \to \mathbb{R}$ a measurable bounded function. Unfortunately, this distribution is not absolutely continuous with respect to the n-dimensional Lebesgue measure λ^n. However, it is possible to make it absolutely continuous with respect to a properly defined measure. After introducing some notation in Section 2, in Section 3 we obtain such a measure and the corresponding density. The density can also be used for parameter estimation, enabling the derivation of a likelihood function, and hence, the application of the maximum likelihood estimation method. In Section 4, we give an equivalent form of this density compatible with the existing literature, together with several examples and particular cases.

2. NOTATION

The following notation will be needed: let $n \geq 2$ be fixed, $I = \{1, 2, ..., n\}$, and let $J \subseteq I$ be a subset with $|J| = k$ elements, $0 \leq k \leq n$; here $|J|$ denotes the cardinality of J. Let $J^c = I \backslash J$ be the complement of J; we emphasize its ordered elements as $J^c = \{j_1 < ... < j_{n-k}\}$.

We denote a line vector by a bold-face letter and its elements by the corresponding italic with a subscript denoting the number of the element; the dimension of this vector will result from the context. The elements of the canonical basis in $\mathbb{R}^n$ will be denoted by $\boldsymbol{e}_i, i = 1, ..., n$. For a non-empty subset $J \subseteq I$ with $|J| = k$, we denote by $\chi_J : \mathbb{R}^{n-k+1} \to \mathbb{R}^n$ the mapping

$$\chi_J(x_0, x_1, ..., x_{n-k}) = \sum_{i=1}^{n-k} x_i \boldsymbol{e}_{j_i} + x_0 \sum_{j \in J} \boldsymbol{e}_j .$$

By $\chi_\varnothing$ we denote the identity function on $\mathbb{R}^n$. For example, for $n = 4$, we have

$$\begin{aligned}
\chi_{\{1\}}(x_0, x_1, x_2, x_3) &= (x_0, x_1, x_2, x_3), \ \chi_{\{2\}}(x_0, x_1, x_2, x_3) = (x_1, x_0, x_2, x_3), \\
\chi_{\{3\}}(x_0, x_1, x_2, x_3) &= (x_1, x_2, x_0, x_3), \ \chi_{\{4\}}(x_0, x_1, x_2, x_3) = (x_1, x_2, x_3, x_0), \\
\chi_{\{1,2\}}(x_0, x_1, x_2) &= (x_0, x_0, x_1, x_2), \ \chi_{\{1,3\}}(x_0, x_1, x_2) = (x_0, x_1, x_0, x_2),
\end{aligned}$$

$$\begin{aligned}
\chi_{\{1,4\}}(x_0,x_1,x_2) &= (x_0,x_1,x_2,x_0)\,, \ \chi_{\{2,3\}}(x_0,x_1,x_2) = (x_1,x_0,x_0,x_2)\,,\\
\chi_{\{2,4\}}(x_0,x_1,x_2) &= (x_1,x_0,x_2,x_0)\,, \ \chi_{\{3,4\}}(x_0,x_1,x_2) = (x_1,x_2,x_0,x_0)\,,\\
\chi_{\{1,2,3\}}(x_0,x_1) &= (x_0,x_0,x_0,x_1)\,, \ \chi_{\{1,2,4\}}(x_0,x_1) = (x_0,x_0,x_1,x_0)\,,\\
\chi_{\{1,3,4\}}(x_0,x_1) &= (x_0,x_1,x_0,x_0)\,, \ \chi_{\{2,3,4\}}(x_0,x_1) = (x_1,x_0,x_0,x_0)\,,\\
\chi_{\{1,2,3,4\}}(x_0) &= (x_0,x_0,x_0,x_0)\,.
\end{aligned}$$

In other words, for $k \geq 2$, the injection χ_J associates a vector with n components to a vector with $n-k+1$ components, making all the components from J equal to x_0; for singletons $J = \{j\}$, χ_J is a permutation, while $\chi_\varnothing$ is the identity.

We also let $A(J) = \{X_j < Z, \text{ for all } j \in J^c, \text{ and } X_j \geq Z, \text{ for all } j \in J\}$, while $\mathbf{Y}(J)$ is the vector $\mathbf{Y}$ on $A(J)$. This means that $(\mathbf{Y}(J))_j = \begin{cases} X_j, & j \in J^c \\ Z, & j \in J \end{cases}$ and $\mathbf{Y}(\varnothing) = \mathbf{X}$. Note that we have $\mathbf{Y}(J) = \chi_J\left(Z, X_{j_1}, ..., X_{j_{n-k}}\right)$.

A density function will be denoted by a lower-case letter, the cumulative distribution function (cdf) by the corresponding capital, and its right tail by a bar on that capital. In this sense, the density of X_i is f_i, the density of Z is f, F_i, F are their cdfs, while $\bar{F}_i, \bar{F}$ are the corresponding right tails (e.g., $\bar{F}(x) = P(Z > x)$).

For a more compact writing, we will use the operators $\wedge$ and $\vee$ to denote the minimum and, respectively, maximum values of a set of elements; e.g., we can rewrite $Y_i = \min(Z, X_i) = Z \wedge X_i$. Moreover, let $x_+ = \max(x, 0)$ and let 1_A denote the indicator function of the subset A, that is, $1_A(u)$ is equal to one if $u \in A$, and zero otherwise.

3. MAIN RESULT

We start by computing $\mathbb{E}\varphi(\mathbf{Y})$, which can be done by breaking the corresponding integral into 2^n smaller ones. More precisely, we use the decomposition

$$\varphi(\mathbf{Y}) = \sum_{J \subseteq I} \varphi(\mathbf{Y}(J))\, 1_{A(J)},$$

from where

$$\mathbb{E}\varphi(\mathbf{Y}) = \sum_{J \subseteq I} \mathbb{E}\left[\varphi(\mathbf{Y}(J))\, 1_{A(J)}\right].$$

Moreover, for a non-empty subset $J \subseteq I$ with $|J| = k$, it is easy to see that denoting $d\mathbf{x} = dx_1...dx_n$, we have

$$\mathbb{E}\left[\varphi\left(\mathbf{Y}\left(J\right)\right)1_{A(J)}\right]=\int_{\mathbb{R}^{n+1}}\varphi\left(\chi_J\left(z,x_{j_1},...,x_{j_{n-k}}\right)\right)f\left(z\right)\left(\prod_{i=1}^{n}f_i\left(x_i\right)\right)$$
$$\times\left(\prod_{j\in J^c}1_{(-\infty,z)}\left(x_j\right)\right)\left(\prod_{j\in J}1_{[z,\infty)}\left(x_j\right)\right)\mathrm{d}\mathbf{x}\mathrm{d}z$$
$$=\int_{\mathbb{R}^{n-k+1}}\varphi\left(\chi_J\left(z,\left(x_j\right)_{j\in J^c}\right)\right)f\left(z\right)\left(\prod_{j\in J^c}f_j\left(x_j\right)1_{(-\infty,z)}\left(x_j\right)\right)$$
$$\times\left(\prod_{j\in J}\bar{F}_j\left(z\right)\right)\left(\prod_{j\in J^c}\mathrm{d}x_j\right)\mathrm{d}z. \tag{3.1}$$

Note that it was possible to reduce the order of the multiple integral from $n+1$ to $n-k+1$ because the integral does not contain the components x_j for $j\in J$, these components being replaced with z. Let us now relabel the $n-k$ components of $\mathbf{x}$ belonging to J^c, namely x_{j_i}, by z_i, and write z_0 instead of z. With this new labeling and with $\mathbf{z}=\left(z_0,z_1,...,z_{n-k}\right)$, formula (3.1) becomes

$$\mathbb{E}\left[\varphi\left(\mathbf{Y}\left(J\right)\right)1_{A(J)}\right] \tag{3.2}$$
$$=\int_{\mathbb{R}^{n-k+1}}\varphi\left(\chi_J\left(\mathbf{z}\right)\right)f\left(z_0\right)\left(\prod_{i=1}^{n-k}f_{j_i}\left(z_i\right)1_{(-\infty,z_0)}\left(z_i\right)\right)\left(\prod_{j\in J}\bar{F}_j\left(z_0\right)\right)\mathrm{d}\mathbf{z}.$$

For example, if $n=4$ and $J=\{1,3\}$, we obtain

$$\mathbb{E}\left[\varphi\left(\mathbf{Y}\left(\{1,3\}\right)\right)1_{A(\{1,3\})}\right]$$
$$=\int_{\mathbb{R}^3}\varphi\left(z,x_2,z,x_4\right)f\left(z\right)f_2\left(x_2\right)1_{(-\infty,z)}\left(x_2\right)f_4\left(x_4\right)1_{(-\infty,z)}\left(x_4\right)\bar{F}_1\left(z\right)$$
$$\times\bar{F}_3\left(z\right)\mathrm{d}x_2\mathrm{d}x_4\mathrm{d}z$$
$$=\int_{\mathbb{R}^3}\varphi\left(\chi_{\{1,3\}}\left(z_0,z_1,z_2\right)\right)f\left(z_0\right)f_2\left(z_1\right)f_4\left(z_2\right)\bar{F}_1\left(z_0\right)\bar{F}_3\left(z_0\right)$$
$$\times1_{\{z_0>z_1\vee z_2\}}\left(z_0,z_1,z_2\right)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_0.$$

This is the integral of $\varphi\left(\chi_{\{1,3\}}\left(\cdot\right)\right)$ with respect to a measure which has the density

$$\rho_{\{1,3\}}\left(z_0,z_1,z_2\right)=f\left(z_0\right)f_2\left(z_1\right)f_4\left(z_2\right)\bar{F}_1\left(z_0\right)\bar{F}_3\left(z_0\right)1_{\{z_0>z_1\vee z_2\}}\left(z_0,z_1,z_2\right).$$

Similarly, we can rewrite (3.2) as

$$\mathbb{E}\left[\varphi\left(\mathbf{Y}\left(J\right)\right)1_{A(J)}\right]=\int_{\mathbb{R}^{n-k+1}}\varphi\left(\chi_J\left(\mathbf{z}\right)\right)\rho_J\left(\mathbf{z}\right)\mathrm{d}\lambda^{n-k+1}\left(\mathbf{z}\right),$$

where the general formula for the density ρ_J, with J a non-empty set, results as

$$\begin{aligned}\rho_J(\mathbf{z}) &= f(z_0)\left(\prod_{i=1}^{n-k} f_{j_i}(z_i)\,1_{(-\infty,z_0)}(z_i)\right)\left(\prod_{j\in J}\bar{F}_j(z_0)\right)\\ &= f(z_0)\left(\prod_{i=1}^{n-k} f_{j_i}(z_i)\right)\left(\prod_{j\in J}\bar{F}_j(z_0)\right)1_{\left\{z_0>\bigvee_{i=1}^{n-k} z_i\right\}}(\mathbf{z}).\end{aligned} \tag{3.3}$$

Finally, for the simplest case $J=\varnothing$ it is easy to see that

$$\rho_\varnothing(z_1,z_2,...,z_n)=\left(\prod_{i=1}^{n} f_i(z_i)\right)\bar{F}\left(\bigvee_{i=1}^{n} z_i\right). \tag{3.4}$$

To conclude, we have the following result.

PROPOSITION 1. *With the above notation, the distribution of* $\mathbf{Y}$ *can be written as*

$$P\circ\mathbf{Y}^{-1}=\sum_{J\subseteq\{1,...,n\},J\neq\varnothing}\left(\rho_J\cdot\lambda^{n-|J|+1}\right)\circ\chi_J^{-1}+\rho_\varnothing\cdot\lambda^n. \tag{3.5}$$

Moreover, we have

$$\mathbb{E}\varphi(\mathbf{Y})=\sum_{J\subseteq\{1,...,n\},J\neq\varnothing}\int_{\mathbb{R}^{n-|J|+1}}(\varphi\circ\chi_J)\,\rho_J\mathrm{d}\lambda^{n-|J|+1}+\int_{\mathbb{R}^n}\varphi\rho_\varnothing\mathrm{d}\lambda^n,$$

where ρ_J *is defined by* (3.3) *and* (3.4).

4. SOME PARTICULAR CASES

4.1. FORMULA (3.5) FOR SMALL n

To better understand these densities, we consider formula (3.5) for $n=2$. This gives

$$P\circ\mathbf{Y}^{-1}=\rho_\varnothing\cdot\lambda^2+\left(\rho_{\{1\}}\cdot\lambda^2\right)\circ\chi_{\{1\}}^{-1}+\left(\rho_{\{2\}}\cdot\lambda^2\right)\circ\chi_{\{2\}}^{-1}+\left(\rho_{\{1,2\}}\cdot\lambda\right)\circ\chi_{\{1,2\}}^{-1},$$

where

$$\begin{aligned}\rho_\varnothing(x,y) &= f_1(x)\,f_2(y)\,\bar{F}(x\vee y), \quad \rho_{\{1\}}(x,y)=f(x)\,f_2(y)\,\bar{F}_1(x)\,1_{\{x>y\}},\\ \rho_{\{2\}}(x,y) &= f(x)\,f_1(y)\,\bar{F}_2(x)\,1_{\{x>y\}},\ \rho_{\{1,2\}}(x)=f(x)\,\bar{F}_1(x)\,\bar{F}_2(x).\end{aligned}$$

Further on, for $n=3$, (3.5) yields

$$\begin{aligned}P\circ\mathbf{Y}^{-1} &= \rho_\varnothing\cdot\lambda^3+\left(\rho_{\{1\}}\cdot\lambda^3\right)\circ\chi_{\{1\}}^{-1}+\left(\rho_{\{2\}}\cdot\lambda^3\right)\circ\chi_{\{2\}}^{-1}\\ &+\left(\rho_{\{3\}}\cdot\lambda^3\right)\circ\chi_{\{3\}}^{-1}+\left(\rho_{\{1,2\}}\cdot\lambda^2\right)\circ\chi_{\{1,2\}}^{-1}+\left(\rho_{\{1,3\}}\cdot\lambda^2\right)\circ\chi_{\{1,3\}}^{-1}\\ &+\left(\rho_{\{2,3\}}\cdot\lambda^2\right)\circ\chi_{\{2,3\}}^{-1}+\left(\rho_{\{1,2,3\}}\cdot\lambda\right)\circ\chi_{\{1,2,3\}}^{-1},\end{aligned}$$

with

$$\begin{aligned}\rho_{\varnothing}(x,y,z) &= f_1(x)f_2(y)f_3(z)\bar{F}(x\vee y\vee z),\\ \rho_{\{1\}}(x,y,z) &= f(x)f_2(y)f_3(z)\bar{F}_1(x)1_{\{x>y\vee z\}},\\ \rho_{\{2\}}(x,y,z) &= f(x)f_1(y)f_3(z)\bar{F}_2(x)1_{\{x>y\vee z\}},\\ \rho_{\{3\}}(x,y,z) &= f(x)f_1(y)f_2(z)\bar{F}_3(x)1_{\{x>y\vee z\}},\\ \rho_{\{1,2\}}(x,y) &= f(x)f_3(y)\bar{F}_1(x)\bar{F}_2(x)1_{\{x>y\}},\\ \rho_{\{1,3\}}(x,y) &= f(x)f_2(y)\bar{F}_1(x)\bar{F}_3(x)1_{\{x>y\}},\\ \rho_{\{2,3\}}(x,y) &= f(x)f_1(y)\bar{F}_2(x)\bar{F}_3(x)1_{\{x>y\}},\\ \rho_{\{1,2,3\}}(x) &= f(x)\bar{F}_1(x)\bar{F}_2(x)\bar{F}_3(x).\end{aligned}$$

Let us now have a look at the particular case with uniform marginal distributions. Let X_i follow an uniform distribution $U[0,a_i], a_i>0, i=1,...,n$, and Z an uniform distribution $U[0,a_0], a_0>0$. Then density (3.3) becomes

$$\rho_J(\mathbf{z}) = \frac{1}{a_0}1_{[0,a_0]}(z_0)\left(\prod_{i=1}^{n-k}\frac{1}{a_{j_i}}1_{[0,a_{j_i}]}(z_i)\right)\left(\prod_{j\in J}\frac{(a_j-z_0)_+}{a_j}\right)1_{\left\{z_0>\bigvee_{i=1}^{n-k}z_i\right\}}(\mathbf{z}),$$

while (3.4) yields

$$\rho_{\varnothing}(z_1,z_2,...,z_n) = \left(\prod_{i=1}^{n}\frac{1}{a_i}1_{[0,a_i]}(z_i)\right)\frac{\left(a_0-\bigvee_{i=1}^{n}z_i\right)_+}{a_0}.$$

Let us detail these formulas for $n=4$. First, we have

$$\rho_{\varnothing}(z_1,z_2,z_3,z_4) = \left(\prod_{i=0}^{4}a_i^{-1}\right)\left(a_0-\bigvee_{i=1}^{4}z_i\right)_+,\ 0\le z_i\le a_i, i=1,..,4.$$

Secondly, for all sets J with $|J|=1$, we have that for $\bigvee_{i=1}^{3}z_i<z_0\le a_0$,

$$\rho_{\{1\}}(z_0,z_1,z_2,z_3) = \left(\prod_{i=0}^{4}a_i^{-1}\right)(a_1-z_0)_+, 0\le z_i\le a_{i+1}, i=1,2,3;$$

$$\rho_{\{2\}}(z_0,z_1,z_2,z_3) = \left(\prod_{i=0}^{4}a_i^{-1}\right)(a_2-z_0)_+, 0\le z_1\le a_1, 0\le z_i\le a_{i+1}, i=2,3;$$

$$\rho_{\{3\}}(z_0,z_1,z_2,z_3) = \left(\prod_{i=0}^{4}a_i^{-1}\right)(a_3-z_0)_+, 0\le z_i\le a_i, i=1,2, 0\le z_3\le a_4;$$

$$\rho_{\{4\}}(z_0,z_1,z_2,z_3) = \left(\prod_{i=0}^{4}a_i^{-1}\right)(a_4-z_0)_+, 0\le z_i\le a_i, i=1,2,3.$$

Thirdly, when $|J|=2$, we have that for $(z_1\vee z_2)<z_0\le a_0$,

$$\rho_{\{1,2\}}(z_0,z_1,z_2)=\left(\prod_{i=0}^{4}a_i^{-1}\right)(a_1-z_0)_+(a_2-z_0)_+,0\le z_1\le a_3,0\le z_2\le a_4;$$

$$\rho_{\{1,3\}}(z_0,z_1,z_2)=\left(\prod_{i=0}^{4}a_i^{-1}\right)(a_1-z_0)_+(a_3-z_0)_+,0\le z_1\le a_2,0\le z_2\le a_4;$$

$$\rho_{\{1,4\}}(z_0,z_1,z_2)=\left(\prod_{i=0}^{4}a_i^{-1}\right)(a_1-z_0)_+(a_4-z_0)_+,0\le z_1\le a_2,0\le z_2\le a_3;$$

$$\rho_{\{2,3\}}(z_0,z_1,z_2)=\left(\prod_{i=0}^{4}a_i^{-1}\right)(a_2-z_0)_+(a_3-z_0)_+,0\le z_1\le a_1,0\le z_2\le a_4;$$

$$\rho_{\{2,4\}}(z_0,z_1,z_2)=\left(\prod_{i=0}^{4}a_i^{-1}\right)(a_2-z_0)_+(a_4-z_0)_+,0\le z_1\le a_1,0\le z_2\le a_3;$$

$$\rho_{\{3,4\}}(z_0,z_1,z_2)=\left(\prod_{i=0}^{4}a_i^{-1}\right)(a_3-z_0)_+(a_4-z_0)_+,0\le z_1\le a_1,0\le z_2\le a_2.$$

When $|J|=3$, we have that for $z_1<z_0\le a_0$,

$$\rho_{\{1,2,3\}}(z_0,z_1)=\left(\prod_{i=0}^{4}a_i^{-1}\right)(a_1-z_0)_+(a_2-z_0)_+(a_3-z_0)_+,0\le z_1\le a_4;$$

$$\rho_{\{1,2,4\}}(z_0,z_1)=\left(\prod_{i=0}^{4}a_i^{-1}\right)(a_1-z_0)_+(a_2-z_0)_+(a_4-z_0)_+,0\le z_1\le a_3;$$

$$\rho_{\{1,3,4\}}(z_0,z_1)=\left(\prod_{i=0}^{4}a_i^{-1}\right)(a_1-z_0)_+(a_3-z_0)_+(a_4-z_0)_+,0\le z_1\le a_2;$$

$$\rho_{\{2,3,4\}}(z_0,z_1)=\left(\prod_{i=0}^{4}a_i^{-1}\right)(a_2-z_0)_+(a_3-z_0)_+(a_4-z_0)_+,0\le z_1\le a_1.$$

And finally, $\rho_{\{1,2,3,4\}}(z_0)=\dfrac{1}{a_0}\displaystyle\prod_{i=1}^{4}\frac{(a_i-z_0)_+}{a_i},0\le z_0\le a_0.$

4.2. A UNIFIED DENSITY

Note that for a subset $J\subseteq I$ with $k=|J|>0$, since χ_J is an injection, we can also write

$$\mathbb{E}\left[\varphi(\mathbf{Y}(J))\,1_{A(J)}\right]=\int_{\mathbb{R}^{n-k+1}}\varphi(\chi_J(\mathbf{z}))\,\rho_J(\mathbf{z})\,\mathrm{d}\lambda^{n-k+1}(\mathbf{z})$$

$$= \int_{\mathbb{R}^{n-k+1}} \varphi\left(\chi_J(\mathbf{z})\right)\left(\rho_J \circ \chi_J^{-1}\right)\left(\chi_J(\mathbf{z})\right) \mathrm{d}\lambda^{n-k+1}(\mathbf{z})$$

$$= \int_{\chi_J\left(\mathbb{R}^{n-k+1}\right)} \varphi(\mathbf{x})\left(\rho_J \circ \chi_J^{-1}\right)(\mathbf{x}) \,\mathrm{d}\left(\lambda^{n-k+1} \circ \chi_J^{-1}\right)(\mathbf{x}),$$

from where, we rewrite (3.5) as

$$P \circ \mathbf{Y}^{-1} = \sum_{J \subseteq \{1,\dots,n\}, J \neq \varnothing} \left(\rho_J \circ \chi_J^{-1}\right) \cdot \left(\lambda^{n-|J|+1} \circ \chi_J^{-1}\right) + \rho_\varnothing \cdot \lambda^n.$$

Therefore, for an event C in the Borel σ-algebra in $\mathbb{R}^n$, we define the following measure

$$\nu(C) = \lambda^n(C) + \sum_{J \subseteq I,\ |J| \geq 2} \left(\lambda^{n-|J|+1} \circ \chi_J^{-1}\right)(C). \tag{4.1}$$

Note that $\lambda^n \ll \nu$. The distribution of $\mathbf{Y}$ is absolutely continuous with respect to ν, with the density given for a subset $J \subseteq I, |J| \geq 2$, by

$$r_J(\mathbf{x}) = \left(\rho_J \circ \chi_J^{-1}\right)(\mathbf{x}),\ \mathbf{x} \in \chi_J\left(\mathbb{R}^{n-|J|+1}\right).$$

Using now (3.3), we obtain that, for $\mathbf{x} \in \chi_J\left(\mathbb{R}^{n-|J|+1}\right)$ and $z = x_j, j \in J$,

$$r_J(\mathbf{x}) = \begin{cases} f(z)\left(\prod_{i=1}^{n-k} f_{j_i}\left(x_{j_i}\right)\right)\left(\prod_{j \in J} \bar{F}_j(z)\right), & z > \bigvee_{i=1}^{n-k} x_{j_i} \\ 0, & \textit{otherwise} \end{cases}. \tag{4.2}$$

We separately consider the cases $|J| = 1$ and $J = \varnothing$, which can be unified since both densities ρ_J and $\rho_\varnothing$ are with respect to the same measure λ^n. Let $J = \{j\}$ and $\mathbf{x} \in \mathbb{R}^n$; then, reasoning as for (4.2), if $x_j > x_i, \forall i \neq j$, (3.3) yields

$$r_{\{j\}}(\mathbf{x}) = f\left(x_j\right)\left(\prod_{i \neq j} f_i\left(x_i\right)\right) \bar{F}_j\left(x_j\right),$$

while, for the same $\mathbf{x}$, (3.4) gives

$$r_\varnothing(\mathbf{x}) = \rho_\varnothing(\mathbf{x}) = \left(\prod_{i=1}^n f_i\left(x_i\right)\right) \bar{F}\left(x_j\right) = f_j\left(x_j\right) \bar{F}\left(x_j\right)\left(\prod_{i \neq j} f_i\left(x_i\right)\right).$$

The last two relations can be easily unified into

$$r(\mathbf{x}) = \left(f\left(x_j\right) \bar{F}_j\left(x_j\right) + f_j\left(x_j\right) \bar{F}\left(x_j\right)\right)\left(\prod_{i \neq j} f_i\left(x_i\right)\right),\ x_j > \bigvee_{i \neq j} x_i. \tag{4.3}$$

Now, looking at (4.2), we notice that $r_J(\mathbf{x}) = 0$, for any $\mathbf{x} \in \chi_J\left(\mathbb{R}^{n-|J|+1}\right)$ that contains a component $x_l, l \in J^c$, such that $x_l \geq x_j, j \in J$. For such an $\mathbf{x}$, there exists another set J' for which $r_{J'}(\mathbf{x})$ is not necessarily 0. Therefore, for compatibility with the existing literature, we unify all densities r_J into

$$r(\mathbf{x}) = \begin{cases} f(z)\left(\prod_{j\in J^c} f_j(x_j)\right)\left(\prod_{j\in J} \bar{F}_j(z)\right), \text{ where } z = \bigvee_{i=1}^{n} x_i, \\ J = \{j \in I \,|x_j = z\} \text{ and } |J| \geq 2 \\ \left(f_j(x_j)\bar{F}(x_j) + f(x_j)\bar{F}_j(x_j)\right)\left(\prod_{i\neq j} f_i(x_i)\right), \text{ if } x_j > x_i, \\ \forall i \in I\backslash\{j\} \text{ (i.e., the maximum is unique and equal to } x_j) \end{cases}. \tag{4.4}$$

4.2.1. **First particular case: multivariate exponential distribution**

If in particular X_i follows an exponential distribution $Exp(\alpha_i), \alpha_i > 0, i = 1, ..., n$, and Z an exponential distribution $Exp(\alpha_0), \alpha_0 > 0$, then $\mathbf{Y}$ will follow the multivariate exponential distribution introduced by Marshall and Olkin [8].

We recall that the density of the exponential distribution $Exp(\alpha)$ is $g(x) = \alpha e^{-\alpha x}$, while its tail function is $\bar{G}(x) = e^{-\alpha x}, x > 0$. Then density (4.4) becomes

$$r(\mathbf{x}) = \begin{cases} \alpha_0\left(\prod_{j\in J^c} \alpha_j\right)\exp\left(-\sum_{i=1}^{n} \alpha_i x_i - \alpha_0 z\right), \; z = \bigvee_{i=1}^{n} x_i, \\ J = \{j \in I \,|x_j = z\} \text{ and } |J| \geq 2 \\ (\alpha_0 + \alpha_j)\left(\prod_{i\neq j} \alpha_i\right)\exp\left(-\sum_{i=1}^{n} \alpha_i x_i - \alpha_0 x_j\right), \; x_j > x_i, \\ \forall i \in I\backslash\{j\} \end{cases}. \tag{4.5}$$

This is the density obtained by Proschan and Sullo [9] and used to estimate the parameters $(\alpha_i)_{i=0,1,\ldots,n}$ by the maximum likelihood method. Unfortunately, the formula of the corresponding measure μ from page 467, Section 3 in [9], makes no sense.

4.2.2. **Second particular case: a multivariate Pareto distribution**

We recall that X is said to follow a Pareto of the second kind distribution, denoted $X \backsim Pa(II)(\mu, \sigma, \alpha)$, if it has the right tail function $\bar{G}(x) = \left(1 + \frac{x-\mu}{\sigma}\right)^{-\alpha}$ and the density $g(x) = \frac{\alpha}{\sigma}\left(1 + \frac{x-\mu}{\sigma}\right)^{-\alpha-1}$, with $x > \mu \in \mathbb{R}$, $\sigma > 0$ and $\alpha > 0$ (see, *e.g.*, [1]).

Based on the fatal shock model, Asimit et al. [2] introduced a new multivariate Pareto distribution by taking $Z \backsim Pa(II)(0,1,\alpha_0)$ and $X_i \backsim Pa(II)(\mu_i,\sigma_i,\alpha_i), i=1,\ldots,n$. However, the construction of this new multivariate Pareto distribution is somehow different, in the sense that the corresponding random vector $\mathbf{Y}$ is defined as

$$Y_i = \min\left(\sigma_i Z + \mu_i, X_i\right),\ i = 1,\ldots,n.$$

Nevertheless, to find a density for $\mathbf{Y}$ and the corresponding measure, we can apply a similar reasoning as for Proposition 1. The main difference consists in the definition of the function χ_J, which in this case must be defined for an $J \subseteq I, |J| = k > 0$, by

$$\chi_J\left(x_0, x_1, ..., x_{n-k}\right) = \sum_{i=1}^{n-k} x_i \boldsymbol{e}_{j_i} + \sum_{j\in J}\left(\sigma_j x_0 + \mu_j\right)\boldsymbol{e}_j.$$

Also,

$$A\left(J\right) = \left\{X_j < \sigma_j Z + \mu_j,\ \text{for all } j\in J^c,\ \text{and } X_j \geq \sigma_j Z + \mu_j,\ \text{for all } j \in J\right\},$$

etc. The measure ν has the same form (4.1) and, for $|J| \geq 2$, $\mathbf{x} \in \chi_J\left(\mathbb{R}^{n-|J|+1}\right)$ and $z = \dfrac{x_j - \mu_j}{\sigma_j}, j \in J$, we have

$$r_J\left(\mathbf{x}\right) = \begin{cases} f\left(z\right)\left(\prod_{i=1}^{n-k} f_{j_i}\left(x_{j_i}\right)\right)\left(\prod_{j\in J}\bar{F}_j\left(\sigma_j z + \mu_j\right)\right),\ z > \bigvee_{i=1}^{n-k}\dfrac{x_{j_i}-\mu_{j_i}}{\sigma_{j_i}} \\ 0,\ \textit{otherwise} \end{cases}.$$

The easiest way to obtain the density corresponding to λ^n (*i.e.*, for the cases $|J| = 1$ and $J = \varnothing$) is by differentiating the cdf of $\mathbf{Y}$ with respect to $x_1, ..., x_n$, as done in [3]. The unified density of $\mathbf{Y}$ results for an $\mathbf{x}$ with $x_i > \mu_i, i = 1, ..., n$, as

$$r(\mathbf{x}) = \begin{cases} \alpha_0\left(1+z\right)^{-\alpha_0-\sum_{j\in J}\alpha_j-1}\left(\prod_{j\in J^c}\dfrac{\alpha_j}{\sigma_j}\left(1+\dfrac{x_j-\mu_j}{\sigma_j}\right)^{-\alpha_j-1}\right), \\ \qquad z = \bigvee_{i=1}^{n}\dfrac{x_i-\mu_i}{\sigma_i}, J = \left\{j\in I \,\middle|\, \dfrac{x_j-\mu_j}{\sigma_j} = z\right\} \text{ and } |J|\geq 2 \\ \dfrac{\alpha_0+\alpha_j}{\sigma_j}\left(1+\dfrac{x_j-\mu_j}{\sigma_j}\right)^{-\alpha_0-\alpha_j-1}\left(\prod_{i\neq j}\dfrac{\alpha_i}{\sigma_i}\left(1+\dfrac{x_i-\mu_i}{\sigma_i}\right)^{-\alpha_i-1}\right), \\ \qquad \dfrac{x_j-\mu_j}{\sigma_j} > \dfrac{x_i-\mu_i}{\sigma_i}, \forall i\neq j \end{cases}.$$

This density was also obtain in a different way by Asimit *et al.* [3]. Moreover, for $\mu_i = \sigma_i \equiv 1, \forall i = 1, ..., n$, it simplifies to the density presented by Hanagal [7].

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