# HERMITE-HADAMARD TYPE INEQUALITIES OF CONVEX FUNCTIONS WITH RESPECT TO A PAIR OF QUASI-ARITHMETIC MEANS

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In this paper, we establish some integral inequalities of Hermite-Hadamard type, in the framework of Borel probability measures.

*AMS 2010 Subject Classification:* 26A51, 26D15.

*Key words:* convexity, quasi-arithmetic mean, Borel probability measure.

The *Hermite-Hadamard inequality* asserts that for every continuous convex function $f$ defined on an interval $[a, b]$ and every Borel probability measure $\mu$ on $[a, b]$ we have

$$f\left(b_\mu\right) \leq \int_a^b f(x)\mathrm{d}\mu(x) \leq \frac{b-b_\mu}{b-a}f(a) + \frac{b_\mu - a}{b-a}f(b), \tag{HH}$$

where

$$b_\mu = \int_a^b x\mathrm{d}\mu(x)$$

is the barycenter of $\mu$. See [3] for details.

The aim of this paper is to prove an analogue of Hermite-Hadamard inequality in the framework of quasi-arithmetic means.

Let $I$ be an interval and $\varphi : I \to \mathbb{R}$ a continuous increasing function. The weighted quasi-arithmetic mean associated to $\varphi$ is defined by the formula

$$M_{[\varphi]}\left(a, b; 1-\lambda, \lambda\right) = \varphi^{-1}\left(\left(1-\lambda\right)\varphi(a) + \lambda\varphi(b)\right),$$

for $a, b \in I$ and $\lambda \in [0, 1]$.

The weighted arithmetic mean

$$A\left(a, b; 1-\lambda, \lambda\right) = \left(1-\lambda\right)a + \lambda b$$

corresponds to $\varphi(x) = x$, and the weighted geometric mean

$$G\left(a, b; 1-\lambda, \lambda\right) = a^{1-\lambda}b^{\lambda}$$

corresponds to $\varphi(x) = \log x$.

Given a pair of continuous increasing functions $\varphi : [a, b] \to \mathbb{R}$ and $\psi : [c, d] \to \mathbb{R}$, a function $f : [a, b] \to [c, d]$ is called $(M_{[\varphi]}, M_{[\psi]})$-*convex* if

$$f\left(M_{[\varphi]}\left(x, y; 1-\lambda, \lambda\right)\right) \le M_{[\psi]}\left(f(x), f(y); 1-\lambda, \lambda\right)$$

for every $x, y \in [a, b]$ and $\lambda \in [0, 1]$.

The theory of $\left(M_{[\varphi]}, M_{[\psi]}\right)$-convex functions can be deduced from the theory of usual convex functions. Indeed, $f$ is a $\left(M_{[\varphi]}, M_{[\psi]}\right)$-convex function if and only if $\psi \circ f \circ \varphi^{-1}$ is convex. This fact allows us to translate results known for convex functions into their counterparts for $\left(M_{[\varphi]}, M_{[\psi]}\right)$-convex functions. We will next consider the case of Hermite-Hadamard inequality. Our approach is based on the concept of push-forward measure.

Given a Borel probability measure $\mu$ (on an interval $[a, b]$), the push-forward of $\mu$ through a continuous map $\varphi : [a, b] \to \mathbb{R}$ is defined by

$$(\varphi \# \mu)(A) = \mu\left(\varphi^{-1}(A)\right)$$

for every Borel subset $A$ of $[\varphi(a), \varphi(b)]$. This measure allows the following change of variable formula

$$\int_a^b f(\varphi(x))\, \mathrm{d}\mu(x) = \int_{\varphi(a)}^{\varphi(b)} f(x) \mathrm{d}\mu\left(\varphi^{-1}(x)\right).$$

The barycenter of $\varphi \# \mu$ is

$$b_{\varphi \# \mu} = \int_{\varphi(a)}^{\varphi(b)} x \mathrm{d}\mu\left(\varphi^{-1}(x)\right) = \int_a^b \varphi(x) \mathrm{d}\mu(x),$$

so if we put

$$\xi = \varphi^{-1}\left(b_{\varphi \# \mu}\right)$$

and

$$\mathcal{M}(\xi) = \frac{b\left(\varphi(b) - \varphi(\xi)\right) - a\left(\varphi(a) - \varphi(\xi)\right)}{\varphi(b) - \varphi(a)},$$

we obtain the identity

$$\frac{\varphi(\xi) - \varphi(a)}{\varphi(b) - \varphi(a)} = \frac{b - \mathcal{M}(\xi)}{b - a}. \tag{1}$$

Lemma 1. *The barycenter of $\varphi \# \mu$ verifies the formula*

$$b_{\varphi \# \mu} = \frac{\mathcal{M}(\xi) - a}{b - a} \varphi(a) + \frac{b - \mathcal{M}(\xi)}{b - a} \varphi(b). \tag{2}$$

*Proof.* In fact

$$b_{\varphi\#\mu} = \frac{\varphi(b) - b_{\varphi\#\mu}}{\varphi(b) - \varphi(a)}\varphi(a) + \frac{b_{\varphi\#\mu} - \varphi(a)}{\varphi(b) - \varphi(a)}\varphi(b)$$
$$= \frac{\mathcal{M}(\xi) - a}{b - a}\varphi(a) + \frac{b - \mathcal{M}(\xi)}{b - a}\varphi(b).$$

due to the identity (1). □

THEOREM 1 [The Hermite-Hadamard inequality for $(M_{[\varphi]}, M_{[\psi]})$-convex functions]. *Let $f : [a, b] \to [c, d]$ be a continuous $(M_{[\varphi]}, M_{[\psi]})$-convex function and $\mu$ be a Borel probability measure on $[a, b]$. Then*

$$\text{(RHH)} \qquad f(\xi) \leq \psi^{-1}\left(\int_a^b \psi\left(f(x)\right) \mathrm{d}\mu(x)\right)$$
$$\text{(LHH)} \qquad \leq M_{[\psi]}\left(f(a), f(b); \frac{\varphi(b) - \varphi(\xi)}{\varphi(b) - \varphi(a)}, \frac{\varphi(\xi) - \varphi(a)}{\varphi(b) - \varphi(a)}\right),$$

*where* $\xi = \varphi^{-1}\left(b_{\varphi\#\mu}\right)$.

*Proof.* We apply the inequality (HH) to $\psi \circ f \circ \varphi^{-1}$. As we have seen,

$$(\psi \circ f)(\xi) = \left(\psi \circ f \circ \varphi^{-1}\right)\left(b_{\varphi\#\mu}\right)$$
$$\leq \int_{\varphi(a)}^{\varphi(b)} \psi\left(f\left(\varphi^{-1}(x)\right)\right) \mathrm{d}\mu\left(\varphi^{-1}(x)\right) = \int_a^b \psi\left(f(x)\right) \mathrm{d}\mu(x)$$
$$\leq \frac{\varphi(b) - b_{\varphi\#\mu}}{\varphi(b) - \varphi(a)}\psi\left(f(a)\right) + \frac{b_{\varphi\#\mu} - \varphi(a)}{\varphi(b) - \varphi(a)}\psi\left(f(b)\right)$$

and the conclusion follows. □

*Remark* 1. Theorem 1 was proved for $\left(A, M_{[\psi]}\right)$-convex functions in [1, Theorem 3.3], under more restrictive conditions. The particular case of $(G, A)$-convex functions was proved in [5], while the case of $(G, G)$-convex functions appeared in [2] and [4].

We will call the function $\Phi$ a support of $f$ if $\psi \circ \Phi \circ \varphi^{-1} = \Psi$, where $\Psi$ is a support line of the convex function $\psi \circ f \circ \varphi^{-1}$.

THEOREM 2. *Let $f : [a, b] \to [c, d]$ be a continuous $(M_{[\varphi]}, M_{[\psi]})$-convex function, $\psi$ concave and $\mu$ be a Borel probability measure on $[a, b]$. Then*

$$\int_a^b f(x)\mathrm{d}\mu(x) \geq f\left(\varphi^{-1}\left(b_{\varphi\#\mu}\right)\right)$$
$$= \sup_{\Phi \text{ is a support of } f} \left\{\psi^{-1}\left(\int_a^b \psi\left(\Phi(x)\right) \mathrm{d}\mu(x)\right)\right\}.$$

*Proof.* The proof is similar to [2, Theorem 3]. Details are left to the reader.

The Hermite-Hadamard type inequalities proved in Theorem 1 are not just consequences of $\left(M_{[\varphi]}, M_{[\psi]}\right)$-convexity, but also characterize it. The converse of Hermite-Hadamard inequality for $(M_{[\varphi]}, M_{[\psi]})$-convex functions reads as follows:

THEOREM 3. *Let $I, J$ be two intervals and $f : I \to J$ a continuous function. Assume that $\varphi : I \to \mathbb{R}$ and $\psi : J \to \mathbb{R}$ are continuous increasing functions. If for every compact subinterval $[a, b]$ of $I$ and for every atomless Borel probability measure $\mu$ on $[a, b]$ the function $f$ satisfies either the inequality* (RHH) *or* (LHH) *then $f$ is $\left(M_{[\varphi]}, M_{[\psi]}\right)$-convex.*

*Proof.* If (RHH) holds, by Jensen's inequality we conclude that $\psi \circ f \circ \varphi^{-1}$ is convex, hence $f$ is $\left(M_{[\varphi]}, M_{[\psi]}\right)$-convex.

It remains to consider that (LHH) holds. We proceed by reductio ad absurdum. Assume that $f$ is not $\left(M_{[\varphi]}, M_{[\psi]}\right)$-convex. Then there exists a subinterval $[x, y] \subset I$ and a number $\varepsilon \in (0, 1)$ such that

$$f\left(M_{[\varphi]}\left(x, y; 1-\varepsilon, \varepsilon\right)\right) > M_{[\psi]}\left(f(x), f(y); 1-\varepsilon, \varepsilon\right). \tag{3}$$

Since $f$ is continuous, the inequality (3) holds on an entire neighbourhood $(\varepsilon_1, \varepsilon_2)$ of $\varepsilon$. We choose $(\varepsilon_1, \varepsilon_2)$ the biggest neighbourhood with this property. Put $a = M_{[\varphi]}\left(x, y; 1-\varepsilon_1, \varepsilon_1\right)$ and $b = M_{[\varphi]}\left(x, y; 1-\varepsilon_2, \varepsilon_2\right)$ $(a < b)$. The continuity of $f$ ensures that

$$f(a) = M_{[\psi]}\left(f(x), f(y); 1-\varepsilon_1, \varepsilon_1\right)$$

and

$$f(b) = M_{[\psi]}\left(f(x), f(y); 1-\varepsilon_2, \varepsilon_2\right).$$

Since we have $(1-t)\,\varepsilon_1 + t\varepsilon_2 \in (\varepsilon_1, \varepsilon_2)$ for every $t$ in $(0, 1)$, we infer from (3) that

$$\begin{aligned}
&f\left(M_{[\varphi]}\left(a, b; 1-t, t\right)\right)\\
&= f\left(M_{[\varphi]}\left(M_{[\varphi]}\left(x, y; 1-\varepsilon_1, \varepsilon_1\right), M_{[\varphi]}\left(x, y; 1-\varepsilon_2, \varepsilon_2\right); 1-t, t\right)\right)\\
&= f\left(M_{[\varphi]}\left(x, y; 1-(1-t)\,\varepsilon_1 - t\varepsilon_2, (1-t)\,\varepsilon_1 + t\varepsilon_2\right)\right)\\
&> M_{[\psi]}\left(f(x), f(y); 1-(1-t)\,\varepsilon_1 - t\varepsilon_2, (1-t)\,\varepsilon_1 + t\varepsilon_2\right)\\
&= M_{[\psi]}\left(M_{[\psi]}\left(f(x), f(y); 1-\varepsilon_1, \varepsilon_1\right), M_{[\psi]}\left(f(x), f(y); 1-\varepsilon_2, \varepsilon_2\right); 1-t, t\right)\\
&= M_{[\psi]}\left(f(a), f(b); 1-t, t\right).
\end{aligned}$$

Thus, it follows

$$\begin{aligned}
&\int_a^b \psi\left(f(x)\right)\mathrm{d}\mu(x)\\
&= \int_a^b \psi\left(f\left(M_{[\varphi]}\left(a, b; \frac{\varphi(b)-\varphi(x)}{\varphi(b)-\varphi(a)}, \frac{\varphi(x)-\varphi(a)}{\varphi(b)-\varphi(a)}\right)\right)\right)\mathrm{d}\mu(x)\\
&> \int_a^b \psi\left(M_{[\psi]}\left(f(a), f(b); \frac{\varphi(b)-\varphi(x)}{\varphi(b)-\varphi(a)}, \frac{\varphi(x)-\varphi(a)}{\varphi(b)-\varphi(a)}\right)\right)\mathrm{d}\mu(x)\\
&= \frac{\varphi(b)-\varphi(\xi)}{\varphi(b)-\varphi(a)}\psi\left(f(a)\right) + \frac{\varphi(\xi)-\varphi(a)}{\varphi(b)-\varphi(a)}\,\psi\left(f(b)\right).
\end{aligned}$$

This is a contradiction, completing the reductio ad absurdum. □

**Acknowledgements.** The first author was supported by CNCSIS Grant 420/2008. The authors are indebted to Professor Constantin P. Niculescu for his support and useful discussions during the preparation of this paper.

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*Received 13 march 2011*

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