SOLUTION OF GENERALIZED DRINFELD-SOKOLOV EQUATIONS BY HOMOTOPY PERTURBATION AND VARIATIONAL ITERATION METHODS

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In this article, we find an approximate analytical solution of the system of generalized Drinfeld-Sokolov Equations (gDSE) by applying two relatively new iterative methods, i.e., the Homotopy Perturbation Method (HPM) and Variational Iteration Method (VIM). From the obtained results, we observed that both methods are effective and quite accurate for solving system of partial differential equations. The most attractive features of these methods lie in their simplicity and easy in implementation. The results obtained from both methods are compared with multiple soliton-like solutions. It is observed that the computed results are in good agreement with the published reference solutions.

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Key words: generalized Drinfeld-Sokolov equations, homotopy perturbation method, variational iteration method.

1. INTRODUCTION

In recent years, application of approximate analytical techniques to solve both linear and nonlinear problems is the most attractive field of research in mathematics and engineering sciences. Researchers are looking for the approximate solutions of the problems which do not have an exact solution or it is very complicated and tedious to find their exact solution. Numerous approximate analytical methods have been discussed in literature, some of them being Adomian decomposition method [1], homotopy analysis method [2], tanh-coth method [3], energy balance method [4], homotopy perturbation method [5, 6, 7, 8], variational iteration method [9, 10, 11, 12] etc. In this article, we implement HPM and VIM to obtain approximate analytical solution of the generalized Drinfeld-Sokolov equations. The advantage of both the methods over other methods is in their accuracy, simplicity and easy approach to the required solution. Recently, these methods have been successfully applied to solve many other linear and nonlinear problems [5, 12]. The main purpose

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of this article is to provide approximate analytical solution to the following problem

$$\phi_t + \phi_{xxx} - 6\phi\phi_x - 6\left(\psi^{\alpha}\right)_x = 0, \quad \psi_t - 2\psi_{xxx} + 6\phi\psi_x = 0 \tag{1}$$

with initial conditions

$$\phi\left(x,0\right) = \frac{-b^2 - 4k^4}{4k^2} + 2k^2\tanh^2\left(kx\right), \tag{2}$$

$$\psi\left(x,0\right) = b\tanh\left(kx\right) \tag{3}$$

by using proposed methods. The multiple soliton-like solution [14] of the system for $\alpha = 2$ is given by

$$\phi\left(x,t\right) = \frac{-b^2 - 4k^4}{4k^2} + 2k^2\tanh^2\left(kx + \frac{3b^2 + 4k^4}{2k}t\right), \tag{4}$$

$$\psi\left(x,t\right) = b\ \tanh\left(kx + \frac{3b + 4k^4}{2k}t\right). \tag{5}$$

2. BASIC IDEA OF HOMOTOPY PERTURBATION METHOD

To explain this method, we consider the following nonlinear differential equation

$$A(v) - g(r) = 0, \quad r \in \Omega, \tag{6}$$

with the boundary conditions

$$B\left(v, \frac{\partial v}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{7}$$

where A is a differential operator, B is boundary operator, $g(r)$ is known analytic function, and Γ is the boundary of the domain Ω. Now, we can divide A into two parts, $L(v)$ and $N(v)$, where $L(v)$ is linear operator and $N(v)$ is nonlinear operator. Therefore, Eq. (6) can be written as

$$L(v) + N(v) - g(r) = 0. \tag{8}$$

By using homotopy technique, we make a homotopy $w(r,q) : \Omega \times [0,1] \to R$ which satisfies

$$H\left(w,q\right) = (1-q)\left[L(w) - L\left(v_o\right)\right] + q\left[A(w) - g(r)\right] = 0, \quad r \in \Omega, \tag{9}$$

where $q \in [0,1]$ and is known as embedding parameter, and v_o is an initial approximation of the Eq. (6) which satisfies the initial condition. From Eq. (9), for $q = 0$ and $q = 1$, we will have

$$H(w,0) = L(w) - L(v_0) = 0, \tag{10}$$

$$H(w,1) = A(w) - g(r) = 0. \tag{11}$$

The variation of q from zero to one is just that of $w(r, q)$ from v_o to $v(r)$. In topology, this is called deformation, while the terms $L(w) - L(v_o)$ and $A(w) - g(r)$ are called homotopy. By using HPM, the solution of Eq. (9) can be written as a power series in q

$$w = w_o + qw_1 + q^2 w_2 + \cdots . \tag{12}$$

Now, setting $q \to 1$, Eq. (12) yields

$$v = \lim_{q\to 1} w = w_o + w_1 + w_2 + \cdots . \tag{13}$$

The unknowns $w_1, w_2, w_3, \ldots$ can be calculated by equating the coefficients of like powers of q in Eq. (9).

3. BASIC IDEA OF VARIATIONAL ITERATION METHOD

This method is based on the general Lagrange multiplier method. It was first proposed by Ji-Huan He as a modification of the general Lagrange multiplier method used in optimization. He solved a wide class of linear and nonlinear differential equations and found that the VIM is an effective, easy and accurate method for finding their approximate solutions [13].

To explain this method, we consider the following general nonlinear differential equation

$$Lw + Nw = f(t), \tag{14}$$

where L is a linear operator, N a nonlinear operator and $f(t)$ is the homogeneous term. According to the VIM procedure, we can construct a correction functional for the given problem as follows

$$w_{n+1}(x) = w_n(x) + \int_0^t \lambda(\xi)\,(Lw_n(\xi) + N\widetilde{w}_n(\xi) - g(\xi))\mathrm{d}\xi, \tag{15}$$

where λ is a Lagrange multiplier, which can be identified optimally through variational iteration theory. The subscript n denotes the n^{th} approximation, w_n is the n^{th} approximate solution, and $\widetilde{w}_n$ denotes a restricted variation, i.e., $\delta\widetilde{w}_n = 0$. The successive approximations $w_{n+1}(x, t), n \geq 0$ of the solution $w(x, t)$ will be obtained by using suitably chosen function w_0 as an initial guess. Finally, the series solution is given as

$$w(x, t) = \lim_{n\to\infty} w_n(x, t). \tag{16}$$

4. SOLUTION OF GENERALIZED DRINFELD-SOKOLOV EQUATIONS

We have to provide an approximate analytical solution of the following problem

$$\phi_t + \phi_{xxx} - 6\phi\phi_x - 6\left(\psi^{\alpha}\right)_x = 0, \quad \psi_t - 2\psi_{xxx} + 6\phi\psi_x = 0. \tag{17}$$

with initial conditions

$$\phi\left(x,0\right) = \frac{-b^2 - 4k^4}{4k^2} + 2k^2 \tanh^2\left(kx\right), \tag{18}$$

$$\psi\left(x,0\right) = b\,\tanh\left(kx\right). \tag{19}$$

The analytical solution of the above problem for $\alpha = 2$ is obtained in [14], and given as following

$$\phi\left(x,t\right) = \frac{-b^2 - 4k^4}{4k^2} + 2k^2 \tanh^2\left(kx + \frac{3b^2 + 4k^4}{2k}t\right), \tag{20}$$

$$\psi\left(x,t\right) = b\;\tanh\left(kx + \frac{3b + 4k^4}{2k}t\right). \tag{21}$$

4.1. APPLICATION OF HOMOTOPY PERTURBATION METHOD

Let us consider Eq. (17) with $\alpha = 2$

$$\phi_t + \phi_{xxx} - 6\phi\phi_x - 6\left(\psi^2\right)_x = 0, \tag{22}$$

$$\psi_t - 2\psi_{xxx} + 6\phi\psi_x = 0. \tag{23}$$

We will use initial conditions as an initial approximation

$$\phi_0\left(x,t\right) = \frac{-b^2 - 4k^4}{4k^2} + 2k^2 \tanh^2\left(kx\right), \quad \psi_0\left(x,t\right) = b\,\tanh\left(kx\right). \tag{24}$$

The constructed homotopy equations for Eq. (22) and Eq. (23) using Eq. (9) are given bellow

$$\left(1-p\right)\left(\dot{v} - \dot{\phi}_0\right) + p\left(\dot{v} + v^{(3)} - 6vv^{(1)} - 6\left(w^2\right)^{(1)}\right) = 0, \tag{25}$$

$$\left(1-q\right)\left(\dot{w} - \dot{\psi}_0\right) + q\left(\dot{w} - 2w^{(3)} + 6vw^{(1)}\right) = 0, \tag{26}$$

where

$$\dot{v} = \frac{\partial v}{\partial t},\;\; \dot{w} = \frac{\partial w}{\partial t},\;\; v^{(1)} = \frac{\partial v}{\partial x},\;\; w^{(1)} = \frac{\partial w}{\partial x},\;\; v^{(3)} = \frac{\partial^3 v}{\partial x^3},\;\; w^{(3)} = \frac{\partial^3 w}{\partial x^3}.$$

Using Eq. (12) in Eq. (25) and comparing the coefficients of like powers of p, we get the following system of equations

$$\dot{v}_0 - \dot{\phi}_0 = 0, \tag{27}$$

$$\dot{v}_1 + v_0^{(3)} - 6v_0v_0^{(1)} - 6\left(w_0^2\right)^{(1)} = 0, \tag{28}$$

$$\dot{v}_2 + v_1^{(3)} - 6v_0v_1^{(1)} - 6v_1v_0^{(1)} - 12\left(w_0w_1\right)^{(1)} = 0, \tag{29}$$

$$\dot{v}_3 + v_2^{(3)} - 6v_0v_2^{(1)} - 6v_1v_1^{(1)} - 6v_2v_0^{(1)} - \left(2w_0w_2 + w_1^2\right)^{(1)} = 0, \tag{30}$$

$$\dot{v}_4 + v_3^{(3)} - 6\left(v_0v_3^{(1)} - v_1v_2^{(1)} - v_2v_1^{(1)} - v_3v_0^{(1)} + 2w_0w_3 + 2w_1w_2\right) = 0, \tag{31}$$

and so on. On solving Eq. (27) to Eq. (31), we have

$$v_0 = \frac{-b^2 - 4k^4}{4k^2} + 2k^2\tanh^2(kx),$$
$$v_1 = -2kt\tanh(kx)\left(-1 + \tanh^2(kx)\right)\left(4k^4 + 3b^2\right),$$
$$v_2 = \frac{t^2}{2}\left(-1 + \tanh^2(kx)\right)\left(4k^4 + 3b^2\right)\left[-4k^4 + \tanh^2(kx)\left(12k^4 + 9b^2\right) - 3b^2\right],$$
$$v_3 = -\frac{t^3}{12k^2}\begin{bmatrix} \left(27b^6 + 180k^4b^4 + 336k^8b^2 + 192k^{12}\right) \\ +\tanh(kx)\left(1296k^5b^4 + 2304k^9b^2 + 1280k^{13} + 216kb^6\right) \\ -\tanh^2(kx)\left(1920b^2k^8 + 1152k^{12} + 108b^6 + 936k^4b^4\right) \\ -\tanh^3(kx)\left(4320k^5b^4 + 8640k^9b^2 + 5120k^{13} + 540kb^6\right) \\ +\tanh^4(kx)\left(2112k^{12} + 81b^6 + 3312k^8b^2 + 1404k^4b^4\right) \\ +\tanh^5(kx)\left(9792k^9b^2 + 4320k^5b^4 + 6144k^{13} + 324kb^6\right) \\ -\tanh^6(kx)\left(1152k^{12} + 648k^4b^4 + 1728k^8b^2\right) \\ -\tanh^7(kx)\left(4356k^9b^2 + 1296k^5b^4 + 2304k^{13}\right) \end{bmatrix},$$

and so on. Similarly, we can calculate $v_4, v_5, v_6\ldots$ up to required degree of accuracy. Hence, the required solution for Eq. (22) is

$$\phi(x,t) = v(x,t) = v_0 + v_1 + v_2 + v_3 + v_4 + \cdots. \tag{32}$$

Now, using Eq. (12) in Eq. (26) and comparing the coefficients of like powers of q, we get the following system of equations

$$\dot{w}_0 + \dot{\psi}_0 = 0, \tag{33}$$

$$\dot{w}_1 - 2w_0^{(3)} + 6v_0w_0^{(1)} = 0, \tag{34}$$

$$\dot{w}_2 - 2w_1^{(3)} + 6v_0w_1^{(1)} + 6v_1w_0^{(1)} = 0, \tag{35}$$

$$\dot{w}_3 - 2w_2^{(3)} + 6v_0w_2^{(1)} + 6v_1w_1^{(1)} + 6v_2w_0^{(1)} = 0, \tag{36}$$

$$\dot{w}_4 - 2w_3^{(3)} - 6v_0w_3^{(1)} - 6v_1w_2^{(1)} - 6v_2w_1^{(1)} - 6v_3w_0^{(1)} = 0, \tag{37}$$

and so on. On solving Eq. (33) to Eq. (37), we have

$$w_0 = b \tanh(kx),$$
$$w_1 = -\frac{t}{2k}\left(-1+\tanh^2(kx)\right)\left(4k^4+3b^2\right),$$
$$w_2 = \frac{bt^2}{4k^2}\tanh(kx)\left(-1+\tanh^2(kx)\right)\left(4k^4+3b^2\right)^2,$$
$$w_3 = -\frac{bt^3}{24k^3}\begin{bmatrix} \left(108k^4b^4+144k^8b^4+64k^{12}+27b^6\right) \\ -\tanh^2(kx)\left(432k^4b^4+256k^{12}+576k^8b^2+108b^6\right) \\ +\tanh^4(kx)\left(192k^{12}+432k^8b^2+324k^4b^4+81b^6\right) \end{bmatrix},$$

and so on. Similarly, we can calculate $w_4, w_5, w_6 \ldots$ up to required degree of accuracy. Hence, the required solution for Eq. (23) is

$$\psi(x,t) = w_0 + w_1 + w_2 + w_3 + w_4 + \cdots . \tag{38}$$

4.2. NUMERICAL RESULTS

Following are the numerical results for $b = 0.001$ and $k = 0.01$. We have presented error analysis of an approximate solution up to the 4^{th} order using HPM.

Table 1. The comparison of the results of the HPM with the analytical solution $\phi(x,t)$

t_n/x_n	0.2	0.4	0.6	0.8	1.0
0.2	1×10^{-12}	3×10^{-12}	4×10^{-12}	5×10^{-12}	7×10^{-12}
0.4	1×10^{-12}	3×10^{-12}	4×10^{-12}	5×10^{-12}	7×10^{-12}
0.6	1×10^{-12}	3×10^{-12}	4×10^{-12}	5×10^{-12}	7×10^{-12}
0.8	1×10^{-12}	3×10^{-12}	4×10^{-12}	5×10^{-12}	7×10^{-12}
1.0	1×10^{-12}	3×10^{-12}	4×10^{-12}	5×10^{-12}	7×10^{-12}

Table 2. The comparison of the results of the HPM with the analytical solution $\psi(x,t)$

t_n/x_n	0.2	0.4	0.6	0.8	1.0
0.2	0.00000	1×10^{-17}	1×10^{-17}	0.00000	3×10^{-17}
0.4	0.00000	0.00000	0.00000	0.00000	2×10^{-17}
0.6	1×10^{-17}	0.00000	1×10^{-17}	0.00000	3×10^{-17}
0.8	0.00000	1×10^{-17}	1×10^{-17}	0.00000	3×10^{-17}
1.0	0.00000	1×10^{-17}	0.00000	0.00000	0.00000

4.3. APPLICATION OF VARIATIONAL ITERATION METHOD

Again consider Eq. (17) with $\alpha = 2$;

$$\phi_t + \phi_{xxx} - 6\phi\phi_x - 6\left(\psi^2\right)_x = 0, \tag{39}$$

$$\psi_t - 2\psi_{xxx} + 6\phi\psi_x = 0. \tag{40}$$

Here also, we will use initial conditions as an initial approximation

$$\phi_0(x,t) = \frac{-b^2 - 4k^4}{4k^2} + 2k^2 \tanh^2(kx), \quad \psi_0(x,t) = b\ \tanh(kx). \tag{41}$$

The constructed correction functionals for Eq. (22) and Eq. (23) are given bellow

$$\phi_{n+1}(x,t) = \phi_n + \int_0^t \lambda_1(\zeta)\left(\frac{\partial\phi_n}{\partial\zeta} + \frac{\partial^3\tilde{\phi}_n}{\partial x^3} - 6\phi_n\frac{\partial\tilde{\phi}_n}{\partial x} - 6\frac{\partial\tilde{\psi}_n^2}{\partial x}\right)\mathrm{d}\zeta, \tag{42}$$

$$\psi_{n+1}(x,t) = \psi_n + \int_0^t \lambda_2(\zeta)\left(\frac{\partial\psi_n}{\partial\zeta} - 2\frac{\partial^3\tilde{\psi}_n}{\partial x^3} + 6\phi_n\frac{\partial\tilde{\psi}_n}{\partial x}\right)\mathrm{d}\zeta, \tag{43}$$

where λ_1 and λ_2 are Lagrange multipliers and $\tilde{\phi}_n(x,\zeta)$ and $\tilde{\psi}_n(x,\zeta)$ are restricted variations, i.e., $\delta\tilde{\phi}_n(x,t) = 0$ and $\delta\tilde{\psi}_n(x,\zeta) = 0$. Now, taking variation δ both sides of Eq. (42) and Eq. (43), we get

$$\delta u_{n+1}(x,t) = \delta\phi_n + \delta\int_0^t \lambda_1(\zeta)\left(\frac{\partial\phi_n}{\partial\zeta} + \frac{\partial^3\tilde{\phi}_n}{\partial x^3} - 6\phi_n\frac{\partial\tilde{\phi}_n}{\partial x} - 6\frac{\partial\tilde{\psi}^2}{\partial x}\right)\mathrm{d}\zeta, \tag{44}$$

$$\delta\psi_{n+1}(x,t) = \delta\psi_n + \delta\int_0^t \lambda_2(\zeta)\left(\frac{\partial\psi_n}{\partial\zeta} - 2\frac{\partial^3\tilde{\psi}_n}{\partial x^3} + 6\phi_n\frac{\partial\tilde{\psi}_n}{\partial x}\right)\mathrm{d}\zeta. \tag{45}$$

From Eq. (44) and Eq. (45), using restricted variations, stationary conditions and integration by parts, the Lagrange multipliers are found to be $\lambda_1 = -1$, $\lambda_2 = -1$. Using these values of λ_1 and λ_2 in Eq. (42) and Eq. (43) respectively, we get the following iterative formulas

$$\phi_{n+1}(x,t) = \phi_n - \int_0^t \left(\frac{\partial\phi_n}{\partial\zeta} + \frac{\partial^3\tilde{\phi}_n}{\partial x^3} - 6\phi_n\frac{\partial\tilde{\phi}_n}{\partial x} - 6\frac{\partial\tilde{\psi}_n^2}{\partial x}\right)\mathrm{d}\zeta, \tag{46}$$

$$\psi_{n+1}(x,t) = \psi_n - \int_0^t \left(\frac{\partial\psi_n}{\partial\zeta} - 2\frac{\partial^3\tilde{\psi}_n}{\partial x^3} + 6\phi_n\frac{\partial\tilde{\psi}_n}{\partial x}\right)\mathrm{d}\zeta. \tag{47}$$

For $n = 0, 1$, Eq. (46) gives following results

$$\phi_1 = -\frac{1}{4k^2}(b^2 + 4k^4) + 2k^2\tanh^2(kx) - \\ -2kt\tanh(kx)(-1 + \tanh^2(kx))(4k^4 + 3b^2),$$

$$\phi_2 = -\frac{1}{4k^2}(b^2 + 4k^4) + 2k^2\tanh^2(kx) - \\ -2kt\tanh(kx)(-1 + \tanh^2(kx))(4k^4 + 3b^2)$$

$$-\frac{1}{2k}t^2\begin{bmatrix} -\left(24b^2k^5 + 16k^9 + 9b^4k\right) \\ +\tanh(kx)\left(36tb^6 - 48tb^4k^4 - 320tb^2k^8 - 256tk^{12}\right) \\ +\tanh^2(kx)\left(64k^9 + 96b^2k^5 + 36b^4k\right) \\ +\tanh^3(kx)\left(1280tk^{12} - 72tb^6 + 528tb^4k^4 + 1792tb^2k^8\right) \\ -\tanh^4(kx)\left(27b^4k + 72b^2k^5 + 48k^9\right) \\ -\tanh^5(kx)\left(2624tb^2k^8 + 1792tk^{12} + 912tb^4k^4 - 36tb^6\right) \\ +\tanh^7(kx)\left(768tk^{12} + 432tb^4k^4 + 1152tb^2k^8\right) \end{bmatrix}.$$

In the same way, we can find $\phi_3, \phi_4, \ldots$ for $n = 2, 3, \ldots$. Similarly, for $n = 0, 1$, Eq. (47) gives the following results

$$\psi_1 = b\ \tanh(kx) - \frac{b}{2k}t\left(-1 + \tanh^2(kx)\right)\left(4k^4 + 3b^2\right),$$

$$\psi_2 = \frac{b}{4k^2}\begin{bmatrix} \left(8tk^5 + 6tkb^2\right) + \tanh(kx)\left(4k^2 - 16t^2k^8 - 9t^2b^4 - 24t^2b^2k^4\right) \\ +\tanh^2(kx)\left(256t^3k^{11} - 6tb^2k + 384t^3b^2k^7 + 144t^3b^4k^3 - 8tk^5\right) \\ +\tanh^3(kx)\left(9t^2b^4 + 24t^2b^2k^4 + 16t^2k^8\right) \\ -\tanh^4(kx)\left(768t^3b^2k^7 + 288t^3b^4k^3 + 512t^3k^{11}\right) \\ +\tanh^6(kx)\left(144t^3b^4k^3 + 256t^3k^{11} + 384t^3b^2k^7\right) \end{bmatrix}.$$

Continuing in the same manner we can calculate $\psi_3, \psi_4, \ldots$, for $n = 2, 3, \ldots$.

4.4. NUMERICAL RESULTS

Following are the numerical results for $b = 0.001$, $k = 0.01$. We have presented error analysis of an approximate solution up to $\phi_2(x, t)$ and $\psi_2(x, t)$ by VIM.

Table 3. The comparison of the results of the VIM with the analytical solution $\phi(x, t)$

t_n/x_n	0.2	0.4	0.6	0.8	1.0
0.2	1.85×10^{-13}	2.61×10^{-13}	3.36×10^{-13}	4.27×10^{-14}	3.79×10^{-13}
0.4	1.85×10^{-13}	2.61×10^{-13}	3.36×10^{-13}	4.27×10^{-14}	3.79×10^{-13}
0.6	1.85×10^{-13}	2.61×10^{-13}	3.36×10^{-13}	4.27×10^{-14}	3.79×10^{-13}
0.8	1.85×10^{-13}	2.61×10^{-13}	3.36×10^{-13}	4.27×10^{-14}	3.79×10^{-13}
1.0	1.85×10^{-13}	2.61×10^{-13}	3.36×10^{-13}	4.27×10^{-14}	3.79×10^{-13}

Table 4. The comparison of the results of the VIM with the analytical solution $\psi(x,t)$

t_n/x_n	0.2	0.4	0.6	0.8	1.0
0.2	1.40×10^{-16}	4.70×10^{-16}	4.90×10^{-16}	1.70×10^{-16}	1.38×10^{-15}
0.4	2.90×10^{-16}	8.60×10^{-16}	7.20×10^{-16}	2.60×10^{-16}	1.59×10^{-15}
0.6	4.40×10^{-16}	1.25×10^{-15}	9.60×10^{-16}	6.80×10^{-16}	1.79×10^{-15}
0.8	6.00×10^{-16}	1.65×10^{-15}	1.19×10^{-15}	1.11×10^{-15}	2.00×10^{-15}
1.0	8.00×10^{-16}	2.00×10^{-15}	1.40×10^{-15}	1.60×10^{-15}	2.20×10^{-15}

5. CONCLUSION

In the present work, we have applied homotopy perturbation method and variational iteration method to find the approximate analytical solution to the generalized Drinfeld-Sokolov equations. As a conclusion, we can note that the results obtained are satisfactory for both methods. The results we obtained here are in a very good agreement with that of analytical solution. The applied methods are simple and straightforward in their implementation and provide reasonably good results, which is clear from the tables [1, 2, 3, 4]. The results show that HPM and VIM are powerful mathematical tools for solving both linear and nonlinear partial differential equations, and therefore, can be widely applied in solving science and engineering problems.

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