REFINEMENTS OF HERMITE-HADAMARD INEQUALITY ON SIMPLICES

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Communicated by the former editorial board

Some refinements of the Hermite-Hadamard inequality are obtained in the case of continuous convex functions defined on simplices.

AMS 2010 Subject Classification: 26A51.

Key words: Hermite-Hadamard inequality, convex function, simplex.

1. INTRODUCTION

The aim of this paper is to provide a refinement of the Hermite-Hadamard inequality on simplices.

Suppose that K is a metrizable compact convex subset of a locally convex Hausdorff space E. Given a Borel probability measure μ on K, one can prove the existence of a unique point $b_\mu \in K$ (called the *barycenter* of μ) such that

$$x'(b_\mu) = \int_K x'(x)\,\mathrm{d}\mu(x),$$

for all continuous linear functionals x' on E. The main feature of barycenter is the inequality

$$f(b_\mu) \le \int_K f(x)\,\mathrm{d}\mu(x),$$

valid for every continuous convex function $f : K \to \mathbb{R}$. It was noted by several authors that this inequality is actually equivalent to the Jensen inequality.

The following theorem, due to G. Choquet, complements this inequality and relates the geometry of K to a given mass distribution.

Theorem 1 (The general form of Hermite-Hadamard inequality). *Let μ be a Borel probability measure on a metrizable compact convex subset K of a locally convex Hausdorff space. Then, there exists a Borel probability measure ν on K which has the same barycenter as μ, is zero outside* $\operatorname{Ext} K$, *and verifies* *the double inequality*

$$f(b_\mu) \leq \int_K f(x)\,\mathrm{d}\mu(x) \leq \int_{\operatorname{Ext} K} f(x)\,\mathrm{d}\nu(x), \tag{1}$$

for all continuous convex functions $f : K \to \mathbb{R}$.

Here $\operatorname{Ext} K$ *denotes the set of all extreme points of* K.

The details can be found in [7], pp. 192–194. See also [6].

In the particular case of simplices one can take advantage of the barycentric coordinates.

Suppose that $\Delta \subset \mathbb{R}^n$ is an n-dimensional simplex of vertices $P_1, \dots, P_{n+1}$. In this case $\operatorname{Ext}\Delta = \{P_1, \dots, P_{n+1}\}$ and each $x \in \Delta$ can represented uniquely as a convex combination of vertices,

$$\sum_{k=1}^{n+1} \lambda_k(x) P_k = x, \tag{2}$$

where the coefficients $\lambda_k(x)$ are nonnegative numbers (depending on x) and

$$\sum_{k=1}^{n+1} \lambda_k(x) = 1. \tag{3}$$

Each function $\lambda_k : x \to \lambda_k(x)$ is an affine function on Δ. This can be easily seen by considering the linear system consisting of the equations (2) and (3).

The coefficients $\lambda_k(x)$ can be computed in terms of Lebesgue volumes (see [1], [5]). We denote by $\Delta_j(x)$ the subsimplex obtained when the vertex P_j is replaced by $x \in \Delta$. Then, one can prove that

$$\lambda_k(x) = \frac{\operatorname{Vol}(\Delta_k(x))}{\operatorname{Vol}(\Delta)}, \tag{4}$$

for all $k = 1, \dots, n+1$ (the geometric interpretation is very intuitive). Here $\operatorname{Vol}(\Delta) = \int_\Delta \mathrm{d}x$.

In the particular case where $\mathrm{d}\mu(x) = \mathrm{d}x/\operatorname{Vol}(\Delta)$ we have $\lambda_k\left(b_{\mathrm{d}x/\operatorname{Vol}(\Delta)}\right) = \frac{1}{n+1}$, for every $k = 1, \dots, n+1$.

The above discussion leads to the following form of Theorem 1 in the case of simplices:

Corollary 1. *Let* $\Delta \subset \mathbb{R}^n$ *be an* n*-dimensional simplex of vertices* $P_1, \dots, P_{n+1}$ *and* μ *be a Borel probability measure on* Δ *with barycenter* b_μ*. Then, for every continuous convex function* $f : \Delta \to \mathbb{R}$,

$$f(b_\mu) \leq \int_\Delta f(x)\,\mathrm{d}\mu(x) \leq \sum_{k=1}^{n+1} \lambda_k(b_\mu) f(P_k).$$

Proof. In fact,

$$\int_{\Delta} f(x)\,\mathrm{d}\mu(x) = \int_{\Delta} f\left(\sum_{k=1}^{n+1} \lambda_k(x) P_k\right) \mathrm{d}\mu(x) \le \int_{\Delta} \sum_{k=1}^{n+1} \lambda_k(x)\, f(P_k)\,\mathrm{d}\mu(x)$$

$$= \sum_{k=1}^{n+1} f(P_k) \int_{\Delta} \lambda_k(x)\ \mathrm{d}\mu(x) = \sum_{k=1}^{n+1} \lambda_k(b_\mu) f\left(P_k\right).$$

On the other hand,

$$\sum_{k=1}^{n+1} \lambda_k(b_\mu) f\left(P_k\right) = \int_{\mathrm{Ext}\,\Delta} f \mathrm{d}\left(\sum_{k=1}^{n+1} \lambda_k(b_\mu)\delta_{P_k}\right)$$

and $\sum_{k=1}^{n+1} \lambda_k(b_\mu)\delta_{P_k}$ is the only Borel probability measure ν concentrated at the vertices of Δ which verifies the inequality

$$\int_{\Delta} f(x)\,\mathrm{d}\mu(x) \le \int_{\mathrm{Ext}\,\Delta} f(x)\,\mathrm{d}\nu(x)$$

for every continuous convex functions $f : \Delta \to \mathbb{R}$. Indeed, ν must be of the form $\nu = \sum_{k=1}^{n+1} \alpha_k \delta_{P_k}$, with $b_\nu = \sum_{k=1}^{n+1} \alpha_k P_k = b_\mu$ and the uniqueness of barycentric coordinates yields the equalities

$$\alpha_k = \lambda_k(b_\mu), \quad \text{for } k = 1, \ldots, n+1. \quad \square$$

It is worth noticing that the Hermite–Hadamard inequality is not just a consequence of convexity, it actually characterizes it. See [9].

The aim of the present paper is to improve the result of Corollary 1, by providing better bounds for the arithmetic mean of convex functions defined on simplices.

2. MAIN RESULTS

We start by extending the following well known inequality concerning the continuous convex functions defined on intervals

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{1}{2}\left(\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right).$$

See [7], p. 52.

THEOREM 2. *Let* $\Delta \subset \mathbb{R}^n$ *be an* n*-dimensional simplex of vertices* $P_1, \ldots,$ P_{n+1}, *endowed with the normalized Lebesgue measure* $\mathrm{d}x/\operatorname{Vol}(\Delta)$. *Then, for* *every continuous convex function* $f : \Delta \to \mathbb{R}$ *and every point* $P \in \Delta$ *we have*

$$\frac{1}{\operatorname{Vol}(\Delta)} \int_\Delta f(x)\,\mathrm{d}x \le \frac{1}{n+1} \left(\sum_{k=1}^{n+1} (1 - \lambda_k(P))\, f(P_k) + f(P) \right). \tag{5}$$

In particular,

$$\frac{1}{\operatorname{Vol}(\Delta)} \int_\Delta f(x)\,\mathrm{d}x \le \frac{1}{n+1} \left(\frac{n}{n+1} \sum_{k=1}^{n+1} f(P_k) + f\left(b_{\mathrm{d}x/\operatorname{Vol}(\Delta)}\right) \right). \tag{6}$$

Proof. Consider the barycentric representation $P = \sum_k \lambda_k(P) P_k \in \Delta$. According to Corollary 1,

$$\frac{1}{\lambda_i(P)\operatorname{Vol}(\Delta)} \int_{\Delta_i(P)} f(x)\,\mathrm{d}x \le \frac{1}{n+1} \left(\sum_{k \ne i} f(P_k) + f(P) \right),$$

where $\Delta_i(P)$ denotes the simplex obtained from Δ by replacing the vertex P_i by P. Notice that (4) yields $\lambda_i(P)\operatorname{Vol}(\Delta) = \operatorname{Vol}(\Delta_i(P))$. Multiplying both sides by $\lambda_i(P)$ and summing up over i, we obtain the inequality (5).

In the particular case when $P = b_{\mathrm{d}x/\operatorname{Vol}(\Delta)}$, all coefficients $\lambda_k(P)$ equal $1/(n+1)$. □

Remark 1. Using [1, Theorem 1] instead of Corollary 1 of the present paper, one may improve the statement of Theorem 3 by the cancellation of the continuity condition. Such statement is proved independently, by a different approach, in [12].

An extension of Theorem 2 is as follows:

THEOREM 3. *Let* $\Delta \subset \mathbb{R}^n$ *be an* n*-dimensional simplex of vertices* $P_1, \ldots, P_{n+1}$ *endowed with the Lebesgue measure and let* $\Delta' \subseteq \Delta$ *a subsimplex of vertices* $P'_1, \ldots, P'_{n+1}$ *which has the same barycenter as* Δ*. Then, for every continuous convex function* $f : \Delta \to \mathbb{R}$*, and every index* $j \in \{1, \ldots, n+1\}$*, we have the estimates*

$$\begin{aligned} \frac{1}{\operatorname{Vol}(\Delta)} \int_\Delta f(x)\,\mathrm{d}x &\le \frac{1}{n+1} \left(\sum_{k \ne j} \sum_i \lambda_i\left(P'_k\right) f(P_i) + f\left(P'_j\right) \right) \\ &\le \frac{1}{n+1} \sum_i f(P_i) \end{aligned} \tag{7}$$

and

$$\frac{1}{\operatorname{Vol}(\Delta)}\int_{\Delta} f(x)\,\mathrm{d}x \geq \sum_i \lambda_i\left(P_j'\right) f\left(\frac{1}{n+1}\left(\sum_{k\neq i} P_k + P_j'\right)\right) \geq f\left(\frac{1}{n+1}\sum_i P_i\right). \tag{8}$$

Proof. We will prove here only the inequalities (7), the proof of (8) being similar.

By Corollary 1, for each index i,

$$\frac{1}{\lambda_i\left(P_j'\right)\operatorname{Vol}(\Delta)}\int_{\Delta_i(P_j')} f(x)\,\mathrm{d}x \leq \frac{1}{n+1}\left(\sum_{k\neq i} f(P_k) + f\left(P_j'\right)\right),$$

where $\Delta_i(P_j')$ denotes the simplex obtained from Δ by replacing the vertex P_i by P_j'. Notice that (4) yields $\lambda_i(P_j')\operatorname{Vol}(\Delta) = \operatorname{Vol}(\Delta_i(P_j'))$. By multiplying both sides by $\lambda_i(P_j')$ and summing over i we obtain

$$\begin{aligned}\frac{1}{\operatorname{Vol}(\Delta)}\int_{\Delta} f(x)\,\mathrm{d}x &\leq \frac{1}{n+1}\left(\sum_i \lambda_i\left(P_j'\right)\sum_{k\neq i} f(P_k) + f\left(P_j'\right)\right)\\ &= \frac{1}{n+1}\left(\sum_i \lambda_i\left(P_j'\right)\left(\sum_k f(P_k) - f(P_i)\right) + f\left(P_j'\right)\right)\\ &= \frac{1}{n+1}\left(\sum_i \left(1-\lambda_i\left(P_j'\right)\right) f(P_i) + f\left(P_j'\right)\right).\end{aligned} \tag{9}$$

Furthermore, since Δ' and Δ have the same barycenter, we have

$$1-\lambda_i\left(P_j'\right) = \sum_{k\neq j}\lambda_i\left(P_k'\right),$$

for all $i = 1,\ldots,n+1$. Then

$$\frac{1}{\operatorname{Vol}(\Delta)}\int_{\Delta} f(x)\,\mathrm{d}x \leq \frac{1}{n+1}\left(\sum_{k\neq j}\sum_i \lambda_i\left(P_k'\right) f(P_i) + f\left(P_j'\right)\right).$$

The fact that Δ' and Δ have the same barycenter and the convexity of the function f yield

$$\begin{aligned}\sum_j f(P_j) &= \sum_k \sum_i \lambda_i\left(P'_k\right) f(P_i)\\ &= \sum_{k\neq j} \sum_i \lambda_i\left(P'_k\right) f(P_i) + \sum_i \lambda_i\left(P'_j\right) f(P_i)\\ &\geq \sum_{k\neq j} \sum_i \lambda_i\left(P'_k\right) f(P_i) + f\left(\sum_i \lambda_i\left(P'_j\right) P_i\right)\\ &= \sum_{k\neq j} \sum_i \lambda_i\left(P'_k\right) f(P_i) + f\left(P'_j\right).\end{aligned}$$

This concludes the proof (7). □

When Δ' as a singleton, the inequality (7) coincides with the inequality (6). Notice that if we omit the barycenter condition then, the inequality (9) translates into (5).

An immediate consequence of Theorem 3 is the following result due to Farissi [3]:

Corollary 2. *Assume that $f : [a, b] \to \mathbb{R}$ is a convex function and $\lambda \in [0, 1]$. Then*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{(1-\lambda) f(a) + \lambda f(b) + f(\lambda a + (1-\lambda) b)}{2} \leq \frac{f(a)+f(b)}{2}$$

and

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x &\geq \lambda f\left(\frac{a+(1-\lambda)a+\lambda b}{2}\right) + (1-\lambda) f\left(\frac{b+(1-\lambda)a+\lambda b}{2}\right)\\ &\geq f\left(\frac{a+b}{2}\right).\end{aligned}$$

Proof. Apply Theorem 3 for $n = 1$ and Δ' the subinterval of endpoints $(1-\lambda) a + \lambda b$ and $\lambda a + (1-\lambda) b$. □

Another refinement of the Hermite-Hadamard inequality in the case of simplices is as follows.

Theorem 4. *Let $\Delta \subset \mathbb{R}^n$ be an n-dimensional simplex of vertices $P_1, \ldots,$ P_{n+1} endowed with the Lebesgue measure and let $\Delta' \subseteq \Delta$ be a subsimplex whose barycenter with respect to the normalized Lebesgue measure $\frac{\mathrm{d}x}{\mathrm{Vol}(\Delta')}$ is $P = \sum_k \lambda_k(P)P_k$. Then, for every continuous convex function $f : \Delta \to \mathbb{R}$,*

$$f(P) \leq \frac{1}{\mathrm{Vol}(\Delta')}\int_{\Delta'} f(x)\,\mathrm{d}x \leq \sum_j \lambda_j(P) f(P_j). \tag{10}$$

Proof. Let P'_k, $k = 1, \dots, n+1$ be the vertices of Δ'. By Corollary 1,

$$f(P) \leq \frac{1}{\operatorname{Vol}(\Delta')} \int_{\Delta'} f(x)\,\mathrm{d}x \leq \frac{1}{n+1} \sum_k f\left(P'_k\right).$$

The barycentric representation of each of the points $P'_k \in \Delta$ gives us

$$\begin{cases} \sum_j \lambda_j \left(P'_k\right) P_j = P'_k \\ \sum_k \lambda_k (P'_j) = 1 \end{cases}.$$

Since f is a convex function,

$$\frac{1}{n+1} \sum_k f\left(P'_k\right) = \frac{1}{n+1} \sum_k f\left(\sum_j \lambda_j \left(P'_k\right) P_j\right)$$
$$\leq \sum_j \left(\frac{1}{n+1} \sum_k \lambda_j \left(P'_k\right)\right) f\left(P_j\right) = \sum_j \lambda_j (P)\, f\left(P_j\right)$$

and the assertion of Theorem 4 is now clear. □

As a corollary of Theorem 4, we get the following result due to Vasić and Lacković [10], and Lupaş [2] (cf. J.E. Pečarić et. al. [8]).

Corollary 3. *Let p and q be two positive numbers and $a_1 \leq a \leq b \leq b_1$. Then, the inequalities*

$$f\left(\frac{pa+qb}{p+q}\right) \leq \frac{1}{2y} \int_{A-y}^{A+y} f(x)\,\mathrm{d}x \leq \frac{pf(a)+qf(b)}{p+q}$$

hold for $A = \frac{pa+qb}{p+q}$, $y > 0$ and all continuous convex functions $f : [a_1, b_1] \to \mathbb{R}$ if and only if

$$y \leq \frac{b-a}{p+q} \min\{p, q\}.$$

The right-hand side of the inequality stated in Theorem 4 can be improved as follows.

Theorem 5. *Suppose that Δ is an n-dimensional simplex of vertices $P_1, \dots, P_{n+1}$ and let $P \in \Delta$. Then, for every subsimplex $\Delta' \subset \Delta$ such that $P = \sum_j \lambda_j (P) P_j = b_{\mathrm{d}x/\operatorname{Vol}(\Delta')}$. Then*

$$\frac{1}{\operatorname{Vol}(\Delta')} \int_{\Delta'} f(x)\,\mathrm{d}x \leq \frac{1}{n+1} \left(n \sum_j \lambda_j(P) f\left(P_j\right) + f(P)\right),$$

for every continuous convex function $f : \Delta \to \mathbb{R}$.

Proof. Let P'_k, $k = 1, \dots, n+1$ be the vertices of Δ'. We denote by Δ'_i the subsimplex obtained by replacing the vertex P'_i by the barycenter P of the normalized measure $\mathrm{d}x / \operatorname{Vol}(\Delta')$ on Δ'.

According to Corollary 1,

$$\begin{aligned}
\frac{1}{\operatorname{Vol}(\Delta'_i)} \int_{\Delta'_i} f(x)\,\mathrm{d}x &\leq \frac{1}{n+1} \sum_{k \neq i} f\left(P'_k\right) + \frac{1}{n+1} f(P) \\
&= \frac{1}{n+1} \sum_{k \neq i} f\left(\sum_j \lambda_j\left(P'_k\right) P_j \right) + \frac{1}{n+1} f(P) \\
&\leq \sum_j \left(\frac{1}{n+1} \sum_{k \neq i} \lambda_j\left(P'_k\right) \right) f\left(P_j\right) + \frac{1}{n+1} f(P) \\
&= \sum_j \left(\lambda_j(P) - \frac{1}{n+1} \lambda_j\left(P'_i\right) \right) f\left(P_j\right) + \frac{1}{n+1} f(P),
\end{aligned}$$

for each index $i = 1, \dots, n+1$. Summing up over i we obtain

$$\begin{aligned}
&\frac{n+1}{\operatorname{Vol}(\Delta')} \int_{\Delta'} f(x)\,\mathrm{d}x \leq \\
&\quad \leq (n+1) \sum_j \lambda_j(P) f\left(P_j\right) - \sum_i \frac{1}{n+1} \sum_j \lambda_j\left(P'_i\right) f\left(P_j\right) + f(P) \\
&\quad = n \sum_j \lambda_j(P) f\left(P_j\right) + f\left(P\right).
\end{aligned}$$

and the proof of the theorem is completed. □

Of course, the last theorem yields an improvement of Corollary 3. This was first noticed in [8, p. 146].

We end this paper with an alternative proof of some particular case of a result established by *A.* Guessab and G. Schmeisser [4, Theorem 2.4]. Before we start, we would like to turn the reader's attention to the paper [12] by S. Wasowicz, where the result we are going to present was obtained by some more general approach (see [11, Theorem 2 and Corollary 4]).

Theorem 6. *Let $\Delta \subset \mathbb{R}^n$ be an n-dimensional simplex of vertices $P_1, \dots, P_{n+1}$ endowed with the Lebesgue measure and let $M_1, \dots, M_m$ be points in Δ such that $b_{\mathrm{d}x/\operatorname{Vol}(\Delta)}$ is a convex combination $\sum_j \beta_j M_j$ of them. Then, for every continuous convex function $f : \Delta \to \mathbb{R}$, the following inequalities hold:*

$$f\left(b_{\mathrm{d}x/\operatorname{Vol}(\Delta)}\right) \leq \sum_{j=1}^{m} \beta_j f\left(M_j\right) \leq \frac{1}{n+1} \sum_{i=1}^{n+1} f\left(P_i\right).$$

Proof. The first inequality follows from Jensen's inequality. In order to establish the second inequality, since we have

$$b_{\mathrm{d}x/\operatorname{Vol}(\Delta)} = \sum_j \beta_j \left(\sum_i \lambda_i (M_j) P_i \right) = \sum_i \left(\sum_j \beta_j \lambda_i (M_j) \right) P_i = \frac{1}{n+1} \sum_i P_i ,$$

we infer that $\sum_j \beta_j \lambda_i (M_j) = \frac{1}{n+1}$, for every i and

$$\sum_j \beta_j f(M_j) = \sum_j \beta_j f \left(\sum_i \lambda_i (M_j) P_i \right) \le \sum_i \left(\sum_j \beta_j \lambda_i (M_j) \right) f(P_i) = \frac{1}{n+1} \sum_i f(P_i) .$$

This completes the proof. □

Acknowledgements. The first author has been supported by CNCSIS Grant 420/2008. The authors are indebted to Szymon Wasowicz for pointing out to them references [4] and [11].

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Received 23 May 2011

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