FUNCTIONALS AND GRADIENT STOCHASTIC FLOWS WITH JUMPS ASSOCIATED WITH NONLINEAR SPDEs

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Classical solutions for some nonlinear SPDEs of parabolic type are found using a stochastic characteristic system with jumps and the corresponding gradient representation of its solution.

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1. INTRODUCTION

There are investigated two problems relying on SDEs with jumps while impossing a mutually commuting condition for the vector fields $\{f_1, f_2, g\} \subseteq \subseteq \left(C_b \cap C_b^1 \cap C^2\right)(\mathbb{R}^n; \mathbb{R}^n)$ driving the motion. It implies a useful integral representation for any solution of the SDE under consideration. In addition, a fundamental system of stochastic first integrals can be constructed provided the conditions of the contraction mapping theorem are satisfied.

A solution of Problem (I) containing the above mentioned subjects is given in Theorem 1 of Section 3.

The solution of Problem (I) is used to associate a non-Markovian SDE and functionals with jumps for which a filtering Problem (II) is solved in Theorem 2 of Section 3.

Section 2 of this paper is dedicated to some preliminaries including an application of Banach fixed point theorem for solving integral stochastic equations with jumps.

The main results (see Theorems 1 and 2) either associate the evolution of some pathwise functionals with nonlinear SPDEs of parabolic type or introduce appropriate parameterized backward parabolic equations.

The general method used here relies on piecewise smooth test functions constructed as fundamental solutions for some quasilinear (Hamilton-Jacobi) equations with jumps. Then, a solution of Problem (I) is defined combining the piecewise smooth test functions with a continuous solution of the diffusion part of the SDE under consideration. This method has much in common with the results contained in [1] where SDEs and SPDEs with continuous trajectories are studied. The results given in [2] use a different approach. Some roots of this paper are contained in the reference [3] and a more general situation including several diffusion vector fields in involution might be consistent with the problems analyzed here.

2. PRELIMINARIES AND FORMULATION OF PROBLEMS

Consider two independent processes $\{(w(t), y(t)) : t \in [0,T]\}$ on the filtered complete probability space $\left\{\Omega, \mathcal{F} \supseteq \left\{\mathcal{F}^t\right\}, P\right\}$ (see $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$, $\mathcal{F}^t = \mathcal{F}_1^t \times \mathcal{F}_2$, $P = P_1 \otimes P_2$), where $\{w(t) \in \mathbb{R} : t \in [0,T]\}$ is a Brownian motion over $\left\{\Omega_1, \mathcal{F}_1 \supseteq \left\{\mathcal{F}_1^t\right\}, P_1\right\}$ and

$$\{y(t) \in [-\gamma, \gamma] : t \in [0,T],\ y(0) = 0\}$$

is a piecewise constant process defined on the probability space $\{\Omega_2, \mathcal{F}_2, P_2\}$. The piecewise constant process $\{y(t)\}$ satisfies

$$y\left(t, \omega_2\right) = y\left(\theta_i\left(\omega_2\right), \omega_2\right) \overset{\text{not}}{=} y_i\left(\omega_2\right), \quad t \in \left[\theta_i\left(\omega_2\right), \theta_{i+1}\left(\omega_2\right)\right),$$

where $0 = \theta_0\left(\omega_2\right) < \theta_1\left(\omega_2\right) < \cdots < \theta_{N-1}\left(\omega_2\right) < \theta_N\left(\omega_2\right) = T$ is a partition such that $y_i\left(\omega_2\right) : \Omega_2 \to \mathbb{R}$ is a $\mathcal{F}_2$-measurable random variable for any $i \in \{0, 1, \ldots, N-1\}$. There are given the smooth vector fields $\{g, f_1, f_2\} \subseteq$ $\subseteq \left(C_b \cap C_b^1 \cap C^2\right)\left(\mathbb{R}^n; \mathbb{R}^n\right)$ and two scalar functions $\{\varphi_1, \varphi_2\} \subseteq \left(C_b^1 \cap C^2\right)\left(\mathbb{R}^n\right)$ such that

$$\{g, f_1, f_2\} \text{ mutually commute w.r.t. the Lie bracket,} \tag{1}$$

$$(\gamma + T) V K = \rho \in [0,1), \text{ where } \{|y(t)| \leq \gamma : t \in [0,T]\}, \tag{2}$$

and

$$V = \sup\left\{\left|\partial_x \varphi_1(x)\right|,\ \left|\partial_x \varphi_2(x)\right| : x \in \mathbb{R}^n\right\},$$
$$K = \sup\left\{\left|f_1(x)\right|,\ \left|f_2(x)\right| : x \in \mathbb{R}^n\right\}.$$

Let $\{\widehat{x}_\varphi(t;\lambda) : t \in [0,T],\ \lambda \in \mathbb{R}^n\}$ be the stochastic flow generated by the following SDE with jumps

$$(3)\qquad \begin{cases} \mathrm{d}_t\widehat{x} = [f_1(\widehat{x}(t-))\,\varphi_1(\lambda)\mathrm{d}t + f_2(\widehat{x}(t-))\,\varphi_2(\lambda)\delta y(t)] + g(\widehat{x}(t-)) \circ \mathrm{d}w(t); \\ \widehat{x}(0) = \lambda \in \mathbb{R}^n,\ t \in [0,T],\ \delta y(t) = y(t) - y(t-),\ \widehat{x}(t-) = \lim\limits_{s\nearrow t} \widehat{x}(s), \end{cases}$$

where Fisk-Stratonovich integral "$\circ$" is computed by

$$(4)\qquad g(\widehat{x}(t-)) \circ \mathrm{d}w(t) = \frac{1}{2}([\partial_x g]\, g)(\widehat{x}(t-))\,\mathrm{d}t + g(\widehat{x}(t-)) \cdot \mathrm{d}w(t)$$

using the Itô integral "$\cdot$".

For each continuity interval $t \in [\theta_i, \theta_{i+1})$ (making an abuse, the variables $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$ are omitted) associate the following system of nonlinear SPDEs of parabolic type

$$(5)\qquad \begin{cases} \mathrm{d}_t\psi(t,x) + [\partial_x\psi(t,x)]\, f_1(x)\varphi_1(\psi(t,x))\mathrm{d}t + \\ \qquad\qquad + [\partial_x\psi(t,x)]\, g(x)\widehat{\circ}\mathrm{d}w(t) = 0; \\ \psi(\theta_i, x) \ = F_2[-\varphi_2(\psi(\theta_i-, x))\,\delta y(\theta_i)](\psi(\theta_i-, x)), \\ \qquad\qquad i \in \{0, 1, \ldots, N-1\}; \\ \psi(0,x) \ = x \in \mathbb{R}^n. \end{cases}$$

The Fisk-Stratonovich integral "$\widehat{\circ}$" in (5) is computed by

$$(6)\qquad h(t,x)\,\widehat{\circ}\,\mathrm{d}w(t) = h(t,x) \cdot \mathrm{d}w(t) - \frac{1}{2}\partial_x h(t,x) g(x)\mathrm{d}t$$

using the Itô integral "$\cdot$" while $F_2(\sigma)[z]$, $\sigma \in \mathbb{R}$, $z \in \mathbb{R}^n$, is the global flow generated by the complete vector field f_2.

Problem (I). Under the hypotheses (1) and (2), a $\mathcal{F}^t$-adapted solution $\lambda = \psi(t,x) \in \mathbb{R}^n$ will exist such that

(7) $\quad \widehat{x}_\varphi(t;\lambda) = x,\ t \in [0,T],\ \psi(0,x) = x \in \mathbb{R}^n;$

(8) $\quad \{\psi(t,x) : t \in [\theta_i, \theta_{i+1}),\ x \in \mathbb{R}^n\}$ is a continuous mapping;

(9) $\quad \{\psi(t,x) : t \in [\theta_i, \theta_{i+1}),\ x \in \mathbb{R}^n\}$ is a second order continuously

differentiable mapping w.r.t. $x \in \mathbb{R}^n$, satisfying the nonlinear SPDE of parabolic type given in (5), $i \in \{0, 1, \ldots, N-1\}$.

Problem (II). Using the unique solution $\{\lambda = \psi(t,x)\}$ of Problem (I), describe the evolution of the conditioned mean values

$$(10)\qquad v_i(t,x) = E_1\{h(z_\psi(T;t,x)) \mid \psi(t,x)\}, \quad t \in [\theta_i, \theta_{i+1}),\ x \in \mathbb{R}^n,$$

for any $h \in C_p^2(\mathbb{R}^n)$ and $i \in \{0, 1, 2, \ldots N-1\}$ where $C_p^2(\mathbb{R}^n)$ consists of second order continuously differentiable functions such that h, $\partial_{x_i} h$, $\partial^2_{x_i x_j} h$ satisfy a polynomial growth condition for $i, j \in \{1, 2, \ldots, n\}$.

Here, $\{z_\psi(s; t, x) : s \in [t, T]\}$ is the unique solution of the non-Markovian SDE with jumps

$$(11)\qquad \begin{cases} \mathrm{d}_s z = [f_1(z(s-))\varphi_1(\psi(t,x))\,\mathrm{d}s + f_2(z(s-))\varphi_2(\psi(t,x))\delta y(s)] + \\ \qquad\qquad +g(z(s-)) \circ \mathrm{d}w(s); \\ z(t) = x,\ s \in [t, T]. \end{cases}$$

Remark 1. Using the hypothesis (1), the unique solution $\{\widehat{x}_\varphi(t; \lambda)\}$ of the SDE (3) can be represented as follows

$$(12)\quad \widehat{x}_\varphi(t;\lambda) = G(w(t)) \circ F_1(\tau_1(t,\lambda)) \circ F_2(\tau_2(t,\lambda))[\lambda], \quad t \in [0,T],\ \lambda \in \mathbb{R}^n,$$

where $G(\sigma)[z]$ and $F_i(\sigma)[z]$ are the global flows generated by the complete vector fields g and f_i, respectively. The following notations are used

$$(13)\qquad \tau_1(t,\lambda) = \varphi_1(\lambda)t, \quad \tau_2(t,\lambda) = \varphi_2(\lambda)y(t)$$

while the integral representation (12) help us to replace $\widehat{x}_\varphi(t;\lambda) = x$ by other integral equations

$$(14)\qquad \lambda = V(t,x;\lambda) := F(-\tau(t,\lambda))\left[G(-w(t))[x]\right],$$

where

$$F(\sigma_1, \sigma_2)[z] \overset{\mathrm{def}}{=} F_1(\sigma_1) \circ F_2(\sigma_2)[z]$$

and

$$\tau(t,\lambda) = (\tau_1(t,\lambda), \tau_2(t,\lambda)).$$

By a direct computation and using the hypothesis (2), we get

$$(15)\qquad |\partial_\lambda V(t,x;\lambda)| \le \rho \in [0,1),\ \text{for any } x, \lambda \in \mathbb{R}^n,\ t \in [0,T],$$

which allows us to use the Banach fixed point theorem for solving the integral equations (14).

In this respect, the unique solution of (14) will be found as a composition

$$(16)\qquad \psi(t,x) = \widehat{\psi}(t, \widehat{z}(t,x)),$$

where $\widehat{z}(t,x) \overset{\mathrm{def}}{=} G(-w(t))[x]$, $t \in [0,T]$, $x \in \mathbb{R}^n$, is a continuous and $\mathcal{F}_1^t$-adapted process.

On the other hand, for each $z \in \mathbb{R}^n$, the piecewise smooth and $\mathcal{F}_2$-measurable process $\left\{\widehat{\psi}(t,z) \in \mathbb{R}^n : t \in [0,T]\right\}$ is found as the unique solution of the following integral equations with jumps

$$(17)\qquad \lambda = \widehat{V}(t,z;\lambda) := F(-\tau(t,\lambda))[z], \quad t \in [0,T],\ z \in \mathbb{R}^n.$$

Here $\tau(t,\lambda) = (\tau_1(t,\lambda), \tau_2(t,\lambda))$ and $F(\sigma_1,\sigma_2)[z]$ are defined in (14).

LEMMA 1. *Under the hypotheses* (1) *and* (2), *there exists a unique solution* $\{\lambda = \widehat{\psi}(t,z) \in \mathbb{R}^n : t \in [0,T],\ z \in \mathbb{R}^n\}$ *of* (17), *which is second order continuously differentiable w.r.t.* $z \in \mathbb{R}^n$. *In adition, the following integral equations with jumps are valid*

$$
(18)\qquad \begin{cases} \widehat{\psi}(t,z) = \widehat{V}\left(t,z;\widehat{\psi}(t-,z)\right),\ t\in[0,T],\ \widehat{\psi}(0,z) = z \in \mathbb{R}^n; \\ \widehat{\psi}(\theta_i,z) = \widehat{V}\left(\theta_i,z;\widehat{\psi}(\theta_i-,z)\right) = \\ \qquad = F_2\left[-\varphi_2\left(\widehat{\psi}(\theta_i-,z)\right)\delta y(\theta_i)\right]\left(\widehat{\psi}(\theta_i-,z)\right), \end{cases}
$$

where $\{\lambda = \widehat{\psi}(\theta_i-,z)\}$ *is the unique solution of the integral equations* $\lambda = \widehat{V}(\theta_i-,z;\lambda)$, $i \in \{0,1,\dots,N-1\}$.

Proof. The unique solution $\{\lambda = \widehat{\psi}(t,z)\}$ satisfying (18) is found as a limit of the standard approximating sequence $\{\lambda_k(t,z)\}_{k\geq 0}$ constructed as follows. Define

$$
(19)\quad \lambda_0(t,z) = z,\ \lambda_{k+1}(t,z) = \widehat{V}(t,z;\lambda_k(t-,z)),\quad k\geq 0,\ t\in[0,T],\ z\in\mathbb{R}^n.
$$

Using the hypothesis (2), we get that $\{\lambda_k(t,z)\}_{k\geq 0}$ is a Cauchy sequence satisfying

$$
(20)\qquad \begin{cases} |\lambda_{k+1}(t,z) - \lambda_k(t,z)| \leq \rho^k \cdot |\lambda_1(t-,z) - \lambda_0(t-,z)|; \\ \widehat{\psi}(t,z) = \lim\limits_{k\to\infty} \lambda_k(t,z),\ \widehat{\psi}(t-,z) = \lim\limits_{k\to\infty} \lambda_k(t-,z), \end{cases}
$$

for any $k \geq 0$, $t \in [0,T]$, and $z \in \mathbb{R}^n$.

In addition, a direct computation leads us to

$$
(21)\qquad \begin{cases} \lambda_1(t-,z) = \widehat{V}(t-,z;z) = \\ \qquad = z - \sum\limits_{i=1}^{2} \int_0^1 f_i\left(F(-\theta\tau(t-;z))[z]\right)\cdot \tau_i(t-;z)\mathrm{d}\theta; \\ |\lambda_1(t-,z) - \lambda_0(t-,z)| = \left|\widehat{V}(t-,z;z) - z\right| \leq \\ \qquad \leq R(\gamma,T,z),\ t\in[0,T], z\in\mathbb{R}^n, \end{cases}
$$

where $R(\gamma,T,z) = [T\,|\varphi_1(z)| + \gamma\,|\varphi_2(z)|]\cdot K$.

Combining (20) and (21), we get the estimate

$$
(22)\qquad \left|\widehat{\psi}(t,z) - z\right| \leq \frac{1}{1-\rho} R(\gamma,T,z),\quad t\in[0,T],\ z\in\mathbb{R}^n,
$$

and by passing $k \to \infty$, from (19), we get that the integral equations (18) hold true.

The proof is complete. □

As a direct consequence of Lemma 1, we obtain

LEMMA 2. *Under the hypotheses* (1) *and* (2), *consider the unique solution* $\{\lambda = \widehat{\psi}(t,z) \in \mathbb{R}^n : t \in [0,T],\ z \in \mathbb{R}^n\}$ *satisfying integral equations* (18). *Then,* $\{\widehat{\psi}(t,z) \in \mathbb{R}^n : t \in [\theta_i, \theta_{i+1}),\ z \in \mathbb{R}^n\}$ *is a continuously differentiable mapping w.r.t.* $z \in \mathbb{R}^n$ (*second order*) *and respectively,* $t \in [\theta_i, \theta_{i+1})$ (*first order*). *In addition, the following quasilinear* (*Hamilton-Jacobi*) *equations with jumps hold true*

$$(23)\qquad \begin{cases} \partial_t \widehat{\psi}(t,z) + \left[\partial_z \widehat{\psi}(t,z) f_1(z)\right] \varphi_1\left(\widehat{\psi}(t,z)\right) = 0,\ t \in [\theta_i, \theta_{i+1})\,; \\ \widehat{\psi}(\theta_i, z) = F_2\left[-\varphi_2\left(\widehat{\psi}(\theta_i -, z)\right) \delta y(\theta_i)\right]\left(\widehat{\psi}\,(\theta_i -, z)\right); \\ \widehat{\psi}(0,z) = z \in \mathbb{R}^n,\ i \in \{0,1,\ldots,N-1\}. \end{cases}$$

Remark 2. Under the hypotheses (1) and (2), a solution for Problem (I) will be

$$(24)\quad \psi(t,x) = \widehat{\psi}\left(t, \widehat{z}(t,x)\right), \quad t \in [\theta_i, \theta_{i+1})\,,\ x \in \mathbb{R}^n,\ i \in \{0,1,\ldots,N-1\},$$

where $\left\{\widehat{\psi}(t,z)\right\}$ is defined in Lemma 2, and the continuous and $\mathcal{F}_1^t$-adapted process $\{\widehat{z}(t,x)\}$ is given by

$$(25)\qquad \widehat{z}(t,x) \overset{\text{def}}{=} G(-w(t))[x] \overset{\text{not}}{=} H(w(t))[x], \quad t \in [0,T],\ x \in \mathbb{R}^n.$$

An application of the standard rule of stochastic derivation leads us to the following SDE

$$(26)\quad \mathrm{d}_t \widehat{z}(t,x) = \partial_\sigma \{H(\sigma)[x]\}\,(\sigma = w(t)) \cdot \mathrm{d}w(t) + \frac{1}{2}\,\partial_\sigma^2 \{H(\sigma)[x]\}\,(\sigma = w(t))\mathrm{d}t,$$

for any $t \in [0,T]$, $x \in \mathbb{R}^n$.

On the other hand, using the identities $H(\sigma) \circ G(\sigma)[\lambda] = \lambda \in \mathbb{R}^n$ and $\lambda = H(\sigma)[x](x = G(\sigma)[\lambda])$, we get

$$(27)\qquad \begin{cases} \partial_\sigma H(\sigma)[x] \quad = -\partial_x \{H(\sigma)[x]\} \cdot g(x),\ \sigma \in \mathbb{R},\ x \in \mathbb{R}^n; \\ \partial_\sigma^2 \{H(\sigma)[x]\} = \partial_\sigma \{\partial_\sigma \{H(\sigma)[x]\}\} = \partial_\sigma \{-\partial_x \{H(\sigma)[x]\} \cdot g(x)\} = \\ \qquad\qquad\qquad = \partial_x \{\partial_x \{H(\sigma)[x]\} \cdot g(x)\} \cdot g(x),\ \sigma \in \mathbb{R},\ x \in \mathbb{R}^n. \end{cases}$$

Combining (26) and (27), we are led to the SPDE of parabolic type satisfied by the continuous process $\{\widehat{z}(t,x)\}$ as follows

LEMMA 3. *Assume* $g \in \left(C_b \cap C_b^1 \cap C^2\right)(\mathbb{R}^n;\mathbb{R}^n)$ *and define the continuous process* $\widehat{z}(t,x) = G\left(-w(t)\right)[x] \overset{not}{=} H(w(t))[x]$, $t \in [0,T]$, $x \in \mathbb{R}^n$ (*see* (25)). *Then, the following SPDE of parabolic type hold true*

$$(28)\qquad \begin{cases} \mathrm{d}_t \widehat{z}(t,x) + [\partial_x \widehat{z}(t,x) \cdot g(x)]\,\widehat{\circ}\mathrm{d}w(t) = 0, \quad t \in [0,T],\ x \in \mathbb{R}^n; \\ \widehat{z}(0,x) = x, \end{cases}$$

where the Fisk-Stratanovich integral "$\widehat{\circ}$" is computed by the formula in (6)*.*

The evolution of $\psi(t,x) \overset{\text{def}}{=} \widehat{\psi}\,(t,\widehat{z}(t,x))$, $t\in[0,T]$ will be described in

LEMMA 4. *Under the hypotheses* (1) *and* (2)*, consider the piecewise continuous and* $\mathcal{F}^t$*-adapted process*

$$\left\{\psi(t,x) \overset{\text{def}}{=} \widehat{\psi}\,(t,\widehat{z}(t,x)) :\ t\in[0,T],\ x\in\mathbb{R}^n\right\}, \tag{29}$$

where $\{\widehat{\psi}(t,z)\}$ *is constructed in Lemma* 2 *and* $\{\widehat{z}(t,x)\}$ *is described in Lemma* 3*.*

Then, the following nonlinear system of SPDEs hold true

$$\begin{cases} \mathrm{d}_t\psi(t,x)+\partial_z\widehat{\psi}\,(t,\widehat{z}(t,x))\,f_1\,(\widehat{z}(t,x))\,\varphi_1\,(\psi(t,x))\,\mathrm{d}t+ \\ \qquad\qquad +\left[\partial_x\psi(t,x)\cdot g(x)\right]\widehat{\circ}\,\mathrm{d}w(t)=0,\ t\in[\theta_i,\theta_{i+1})\,; \\ \psi\,(\theta_i,x)\ =\widehat{\psi}\,(\theta_i,\widehat{z}\,(\theta_i,x))=F_2\left[-\varphi_2\,(\psi\,(\theta_i-,x))\,\delta y\,(\theta_i)\right](\psi\,(\theta_i-,x))\,; \\ \psi(0,x)\ \ =x\in\mathbb{R}^n,\ i\in\{1,2,\ldots,N-1\}, \end{cases} \tag{30}$$

where the Fisk-Stratanovich integral "$\widehat{\circ}$" is computed by the formula in (6)*.*

3. MAIN RESULTS (SOLUTIONS FOR PROBLEMS (I) AND (II))

With the same notations of Section 2, a complete description of the piecewise continuous process $\psi(t,x) \overset{\text{def}}{=} \widehat{\psi}\,(t,\widehat{z}(t,x))$, $t\in[0,T]$, $x\in\mathbb{R}^n$ (see Lemma 4) will be given similarly as in SPDE's formula (5) mentioned in Problem (I).

THEOREM 1 (solution for Problem (I)). *Under the hypotheses* (1) *and* (2)*, consider the piecewise continuous and* $\mathcal{F}^t=\mathcal{F}_1^t\times\mathcal{F}_2$*-adapted process* $\psi(t,x)=\widehat{\psi}\,(t,\widehat{z}(t,x))$*,* $t\in[0,T]$*,* $x\in\mathbb{R}^n$*, defined in Lemma* 4*. Then, the nonlinear system of SPDEs given in* (30) *is equivalent with the system of SPDEs of parabolic type defined in* (5) *of Problem* (I)*.*

Proof. By hypothesis, the conclusions of Lemma 4 are valid and the nonlinear system (30) can be replaced by (5) of Problem (I) provided the following computation is performed. Notice that the middle term of (30) must be rewritten as in (5) and using the hypothesis (1), we get the following identities

$$\left[\partial_x\widehat{z}(t,x)\right]^{-1} f_1\,(\widehat{z}(t,x)) = f_1(x),\quad t\in[0,T],\ x\in\mathbb{R}^n. \tag{31}$$

Then, rewrite the middle term of (30) as follows

$$\partial_z\widehat{\psi}\,(t,\widehat{z}(t,x))\,f_1\,(\widehat{z}(t,x)) = \partial_x\psi(t,x)\left[\partial_x\widehat{z}(t,x)\right]^{-1} f_1\,(\widehat{z}(t,x)) \tag{32}$$

and (31) leads us to

$$\partial_z \widehat{\psi}\,(t, \widehat{z}(t,x))\, f_1\,(\widehat{z}(t,x)) = \partial_x \psi(t,x) f_1(x), \quad t \in [0,T],\ x \in \mathbb{R}^n, \tag{33}$$

which allows us to replace (30) by (5) of Problem (I).

The proof is complete. □

A solution of Problem (II) will rely on the integral representation of solutions satisfying the SDE (10). More precisely, it holds

$$\begin{aligned} z_\psi(T;t,x) =& G(w(T)-w(t)) \circ F_2\,(\varphi_2\,(\psi(t,x))\,[y(T)-y(t)]) \circ \\ & \circ F_1\,(\varphi_1\,(\psi(t,x))\,(T-t))\,[x], \end{aligned} \tag{34}$$

for any $0 \le t < T$, $x \in \mathbb{R}^n$, where $\lambda = \psi(t,x) \in \mathbb{R}^n$ has been obtained in Theorem 1.

Remark 3. Notice that $z_\psi(T;t,x)$ in (34) and any $h\,(z_\psi(T;t,x))$, $h \in C_p^2\,(\mathbb{R}^n)$ are continuous mappings of the following independent random vectors $z_1 := w(T) - w(t)$ (z_1 is independent of $\mathcal{F}^t = \mathcal{F}_1^t \times \mathcal{F}_2$) and respectively, $z_2 := \psi(t,x) \in \mathbb{R}^n$ (z_2 is $\mathcal{F}^t$-adapted). It suggests to compute the conditioned mean values

$$\begin{aligned} v_i(t,x) = E_1\,\{h\,(z_\psi(T;t,x)) \mid \psi(t,x)\}\,, \quad t \in [\theta_i, \theta_{i+1})\,,\ x \in \mathbb{R}^n, \\ i \in \{0,1,\ldots,N-1\} \end{aligned} \tag{35}$$

(see (10) of Problem (II)), using the parameterized functional $u_i(t,x;\lambda)$ given by

$$u_i(t,x;\lambda) = E_1 h\,(z_\lambda(T;t,x))\,, \quad t \in [\theta_i, \theta_{i+1})\,,\ x \in \mathbb{R}^n,\ \lambda \in \mathbb{R}^n. \tag{36}$$

Here, $z_\lambda(T;t,x)$ is obtained from (34) by replacing the random vector $z_2 = \psi(t,x)$ with $\lambda \in \mathbb{R}^n$.

Using (36) we write (35) as follows

$$v_i(t,x) = u_i\,(t,x;\psi(t,x))\,,\ t \in [\theta_i, \theta_{i+1})\,, x \in \mathbb{R}^n,\ i \in \{0,1,\ldots,N-1\}. \tag{37}$$

In addition, $\{u_i(t,x;\lambda) : t \in [\theta_i, \theta_{i+1})\,,\ x \in \mathbb{R}^n\}$ satisfies a backward parabolic equation (Kolmogorov equation), for each parameter $\lambda \in \mathbb{R}^n$. In particular, for $i = N-1$ we get

$$\begin{cases} u_{N-1}(T,x;\lambda) = h(x), & x \in \mathbb{R}^n; \\ \partial_t u_{N-1}(t,x;\lambda) + L_\lambda\,(u_{N-1})\,(t,x;\lambda) = 0, & t \in [\theta_{N-1}, T)\,,\ x \in \mathbb{R}^n. \end{cases} \tag{38}$$

Here, the parabolic operator L_λ is defined by

$$L_\lambda(u)(x) = \langle \partial_x u(x),\ \varphi_1(\lambda) f_1(x) \rangle + \frac{1}{2} \langle [\partial_x \langle \partial_x u(x),\ g(x) \rangle]\,,\ g(x) \rangle. \tag{39}$$

Generally, $u_i(t,x;\lambda)$ satisfies a similar Kolmogorov equation

$$(40)\qquad \begin{cases} \partial_t u_i(t,x;\lambda) + L_\lambda(u_i)(t,x;\lambda) = 0,\ t\in[\theta_i,\theta_{i+1}),\ x\in\mathbb{R}^n; \\ u_i(\theta_{i+1}-,x;\lambda) = E_1 h\left(z_\lambda\left(T;\theta_{i+1}-,x\right)\right), \end{cases}$$

where $z_\lambda(T;\theta_{i+1}-,x) = F_2\left(\delta y\left(\theta_{i+1}\right)\varphi_2(\lambda)\right)\left[z_\lambda\left(T;\theta_{i+1},x\right)\right]$.

We conclude remarks and computations obtained above as the solution of Problem (II).

THEOREM 2 (solution of Problem (II)). *Under the hypotheses* (1) *and* (2), *consider the piecewise continuous and* $\mathcal{F}^t = \mathcal{F}_1^t \times \mathcal{F}_2$*-adapted process* $\{\psi(t,x)\}$ *defined in Theorem* 1. *Associate the functionals* $\{v_i(t,x\},\ i\in\{0,1,\ldots,N-1\}$, *as in* (10) (*see Problem* (II)). *Consider the finite sequence of parameterized backward parabolic equations and their solutions* $u_i(t,x;\lambda)$, $t\in[\theta_i,\theta_{i+1})$, $x\in\mathbb{R}^n$, $i\in\{0,1,\ldots,N-1\}$, *as in* (36), (38) *and* (39). *Then, a solution of Problem* (II) *is given by* (*see* (37))

$$(41)\quad v_i(t,x) = u_i(t,x;\psi(t,x))),\ t\in[\theta_i,\theta_{i+1}),\ x\in\mathbb{R}^n,\ i\in\{0,1,\ldots,N-1\}.$$

Final Remark. The analysis given here relies on the assumption

$$\{g,f_1,f_2\}\subseteq C_b\left(\mathbb{R}^n,\mathbb{R}^n\right)$$

(g, f_1 and f_2 are bounded functions).

In the case that $g\in\left(C_b^1\cap C^2\right)\left(\mathbb{R}^n,\mathbb{R}^n\right)$ and $g\notin C_b\left(\mathbb{R}^n,\mathbb{R}^n\right)$ then, an arbitrary stopping time

$$\widehat{\tau} = \inf\{t\in[0,T] : |w(t)|\ge N\}$$

must be introduced. The solution of Problem (I) will satisfy a nonlinear SPDE using the arbitrary stopping time $\widehat{\tau}$ while the Fisk-Stratonovich integral "$\widehat{\circ}$" in (5) must be computed as follows

$$h(t,x)\widehat{\circ}\mathrm{d}w(t) = \chi_{\widehat{\tau}}(t)h(t,x)\cdot\mathrm{d}w(t) - \frac{1}{2}\left[\chi_{\widehat{\tau}}(t)\partial_x h(t,x)g(x)\right]\mathrm{d}t,$$

where $\chi_{\widehat{\tau}}(t)=1$, for $\widehat{\tau}\ge t$, and $\chi_{\widehat{\tau}}(t)=0$ for $\widehat{\tau}<t$.

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