AN EXTENSION
OF CHEBYSHEV'S ALGEBRAIC INEQUALITY

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We extend Chebyshev's algebraic inequality to the framework of non-monotonic functions.

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Chebyshev's algebraic inequality, also known as the covariance inequality, is an important tool in economics, finance, and decision making. The *covariance* of two functions $f, g \in L^2([a,b])$ is defined by

$$cov(f,g) = \mathcal{E}\left((f-\mathcal{E}(f))(g-\mathcal{E}(g))\right) = \mathcal{E}\left(fg\right) - \mathcal{E}(f)\mathcal{E}(g)),$$

where $\mathcal{E}(f) = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$ represents the *expectation* of f. Chebyshev's inequality states that $cov(f,g) \geq 0$ if f and g are monotonic in the same direction and $cov(f,g) \leq 0$ if f and g are monotonic of opposite direction.

Most of the existing literature concerning this inequality is reviewed in the excellent monograph of Mitrinović, Pečarić and Fink [2]. Recently, Niculescu and Pečarić [3] have shown the (logical) equivalence of Chebyshev's algebraic inequality with another classical result, Jensen's inequality.

As was noticed by K.A. Andréief in 1883 (see [2], p. 243), Chebyshev's inequality is a direct consequence of the identity

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x = \left(\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right)\left(\frac{1}{b-a}\int_a^b g(x)\mathrm{d}x\right) +$$
$$+\frac{1}{2\left(b-a\right)^2}\int_a^b\int_a^b \left(f(x)-f(y)\right)\left(g(x)-g(y)\right)\mathrm{d}x\mathrm{d}y.$$

This also shows that monotonicity in the same direction of the given pair of functions can be weakened by assuming only their *synchronicity*, that is,

$$(f(x)-f(y))\,(g(x)-g(y)) \geq 0 \quad \text{for a.e. } x, y \in [a,b].$$

Here, as usually, a.e. stands for almost everywhere.

The aim of the present paper is to show that Chebyshev's inequality still works when synchronicity of the given pair of functions is replaced by the synchronicity of the functions corrected by their averages.

Given a real-valued function h belonging to the Lebesgue space $L^1([a,b])$, its *average*, $\mathcal{M}(h)$, is defined by the formula

$$\mathcal{M}(h)(x) = \frac{1}{x-a}\int_a^x h(t)\mathrm{d}t, \text{ for } x \in (a,b].$$

Clearly, if h is monotone, then $\mathcal{M}(h)$ is also monotone (in the same direction). However, if h is not monotone, then the intervals of monotonicity of h and its average could be different. For example, the function $x+\sqrt{2}\sin x$ is increasing on the interval $[0,\frac{3\pi}{4}]$ and decreasing on the interval $[\frac{3\pi}{4},\frac{5\pi}{4}]$, while its average, $\frac{x}{2}+\frac{\sqrt{2}(1-\cos x)}{x}$, is increasing on $(0,\infty)$. The same phenomenon occurs for the family of oscillating functions $x^{\alpha}(k+\sin x)$, for $\alpha > 0$ and $k \geq \frac{\alpha+2}{\alpha}$.

Our extension of Chebyshev's inequality is as follows:

THEOREM 1. *Suppose that f and g are two real-valued functions belonging to $L^{\infty}([a,b])$. If*

$$(S) \qquad \left(f(x)-\frac{1}{x-a}\int_a^x f(t)\mathrm{d}t\right)\left(g(x)-\frac{1}{x-a}\int_a^x g(t)\mathrm{d}t\right) \geq 0,$$

for a.e. $x \in [a,b]$, then

$$(C) \qquad \frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \geq \left(\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right)\left(\frac{1}{b-a}\int_a^b g(x)\mathrm{d}x\right).$$

The condition (S) is fulfilled when the averages of f and g are both nondecreasing or both nonincreasing a.e. on $(a,b]$. Indeed, according to the Lebesgue differentiation theorem,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x-a}\int_a^x h(t)\mathrm{d}t\right) = \frac{(x-a)h(x)-\int_a^x h(t)\mathrm{d}t}{(x-a)^2},$$

and the derivative of a monotone function is either almost everywhere ≥ 0, or almost everywhere ≤ 0. See [1], Theorem 18.3, p. 275.

When the averages have opposite monotonicity, then the inequality (C) works in the reverse way.

Theorem 1 allows us to motivate many curious inequalities. For example, according to the above discussion, the following inequality holds true for every $a \geq 0$,

$$\int_0^a t\left(t+\sqrt{2}\sin t\right)\mathrm{d}t \geq \frac{a}{2}\int_0^a \left(t+\sqrt{2}\sin t\right)\mathrm{d}t,$$

that is,

$$\frac{1}{12}a^3 - a\sqrt{2}\cos^2\frac{a}{2} + \sqrt{2}\sin a \geq 0 \quad \text{for } a \geq 0.$$

The proof of Theorem 1 follows immediately from the case $x = b$ of an identity involving averages:

Lemma 1. *If f and g are two functions as in the hypotheses of Theorem 1, then for every $x \in (a, b]$,*

$$\frac{1}{x-a}\int_a^x f(t)g(t)\mathrm{d}t = \left(\frac{1}{x-a}\int_a^x f(s)\mathrm{d}s\right)\left(\frac{1}{x-a}\int_a^x g(s)\mathrm{d}s\right) +$$
$$+\frac{1}{x-a}\int_a^x \left(f(t) - \frac{1}{t-a}\int_a^t f(s)\mathrm{d}s\right)\left(g(t) - \frac{1}{t-a}\int_a^t g(s)\mathrm{d}s\right)\mathrm{d}t,$$

that is,

$$(A) \qquad \mathcal{M}(fg) = \mathcal{M}(f)\mathcal{M}(g) + \mathcal{M}\left(\left(f - \mathcal{M}(f)\right)\left(g - \mathcal{M}(g)\right)\right).$$

Proof. The special case where f and g are both continuously differentiable, is simply a consequence of the repeated application of the method of integration by parts

$$\int_a^x f(t)g(t)\mathrm{d}t = f(t)\int_a^t g(s)\mathrm{d}s\Big|_{t=a}^{t=x} - \int_a^x f'(t)\left(\int_a^t g(s)\mathrm{d}s\right)\mathrm{d}t =$$
$$= f(x)\int_a^x g(s)\mathrm{d}s - \int_a^x (t-a)f'(t)\left(\frac{1}{t-a}\int_a^t g(s)\mathrm{d}s\right)\mathrm{d}t =$$
$$= f(x)\int_a^x g(s)\mathrm{d}s - \left(\frac{1}{x-a}\int_a^x g(s)\mathrm{d}s\right)\left((x-a)f(x) - \int_a^x f(s)\mathrm{d}s\right) +$$
$$+\int_a^x \frac{1}{(t-a)^2}\left((t-a)f(t) - \int_a^t f(s)\mathrm{d}s\right)\left((t-a)g(t) - \int_a^t g(s)\mathrm{d}s\right)\mathrm{d}t =$$
$$= \left(\frac{1}{x-a}\int_a^x g(s)\mathrm{d}s\right)\left(\int_a^x f(s)\mathrm{d}s\right) +$$
$$+\int_a^x \frac{1}{(t-a)^2}\left((t-a)f(t) - \int_a^t f(s)\mathrm{d}s\right)\left((t-a)g(t) - \int_a^t g(s)\mathrm{d}s\right)\mathrm{d}t.$$

The general case is covered by the theory of absolutely continuous functions. They are precisely the functions $h : [a, b] \to \mathbb{R}$ that admit representations of the form

$$h(x) = h(a) + \int_a^x \varphi(t)\mathrm{d}t, \quad x \in [a,b],$$

with g a suitable Lebesgue integrable function on $[a,b]$. Necessarily, h' exists almost everywhere and $h' = \varphi$ in $L^1([a,b])$. See [1], Theorem 18.17, page 286.

The identity (A) can be obtained from the formula of integration by parts for absolutely continuous functions as follows

$$\begin{aligned}\int_a^x f(t)\left(\frac{1}{t-a}\int_a^t g(s)\mathrm{d}s\right)\mathrm{d}t &= \int_a^x \left(\frac{d}{\mathrm{d}t}\int_a^t f(s)\mathrm{d}s\right)\left(\frac{1}{t-a}\int_a^t g(s)\mathrm{d}s\right)\mathrm{d}t = \\ &= \int_a^x \left(\int_a^t f(s)\mathrm{d}s\right)\left(\frac{1}{t-a}\int_a^t g(s)\mathrm{d}s\right)\mathrm{d}t + \\ &+ \int_a^x \left(\frac{1}{t-a}\int_a^t f(s)\mathrm{d}s\right)\left(\frac{1}{t-a}\int_a^t g(s)\mathrm{d}s\right)\mathrm{d}t - \\ &- \int_a^x g(t)\left(\frac{1}{t-a}\int_a^t f(s)\mathrm{d}s\right)\mathrm{d}t. \quad \square\end{aligned}$$

Lemma 1 easily yields the identity

$$\mathcal{M}(fgh) = \mathcal{M}(f)\mathcal{M}(g)\mathcal{M}(h) + \mathcal{M}\left(\left(f - \mathcal{M}(f)\right)\left(g - \mathcal{M}(g)\right)\right)\mathcal{M}\left(h - \mathcal{M}(h)\right),$$

for triplets f, g, h in $L^\infty([a,b])$. This can be extended by mathematical induction to all finite families of functions belonging to $L^\infty([a,b])$, and implies a suitable extension of Theorem 1 for n-tuples of functions verifying the n-analogue of the condition (S).

Theorem 1 can be extended (mutatis mutandis) to weighted measures of the form $p(t)\mathrm{d}t$ (for p a positive continuous function), by replacing the averages $\mathcal{M}(h)$ by the averages $\mathcal{M}_{p(t)\mathrm{d}t}(h)$, defined by the formula

$$\mathcal{M}_{p(t)\mathrm{d}t}(h)(x) = \int_a^x h(t)p(t)\mathrm{d}t \Big/ \int_a^x p(t)\mathrm{d}t, \quad \text{for } x \in (a,b].$$

The discrete analogue of this extension also works.

Theorem 2. *If $(w_k)_{k=1}^n$ is a family of positive numbers and $(x_k)_{k=1}^n$ and $(y_k)_{k=1}^n$ are two sequences of real numbers such that their averages*

$$x_1, \frac{w_1x_1 + w_2x_2}{w_1 + w_2}, \ldots, \frac{w_1x_1 + w_2x_2 + \cdots + w_nx_n}{w_1 + w_2 + \cdots + w_n},$$

and

$$y_1, \frac{w_1y_1 + w_2y_2}{w_1 + w_2}, \ldots, \frac{w_1y_1 + w_2y_2 + \cdots + w_ny_n}{w_1 + w_2 + \cdots + w_n},$$

are synchronous (for example, if both are monotone, in the same direction), then

$$\left(\sum_{k=1}^{n} w_k\right)\left(\sum_{k=1}^{n} w_k x_k y_k\right) \geq \left(\sum_{k=1}^{n} w_k x_k\right)\left(\sum_{k=1}^{n} w_k y_k\right).$$

As an immediate consequence we obtain the following generalization of the key result used by Simonovits [6] in his approach of the ranking the social assurance systems:

COROLLARY 1. *Let* $(\alpha_k)_{1\leq k\leq n}$, $(u_k)_{1\leq k\leq n}$ *be two positive sequences and* $(\beta_k)_{1\leq k\leq n}$, $(v_k)_{1\leq k\leq n}$ *be two nonnegative sequences. If the weighted averages of* $\left(\frac{\beta_k}{\alpha_k}\right)_{1\leq k\leq n}$ *and* $\left(\frac{v_k}{u_k}\right)_{1\leq k\leq n}$ *(with weights* $w_k = \alpha_k u_k$*) are both monotone in the same direction, then*

$$\frac{\sum_{k=1}^{n} \alpha_k v_k}{\sum_{k=1}^{n} \alpha_k u_k} \leq \frac{\sum_{k=1}^{n} \beta_k v_k}{\sum_{k=1}^{n} \beta_k u_k}.$$

The *proof* follows from Theorem 2 applied to $x_k = \beta_k/\alpha_k$, $y_k = v_k/u_k$ and $w_k = \alpha_k u_k$. $\square$

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