ON SOME INEQUALITIES AND HANKEL MATRICES INVOLVING PELL, PELL-LUCAS NUMBERS

SERPIL HALICI

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In [4], the authors defined Toeplitz and Hankel matrices with Pell numbers and gave bounds for the spectral norms of them. In this study, we define Hankel matrices involving the Pell, Pell-Lucas and modified Pell sequences and investigate some properties of them. Moreover, we calculate certain norms of above mentioned matrices. Also, we give upper and lower bounds for spectral norm of Hankel matrix involving Pell, Pell-Lucas, modified Pell numbers.

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1. INTRODUCTION

In the recent years, some considerable works have been done on the norms of some special matrices. In particular, the norms of Toeplitz and Hankel matrices involving Fibonacci and Lucas numbers were investigated in [1] and [2]. In [10], the circulant matrices involving Fibonacci and Lucas numbers were studied and lower and upper bounds for the spectral norms of these matrices were also given. In [4], the authors gave bounds for the spectral norms of Toeplitz and Hankel matrices with Pell numbers. In this study, we define Hankel matrices involving the Pell, Pell-Lucas and modified Pell numbers of the form

(1) $$A = (a_{ij}), \quad a_{ij} = P_{i+j-1};$$

(2) $$B = (b_{ij}), \quad b_{ij} = Q_{i+j-1};$$

(3) $$C = (c_{ij}), \quad c_{ij} = q_{i+j-1};$$

respectively, and we calculate the Euclidean, column and row norms involving these numbers. Also, we give bounds for the spectral norms of the matrices involving Pell, Pell-Lucas and modified Pell numbers, that is, the bounds of spectral norm related to these numbers. We find a new formula which is different from the formula given in the reference [4], which refers to matrices with different indexing. And this formula involves not only Pell numbers but also Pell-Lucas, modified Pell numbers.

Now, we give some fundamental knowledge related to our study. The Pell sequence is defined by the following recursive relation, for $n \geq 2$

$$P_n = 2P_{n-1} + P_{n-2}, \tag{4}$$

where $P_0 = 0$ and $P_1 = 1$ [3]. And the Pell-Lucas sequence is

$$Q_n = 2Q_{n-1} + Q_{n-2}, \tag{5}$$

where $Q_0 = 2$ and $Q_1 = 2$. Similarly, the modified Pell sequence is defined as follows.

$$q_n = 2q_{n-1} + q_{n-2}, \quad q_0 = 1 = q_1. \tag{6}$$

Extension to negative values of n may be made, but here we do not care about it. If we start from $n = 0$, then the first elements of the Pell, Pell-Lucas and modified Pell sequences are given by

n	0	1	2	3	4	5	6	7	8	9	$\cdots$
P_n	0	1	2	5	12	29	70	169	408	985	$\cdots$
Q_n	2	2	6	14	34	82	198	478	1154	2786	$\cdots$
q_n	1	1	3	7	17	41	99	239	577	1393	$\cdots$

Some important elementary relationships involving P_n, Q_n and q_n follow without difficulty with the aid of the Binet formulas.

$$\sum_{k=1}^{n} P_k^2 = \frac{P_n P_{n+1}}{2}, \tag{7}$$

$$\sum_{k=1}^{n-1} P_k P_{k+1} = \frac{P_{2n+1} - 2P_{n+1}P_n - 1}{4}, \tag{8}$$

$$2P_{n-1}P_n + P_{n-1}^2 - P_n^2 = (-1)^n, \tag{9}$$

$$P_{2n+1} = P_n^2 + P_{n+1}^2, \tag{10}$$

$$P_n^2 + P_{n+1}P_{n-1} = \frac{1}{4}Q_n^2, \tag{11}$$

$$P_n^2 = \frac{1}{8}\left(Q_{2n} - 2(-1)^n\right), \tag{12}$$

$$\sum_{k=1}^{n} Q_k^2 = \frac{Q_{2n+1} + 2(-1)^n - 4}{2}, \quad \sum_{k=1}^{n} Q_{2k+1} = \frac{Q_{2n+2} - 6}{2}, \tag{13}$$

$$(P_{n+1} - P_n)^2 = 2P_n^2 + (-1)^n, \tag{14}$$

(15) $$4P_n = Q_n + Q_{n-1}, \quad P_{n+1} - P_n = q_n, \quad P_{n-1} + P_n = q_n.$$

The Hankel matrix is an $n \times n$ matrix

$$H_n = (h_{ij})_{i,j=1}^n ,$$

where $h_{ij} = h_{i+j-1}$, i.e., a matrix of the form

$$H_n = \begin{bmatrix} h_1 & h_2 & h_3 & \cdots & h_{n-1} & h_n \\ h_2 & h_3 & h_4 & \cdots & h_n & h_{n+1} \\ h_3 & h_4 & h_5 & \cdots & h_{n+1} & h_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{n-1} & h_n & h_{n+1} & \cdots & h_{2n-3} & h_{2n-2} \\ h_n & h_{n+1} & h_{n+2} & \cdots & h_{2n-2} & h_{2n-1} \end{bmatrix}.$$

Let $A = (a_{ij})$ be an $m \times n$ matrix. The Frobenius or Euclidean norm of A is defined as

$$\|A\|_F = \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 \right)^{\frac{1}{2}}$$

and also the spectral norm of A is

$$\|A\|_2 = \sqrt{\max_{1 \le i \le n} |\lambda_i|},$$

respectively, where the numbers λ_i are the eigenvalues of matrix $A^H A$. The matrix A^H is the conjugate transpose of the matrix A. The column and row norm of matrix A are defined by

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^m |a_{ij}| , \quad \|A\|_\infty = \max_{1 \le i \le m} \sum_{j=1}^n |a_{ij}| ,$$

respectively [8]. For the matrices $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$, the Hadamard product of these matrices is defined as

$$A \circ B = (a_{ij} b_{ij}) .$$

For the $m \times n$ matrix A, the following inequality is satisfied (see [8]).

(16) $$\frac{1}{\sqrt{\min(m,n)}} \|A\|_F \le \|A\|_2 \le \|A\|_F .$$

The maximum column length norm $c_1(\cdot)$ and maximum row length norm $r_1(\cdot)$ for the matrix $A = (a_{ij})_{m \times n}$ are defined by

(17) $$c_1(A) \equiv \max_j \sqrt{\sum_i |a_{ij}|^2} = \max_j \left\| [a_{ij}]_{i=1}^m \right\|_F ,$$

and

$$r_1(A) \equiv \max_i \sqrt{\sum_j |a_{ij}|^2} = \max_i \left\| [a_{ij}]_{j=1}^n \right\|_F , \tag{18}$$

respectively.

In [7], the author showed that if $A \circ B = C$, then

$$\|C\|_2 \leq r_1(A)\, c_1(B). \tag{19}$$

2. MAIN THEOREMS

In this section, we give the Euclidean, spectral, column and row norms of the Hankel matrices with Pell, Pell-Lucas and modified Pell numbers.

THEOREM 1. *If A is an $n \times n$ matrix $A = (a_{ij})$ with $a_{ij} = P_{i+j-1}$, then we have*

$$\|A\|_F = \frac{1}{2}\sqrt{P_{2n}^2 - 2P_n^2 + \xi},$$

or equivalently

$$\|A\|_F = \frac{1}{2\sqrt{2}}\sqrt{q_{2n}^2 - 2q_n^2 + 1},$$

where $\|\cdot\|_F$ is the Frobenius norm, P_n denotes the nth Pell number and

$$\xi = \begin{cases} 2 & \text{if } n \text{ odd}, \\ 0 & \text{if } n \text{ even}. \end{cases}$$

Proof. Since

$$A = \begin{bmatrix} P_1 & P_2 & P_3 & \cdots & P_{n-1} & P_n \\ P_2 & P_3 & P_4 & \cdots & P_n & P_{n+1} \\ P_3 & P_4 & P_5 & \cdots & P_{n+1} & P_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{n-1} & P_n & P_{n+1} & \cdots & P_{2n-3} & P_{2n-2} \\ P_n & P_{n+1} & P_{n+2} & \cdots & P_{2n-2} & P_{2n-1} \end{bmatrix}$$

and

$$\sum_{k=1}^{n} P_k^2 = \frac{P_n P_{n+1}}{2}, \quad \sum_{k=1}^{n-1} P_k P_{k+1} = \frac{P_{2n+1} - 2P_n P_{n+1} - 1}{4},$$

we have

$$\|A\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}} = \left(\sum_{k=1}^{n} P_k^2 + \sum_{k=2}^{n+1} P_k^2 + \cdots + \sum_{k=n}^{2n-1} P_k^2 \right)^{\frac{1}{2}},$$

$$\|A\|_F = \left(\left(\sum_{k=1}^{n} P_k^2 + \sum_{k=1}^{n+1} P_k^2 + \cdots + \sum_{k=1}^{2n-1} P_k^2 \right) - \left(\sum_{k=1}^{n-1} \sum_{i=1}^{k} P_i^2 \right) \right)^{\frac{1}{2}},$$

$$\|A\|_F = \left(\frac{1}{2} \left(P_n P_{n+1} + P_{n+1} P_{n+2} + \cdots + P_{2n-1} P_{2n} \right) - \frac{1}{2} \sum_{k=1}^{n-1} P_k P_{k+1} \right)^{\frac{1}{2}},$$

$$\|A\|_F = \left(\frac{1}{2} \left(\sum_{k=1}^{2n-1} P_k P_{k+1} - 2 \sum_{k=1}^{n-1} P_k P_{k+1} \right) \right)^{\frac{1}{2}},$$

$$\|A\|_F = \left(\frac{P_{2n}^2 + P_{2n+1}^2 - 2P_{2n}P_{2n+1} - 2\left(P_n^2 + P_{n+1}^2 - 2P_nP_{n+1}\right) + 1}{8} \right)^{\frac{1}{2}},$$

$$\|A\|_F = \frac{1}{2} \left(P_{2n}^2 - 2P_n^2 + 1 - (-1)^n \right)^{\frac{1}{2}}.$$

Thus, we obtain the desired equality. Likewise, from the equation $P_{n+1} - P_n = q_n$, it can be seen that

$$\|A\|_F = \frac{1}{2\sqrt{2}} \sqrt{q_{2n}^2 - 2q_n^2 + 1}.$$

Thus, the proof is completed. □

In the following theorem, the column and row norms of the matrix A are given in terms of the modified Pell numbers.

Theorem 2. *Let A be an $n \times n$ matrix with $a_{ij} = P_{i+j-1}$. Then we have*

$$\|A\|_1 = \|A\|_\infty = \frac{1}{2} \left(q_{2n} - q_n \right),$$

where $\|A\|_1$, $\|A\|_\infty$ are the column and row norms, respectively.

Proof. From the definition of the matrix A, we can write

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}| = \max_{1 \le j \le n} \left\{ |a_{1j}| + |a_{2j}| + \cdots + |a_{nj}| \right\},$$

$$\|A\|_1 = P_n + P_{n+1} + \cdots + P_{2n-1},$$

$$\|A\|_1 = \sum_{i=1}^{2n-1} P_i - \sum_{i=1}^{n-1} P_i.$$

Thus, we get

$$\|A\|_1 = \frac{(P_{2n} + P_{2n-1} - 1) - (P_n + P_{n-1} - 1)}{2} = \frac{1}{2} \left(q_{2n} - q_n \right).$$

Similarly, the row norm of matrix A can be computed as

$$\|A\|_\infty = \max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}| = \sum_{k=n}^{2n-1} P_k = \frac{1}{2}\left(q_{2n}-q_n\right),$$

as desired. □

Now, we give bounds for spectral norm of the matrix A involving Pell and modified Pell numbers. Note that this bound is different from the one in [4] because there the indexing starts from 0, while here it starts from 1.

THEOREM 3. *If A is an $n\times n$ matrix $A=(a_{ij})$ with $a_{ij}=P_{i+j-1}$, then we have*

$$\frac{1}{2\sqrt{n}}\sqrt{P_{2n}^2-2P_n^2+\xi}\le$$

$$\le \|A\|_2 \le \frac{1}{2}\sqrt{\left(P_{2n}P_{2n-1}-P_nP_{n-1}\right)\left(P_{2n-2}P_{2n-1}-P_nP_{n-1}+2\right)}$$

and

$$\frac{1}{2\sqrt{2n}}\sqrt{q_{2n}^2-2q_n^2+1}\le$$

$$\le \|A\|_2 \le \frac{1}{8}\sqrt{\left(q_{4n-1}-q_{2n-1}+\xi\right)\left(q_{4n-3}-q_{2n-1}+\xi+8\right)}.$$

Proof. From Theorem 1 and the inequality (16), we can write

$$\frac{1}{2\sqrt{n}}\sqrt{P_{2n}^2-2P_n^2+\xi}\le\|A\|_2 \quad\text{and}\quad \frac{1}{2\sqrt{2n}}\sqrt{q_{2n}^2-2q_n^2+1}\le\|A\|_2\,.$$

On the other hand, let us define two new matrices $U_n=(u_{ij})_{i,j=1}^n$, where

$$u_{ij}=\begin{cases} P_{i+j-1} & i\le j,\\ 1 & i>j,\end{cases}$$

and $V_n=(v_{ij})_{i,j=1}^n$, where

$$v_{ij}=\begin{cases} P_{i+j-1} & i> j,\\ 1 & i\le j.\end{cases}$$

That is, we write

$$U_n=(u_{ij})=\begin{bmatrix} P_1 & P_2 & P_3 & \cdots & P_{n-1} & P_n\\ 1 & P_3 & P_4 & \cdots & P_n & P_{n+1}\\ 1 & 1 & P_5 & \cdots & P_{n+1} & P_{n+2}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & 1 & 1 & \cdots & P_{2n-3} & P_{2n-2}\\ 1 & 1 & 1 & \cdots & 1 & P_{2n-1}\end{bmatrix}$$

and

$$V_n = (v_{ij}) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ P_2 & 1 & 1 & \cdots & 1 \\ P_3 & P_4 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_n & P_{n+1} & P_{n+2} & \cdots & 1 \end{bmatrix},$$

respectively. It can be easily seen that $A = U_n \circ V_n$. Thus, we obtain that

$$r_1(U_n) = \max_i \sqrt{\sum_j |u_{ij}|^2} = \sqrt{\sum_{i=n}^{2n-1} P_i^2} =$$

$$= \sqrt{\sum_{i=1}^{2n-1} P_i^2 - \sum_{i=1}^{n-1} P_i^2} = \sqrt{\frac{P_{2n-1}P_{2n} - P_{n-1}P_n}{2}}$$

and

$$c_1(V_n) = \max_j \sqrt{\sum_i |v_{ij}|^2} = \sqrt{1 + \frac{P_{2n-2}P_{2n-1} - P_{n-1}P_n}{2}}.$$

Then, using the inequality $\|C\|_2 \leq r_1(A)c_1(B)$ we get

$$\|A\|_2 \leq \frac{1}{2}\sqrt{(P_{2n}P_{2n-1} - P_nP_{n-1})(P_{2n-2}P_{2n-1} - P_nP_{n-1} + 2)}. \quad (20)$$

If we use the inequality (20) and the inequality $\frac{1}{2\sqrt{n}}\sqrt{P_{2n}^2 - 2P_n^2 + \xi} \leq \|A\|_2$, then we obtain that

$$\frac{1}{2\sqrt{n}}\sqrt{P_{2n}^2 - 2P_n^2 + \xi} \leq \|A\|_2 \leq$$

$$\leq \frac{1}{2}\sqrt{(P_{2n}P_{2n-1} - P_nP_{n-1})(P_{2n-2}P_{2n-1} - P_nP_{n-1} + 2)}.$$

In a similar way the other inequality can be easily seen. Thus, the proof is completed. □

In the following theorem, we give the Euclidean (Frobenius) norm of the matrix involving Pell-Lucas numbers.

THEOREM 4. *If B is an $n \times n$ matrix with $b_{ij} = Q_{i+j-1}$, then we have*

$$\|B\|_F = \frac{1}{2}\sqrt{Q_{4n} - 2Q_{2n} - 2 + 4(-1)^n},$$

or

$$\|B\|_F = \sqrt{2\left(P_{2n}^2 - 2P_n^2\right)},$$

where $\|\cdot\|_F$ is the Frobenius norm and Q_n denotes the nth Pell-Lucas number.

Proof. From the definition of Frobenius norm we can write the following equations for the matrix B

$$\|B\|_F = \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|^2\right)^{\frac{1}{2}} = \left(\sum_{k=1}^{n}Q_k^2 + \sum_{k=2}^{n+1}Q_k^2 + \cdots + \sum_{k=n}^{2n-1}Q_k^2\right)^{\frac{1}{2}},$$

$$\|B\|_F = \left(\left(\sum_{k=1}^{n}Q_k^2 + \sum_{k=1}^{n+1}Q_k^2 + \cdots + \sum_{k=1}^{2n-1}Q_k^2\right) - \left(\sum_{k=1}^{n-1}\sum_{i=1}^{k}Q_i^2\right)\right)^{\frac{1}{2}}.$$

If we use the equations given in (13), then we obtain

$$\|B\|_F = \left(\frac{1}{2}\left[\frac{Q_{4n}-6}{2} + 2\frac{-1-(-1)^{2n}}{2} + 4\frac{1+(-1)^n}{2} - Q_{2n} + 6 - 4\right]\right)^{\frac{1}{2}}.$$

That is,

$$\|B\|_F = \frac{1}{2}\sqrt{Q_{4n} - 2Q_{2n} - 2 + 4\,(-1)^n}.$$

Similarly, if we use the equations $P_n + P_{n-1} = q_n = \frac{Q_n}{2}$ then we have

$$\|B\|_F = \sqrt{2\left(P_{2n}^2 - 2P_n^2\right)}.$$

So, the proof is completed. □

The following corollary determines the column and row norms of matrix B and its proof can be easily seen.

COROLLARY 1. *If B is an $n\times n$ matrix with $b_{ij} = Q_{i+j-1}$, then we have*

$$\|B\|_1 = \|B\|_\infty = \frac{1}{2}\left(Q_{2n} + Q_{2n-1} - Q_n - Q_{n-1}\right) = 2\left(P_{2n} - P_n\right).$$

THEOREM 5. *If B is an $n\times n$ matrix with $b_{ij} = Q_{i+j-1}$, then we have*

$$\frac{1}{2\sqrt{n}}\sqrt{Q_{4n} - 2Q_{2n} - 2 + 4\,(-1)^n} \le \|B\|_2 \le$$
$$\le \frac{1}{2}\sqrt{\left(Q_{4n} - Q_{2n}\right)\left(Q_{4n-2} - Q_{2n} + 2\right)},$$

and

$$\sqrt{\frac{2}{n}(P_{2n}^2 - 2P_n^2)} \le \|B\|_2 \le \sqrt{\left(4(P_{2n}^2 - P_n^2) + \xi\right)\left(4(P_{2n-1}^2 - P_n^2) + (-1)^{n+1}\right)}.$$

Proof. Let us define two matrices $R_n = (r_{ij})_{i,j=1}^n$ and $S_n = (s_{ij})_{i,j=1}^n$ such that

$$r_{ij} = \begin{cases} Q_{i+j-1} & i \le j, \\ 1 & i > j \end{cases}$$

and

$$s_{ij} = \begin{cases} Q_{i+j-1} & i > j, \\ 1 & i \leq j, \end{cases}$$

respectively. Using the inequality $\|C\|_2 \leq r_1(A)c_1(B)$, we can write

$$\|B\|_2 \leq r_1(R_n)\, c_1(S_n).$$

From the definitions of $r_1(R_n)$ and $c_1(S_n)$ we have that

$$r_1(R_n) = \max_i \sqrt{\sum_j |r_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |r_{nj}|^2},$$

$$r_1(R_n) = \sqrt{\sum_{i=n}^{2n-1} Q_i^2} = \sqrt{\sum_{i=1}^{2n-1} Q_i^2 - \sum_{i=1}^{n-1} Q_i^2} = \sqrt{\frac{Q_{4n} - Q_{2n}}{2}},$$

and

$$c_1(S_n) = \max_j \sqrt{\sum_i |s_{ij}|^2} = \sqrt{1 + \sum_{i=n}^{2n-2} Q_i^2} = \sqrt{\frac{Q_{4n-2} - Q_{2n} + 2}{2}}.$$

Thus, the upper bound of B can be found as

$$\|B\|_2 \leq \frac{1}{2}\sqrt{(Q_{4n} - Q_{2n})(Q_{4n-2} - Q_{2n} + 2)}.$$

On the other hand, using the inequality $\frac{1}{\sqrt{n}}\|A\|_F \leq \|A\|_2 \leq \|A\|_F$, we get

$$\sqrt{\frac{2}{n}\left(P_{2n}^2 - 2P_n^2\right)} \leq \|B\|_2 \leq \sqrt{\left(4\left(P_{2n}^2 - P_n^2\right) + \xi\right)\left(4\left(P_{2n-1}^2 - P_n^2\right) + (-1)^{n+1}\right)}.$$

Thus, the proof is completed. □

The following corollary is obvious if the norm axiom $\|\alpha A\| = |\alpha|\,\|A\|$ and the identity $Q_n = 2q_n$ are used.

Corollary 2. *For the matrices* $C = (c_{ij})_{n\times n}$, $c_{ij} = q_{i+j-1}$, *and* $B = (b_{ij})_{n\times n}$, $b_{ij} = Q_{i+j-1}$, *we have the following results:*

(i) $\|C\|_F = \dfrac{1}{2\sqrt{2}}\sqrt{q_{4n} - q_{2n} - 1 + 2(-1)^n}$ *or* $\|C\|_F = \sqrt{\dfrac{1}{2}\left(P_{2n}^2 - P_n^2\right)}$.

(ii) $\|C\|_1 = \|C\|_\infty = \dfrac{1}{2}(q_{2n} + q_{2n-1} - q_n - q_{n-1}) = P_{2n} - P_n$.

(iii) $$\frac{1}{2\sqrt{2n}}\sqrt{q_{4n}-2q_{2n}-1+2(-1)^n}\leq\|C\|_2\leq\sqrt{(q_{4n}-q_{2n})(q_{4n-2}-q_{2n}+1)}.$$

(iv) $$\frac{1}{\sqrt{2n}}\sqrt{\left(P_{2n}^2-2P_n^2\right)}\leq\|B\|_2\leq$$

$$\leq\frac{1}{2}\sqrt{\left(4\left(P_{2n}^2-P_n^2\right)+\xi\right)\left(4\left(P_{2n-1}^2-P_n^2\right)+(\ -1)^{n+1}\right)}.$$

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Sakarya University
Faculty of Sciences
Department of Mathematics
54187 Sakarya, Turkey
serpilhalici67@gmail.com