A CHEMOTAXIS MODEL IN A STRATIFIED DOMAIN

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The purpose of this paper is to study a mathematical model of reaction-diffusion with chemotaxis that may describe a process of bioremediation of a polluted medium. We shall prove the existence of an asymptotic solution developed with respect to certain small parameters of the problem.

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1. INTRODUCTION

We deal with the study of a mathematical model of reaction-diffusion with chemotaxis that may describe a process of bioremediation of a medium polluted with a pollutant of concentration $c(t,x)$ by an action of a bacteria of density $b(t,x)$ which is able to destroy the pollutant.

A mathematical reaction-diffusion model of chemotaxis is expressed by a system of equations which describe the movement of some microorganisms (bacteria in our case) whose density is denoted by b in response to chemical gradients emitted by the chemoattractant c (in our case the pollutant). For surveys on this subject we refer to [6], [7], [8], [14], [16]. Generally, a chemotactic system consists of two equations for b and c with initial conditions

$$\frac{\partial b}{\partial t} - \nabla \cdot (D\,(b,c)\,\nabla b) + \nabla \cdot (K\,(b,c)\,b\nabla c) = g\,(b,c) - h\,(b,c)\,, \tag{1}$$

$$b(0,\xi) = b_0(\xi), \tag{2}$$

$$\frac{\partial c}{\partial t} - \nabla \cdot (\delta\,(b,c)\,\nabla c) = \varphi\,(b,c)\,, \tag{3}$$

$$c(0,\xi) = c_0(\xi), \tag{4}$$

and boundary conditions.

In the previous equations $D(b,c)$ and $\delta(b,c)$ represent the diffusion coefficients of the attracted population b and chemoattractant c respectively $g\,(b,c)$ and $h\,(b,c)$ are functions describing the rates of growth and death of b and $\varphi\,(b,c)$ is the function describing the degradation of the chemoattractant. We still denote

$$f(b,c) = g\,(b,c) - h\,(b,c)\,. \tag{5}$$

If $f\,(b,c)$ is positive the rate of growth of bacteria $g\,(b,c)$ is greater than its mortality rate $h\,(b,c)$ and if $f\,(b,c)$ is negative the degradation of b is dominant against its growth. The function K characterizes the chemotactic sensitivity. In literature especially the particular models with special forms for D, K, f or φ have been studied (see the surveys [14], [16]).

The model proposed in this paper focuses on the case in which the kinetic term and the chemotactic sensitivity have a weak influence on the flow.

The chemotaxis model will be set in a stratified 3D domain which can be viewed as a sequence of layers along a space coordinate, in each layer certain problem parameters having constant values, different from one layer to the other. The mathematical model is given by a system of n nonlinear parabolic equations which are made dimensionless, form which displays a dimensionless small parameter (or parameters) ε. We adopt a perturbation procedure (see e.g., [4]), namely we do an asymptotic analysis by developing all the functions with respect to the powers of the small parameters and retain two systems for the 0-order and 1-order of approximations. The existence and uniqueness of a global in time solution for the asymptotic model are studied within the framework of the evolution equations with m-accretive operators in Hilbert spaces, under certain assumptions for the nonlinear functions f and K.

2. THE MODEL FORMULATION

We consider that the spatial 3D domain is

$$\Omega = \{\xi = (x,y,z) \in \mathbb{R}^3;\ x \in (0,L),\ \xi' = (y,z) \in \Omega_2\},$$

where Ω_2 is an open bounded subset of $\mathbb{R}^2$ with a sufficient regular boundary (e.g., of class C^2). We assume that the domain Ω is composed of n parallel layers (x_{i-1}, x_i) along the Ox axis. The separation of the layers being due to the fact that certain parameters of the problem have constant values in the layer i, i.e., they do not depend on the variable x in (x_{i-1}, x_i).

We consider that in each layer i the chemotaxis process is modeled by the equations (1) and (3) written for the b_i and the chemoattractant c_i. In our model, we shall consider a particular form of the function $\varphi\,(b,c)$ encountered also in other studies (see [10]). However, there will be an essential difference with respect to that model, because it will be considered that the chemoattractant diffusion is positive. Let us consider

$$\varphi_i\,(b_i, c_i) = -\frac{\beta_{1i} c_i}{1 + \beta_{2i} c_i} b_i. \tag{6}$$

Thus, in each layer i, $i=\overline{1,n}$, we consider constant values for D_i, δ_i, β_{1i}, β_{2i} with

$$D_i > 0,\ \delta_i > 0,\ \beta_{1i} \geq 0,\ \beta_{2i} \geq 0. \tag{7}$$

These values as well as the expressions of the functions f_i, K_i which do not depend explicitly on x are different from one layer to another. Therefore, the domain Ω consists of n subdomains Ω_i, having the boundaries

$$\partial\Omega_i = \Gamma_{i-1} \cup \Gamma_i \cup \Gamma_i^{lat}, \quad i = 1, \ldots, n,$$

where Γ_i^{lat} are the lateral boundary of Ω_i and $\Gamma_i = \{x = x_i\}$, $i = 0, \ldots, n$. The surfaces Γ_0 and Γ_n are the external boundaries, while Γ_i with $i = 1, \ldots, n-1$ are the boundaries between layers. We denote

$$Q_i := (0,T) \times \Omega_i, \quad \Sigma_i := (0,T) \times \Gamma_i, \quad \Sigma_i^{lat} := (0,T) \times \Gamma_i^{lat}, \quad i = 1, \ldots, n.$$

The interaction between the layers is established by transmission conditions for b_i and c_i, i.e., the continuity of the solutions and fluxes. We assume that the system is closed for bacteria, namely the flux across the exterior frontiers is zero. For the chemoattractant we can require homogeneous Dirichlet conditions on the external boundaries (the pollutant does not reach the boundaries).

With these considerations, we propose as mathematical model the following system

$$\frac{\partial b_i}{\partial t} - D_i \Delta b_i + \nabla \cdot \left[b_i K_i\left(b_i, c_i\right) \nabla c_i \right] = f_i\left(b_i, c_i\right) \text{ in } Q_i, \tag{8}$$

$$\frac{\partial c_i}{\partial t} = \delta_i \Delta c_i - \frac{\beta_{1i} c_i}{1 + \beta_{2i} c_i} b_i \text{ in } Q_i, \tag{9}$$

$$b_i(0,\xi) = b_{i,0}(\xi) \text{ in } \Omega_i, \tag{10}$$

$$c_i(0,\xi) = c_{i,0}(\xi) \text{ in } \Omega_i, \tag{11}$$

for all $i = \overline{1,n}$, where $c_{i,0}$ and $b_{i,0}$ are initial conditions for c_i and b_i. At the interface between two layers we have the conditions

$$\begin{aligned} & -D_i \frac{\partial b_i}{\partial x} + b_i K_i\left(b_i, c_i\right) \frac{\partial c_i}{\partial x} = \\ & = -D_{i+1} \frac{\partial b_{i+1}}{\partial x} + b_{i+1} K_{i+1}\left(b_{i+1}, c_{i+1}\right) \frac{\partial c_{i+1}}{\partial x} \text{ on } \Sigma_i, \end{aligned} \tag{12}$$

$$b_i = b_{i+1} \text{ on } \Sigma_i, \tag{13}$$

$$c_i = c_{i+1} \text{ on } \Sigma_i, \tag{14}$$

$$\delta_i \frac{\partial c_i}{\partial x} = \delta_{i+1} \frac{\partial c_{i+1}}{\partial x} \text{ on } \Sigma_i, \tag{15}$$

for $i=\overline{1,n-1}$ together with the boundary conditions on the exterior horizontal and lateral boundaries

$$-D_1\frac{\partial b_1}{\partial x}+b_1K_1\left(b_1,c_1\right)\frac{\partial c_1}{\partial x}=0 \text{ on } \Sigma_0, \tag{16}$$

$$-D_n\frac{\partial b_n}{\partial x}+b_nK_n\left(b_n,c_n\right)\frac{\partial c_n}{\partial x}=0 \text{ on } \Sigma_n, \tag{17}$$

$$\nabla b_i\cdot\nu=0 \text{ on } \Sigma_i^{lat},\quad i=\overline{1,n}, \tag{18}$$

$$c_1=0 \text{ on } \Sigma_0, \tag{19}$$

$$c_n=0 \text{ on } \Sigma_n, \tag{20}$$

$$c_i=0 \text{ on } \Sigma_i^{lat},\quad i=\overline{1,n}. \tag{21}$$

Here ν is the unit outer normal to Γ_i^{lat} and $\frac{\partial}{\partial v}$ is the normal derivative.

In order to write the dimensionless system, we consider characteristic values denoted by index "a": L_a for length, T_a for time, b_a, c_a for concentrations c_i and b_i, respectively, D_a, δ_a for the diffusion coefficients, K_a for the chemotactic reaction, f_a for the rate of variation of b_i, β_{1a} and β_{2a} for the kinetic coefficients and we introduce the relations

$$\xi=\xi^*L_a,\quad t=t^*T_a,\quad b_i=b_i^*b_a,\quad c_i=c_i^*c_a,\quad D_i=D_i^*D_a,$$
$$\delta_i=\delta_i^*\delta_a,\quad K_i=K_i^*K_a,\quad f_i=f_i^*f_a,\quad \beta_{1i}=\beta_{1i}^*\beta_{1a},\quad \beta_{2i}=\beta_{2i}^*\beta_{2a},$$

where the superscript "$*$" denotes dimensionless quantities. They are replaced in the dimensional system and we get the system equations in dimensionless form

$$\frac{\partial b_i^*}{\partial t^*}-\overline{D}D_i^*\Delta b_i^*+\overline{K}\nabla\cdot\left[b_i^*K_i^*\left(b_i^*,c_i^*\right)\nabla c_i^*\right]=\overline{f}f_i^*\left(b_i^*,c_i^*\right) \text{ in } Q_i^*, \tag{22}$$

$$\frac{\partial c_i^*}{\partial t^*}=\overline{\delta}\delta_i^*\Delta c_i^*-\frac{\overline{\beta_1}\beta_{1i}^*c_i^*}{1+\overline{\beta_2}\beta_{2i}^*c_i^*}b_i^* \text{ in } Q_i^*, \tag{23}$$

$$b_i^*(0,\xi^*)=b_{i,0}^*\left(\xi^*\right),\quad \xi^*\in\Omega_i^*, \tag{24}$$

$$c_i^*(0,\xi^*)=c_{i,0}^*\left(\xi^*\right),\quad \xi^*\in\Omega_i^*, \tag{25}$$

for all $i=\overline{1,n}$, where $\overline{D}$, $\overline{K}$, $\overline{f}$, $\overline{\delta}$ are dimensionless parameters given by

$$\overline{D}=\frac{T_a}{L_a^2}D_a,\quad \overline{\delta}=\frac{T_a}{L_a^2}\delta_a,\quad \overline{K}=\frac{c_aT_a}{L_a^2}K_a,$$
$$\overline{f}=\frac{T_a}{b_a}f_a,\quad \overline{\beta_1}=b_aT_a\beta_{1a},\quad \overline{\beta_2}=c_a\beta_{2a}.$$

The dimensionless boundary conditions are

$$-\overline{D}D_i^*\frac{\partial b_i^*}{\partial x^*}+\overline{K}b_i^*K_i^*\left(b_i^*,c_i^*\right)\frac{\partial c_i^*}{\partial x^*}= \tag{26}$$

$$=-\overline{D}D_{i+1}^*\frac{\partial b_{i+1}^*}{\partial x^*}+\overline{K}b_{i+1}^*K_{i+1}^*\left(b_{i+1}^*,c_{i+1}^*\right)\frac{\partial c_{i+1}^*}{\partial x^*}\text{ on }\Sigma_i^*,$$

$$b_i^*=b_{i+1}^*\text{ on }\Sigma_i^*, \tag{27}$$

$$c_i^*=c_{i+1}^*\text{ on }\Sigma_i^*, \tag{28}$$

$$\delta_i^*\frac{\partial c_i^*}{\partial x^*}=\delta_{i+1}^*\frac{\partial c_{i+1}^*}{\partial x^*}\text{ on }\Sigma_i^*, \tag{29}$$

for $i=\overline{1,n-1}$ and

$$-\overline{D}D_1^*\frac{\partial b_1^*}{\partial x^*}+\overline{K}b_1^*K_1^*\left(b_1^*,c_1^*\right)\frac{\partial c_1^*}{\partial x^*}=0\text{ on }\Sigma_0^*, \tag{30}$$

$$-\overline{D}D_n^*\frac{\partial b_n^*}{\partial x^*}+\overline{K}b_n^*K_n^*\left(b_n^*,c_n^*\right)\frac{\partial c_n^*}{\partial x^*}=0\text{ on }\Sigma_n^*, \tag{31}$$

$$\nabla b_i^*\cdot\nu=0\text{ on }\Sigma_i^{*lat},\quad i=\overline{1,n}, \tag{32}$$

$$c_1^*=0\text{ on }\Sigma_0^*, \tag{33}$$

$$c_n^*=0\text{ on }\Sigma_n^*, \tag{34}$$

$$c_i^*=0\text{ on }\Sigma_i^{*lat},\quad i=\overline{1,n}. \tag{35}$$

To simplify the writing, the superscript "$*$" will be no longer indicated.

2.1. **Hypotheses**

In the system (22)–(35) we assume that the influence of the kinetic term and chemotactic coefficient are of ε-order with respect to the other dimensionless parameters and we set

$$\overline{\beta_2}=\varepsilon,\quad \overline{K}=\varepsilon. \tag{36}$$

The other parameters $\overline{D}$, $\overline{f}$, $\overline{\delta}$, $\overline{\beta_1}$ are assumed of $O\left(1\right)$.

We make the following hypotheses, for all $i=\overline{1,n}$:

i$_1$) $b_{i,0}\geq 0$ and there exists an i such that $b_{i,0}>0$;

i$_2$) $c_{i,0}\geq 0$ and there exists an i such that $c_{i,0}>0$;

i$_3$) $D_i\geq D_0>0$ in Ω_i with $D_0=\min\limits_{i=\overline{1,n}}D_i$;

i$_4$) $\delta_i\geq\delta_0>0$ in Ω_i with $\delta_0=\min\limits_{i=\overline{1,n}}\delta_i$;

i$_5$) $(r_1,r_2)\to K_i(r_1,r_2)$ are bounded in absolute value, i.e., $|K_i(r_1,r_2)|\leq K_M$ for any $r_1,r_2\in\mathbb{R}$.

We observe that generally, equations with nonlinear terms f_i do not admit global solutions in time (see [11], [12]). In this article we consider the next form for f_i for which we shall prove the existence of a global solution in time

$$f_i(r_1,r_2) = -k_i r_1 + \varepsilon \widetilde{f}_i(r_1,r_2), \tag{37}$$

where k_i are positive constants for $i=\overline{1,n}$ with $k_0 = \min\limits_{i=\overline{1,n}} k_i$. We consider that

i$_6$) $(r_1,r_2) \to |\widetilde{f}_i(r_1,r_2)|$ are bounded for any $r_1, r_2 \in \mathbb{R}$.

2.2. ε^0-order and ε^1-order approximations

We write the series expansions of all functions with respect to the small parameters $\overline{\beta_2} = \overline{K} = \varepsilon$. We have

$$\begin{aligned} b_i(t,\xi) &= b_i^0(t,\xi) + \varepsilon b_i^1(t,\xi) + \cdots, \\ c_i(t,\xi) &= c_i^0(t,\xi) + \varepsilon c_i^1(t,\xi) + \cdots, \\ K_i\left(b_i,c_i\right) &= K_i\left(b_i^0,c_i^0\right) + \varepsilon \left(K_i\right)_{b_i}\left(b_i^0,c_i^0\right) b_i^1 + \varepsilon \left(K_i\right)_{c_i}\left(b_i^0,c_i^0\right) c_i^1 + \cdots, \\ f_i\left(b_i,c_i\right) &= f_i\left(b_i^0,c_i^0\right) + \varepsilon \left(f_i\right)_{b_i}\left(b_i^0,c_i^0\right) b_i^1 + \varepsilon \left(f_i\right)_{c_i}\left(b_i^0,c_i^0\right) c_i^1 + \cdots, \end{aligned}$$

where $(K_i)_{b_i}$, $(K_i)_{c_i}$, $(f_i)_{b_i}$, $(f_i)_{c_i}$ represent the derivatives of K_i and f_i with respect to b_i and c_i.

We replace these series in the system (22)–(35) and by equaling the coefficients of the powers of ε^0 and ε^1 we deduce the systems corresponding to the ε^0-order and ε^1-order approximations, without writing the symbol "$*$", as specified before.

So for the ε^0-order approximation we get

$$\frac{\partial b_i^0}{\partial t} - \overline{D} D_i \Delta b_i^0 = -\overline{f} k_i b_i^0 \text{ in } Q_i, \tag{38}$$

$$\frac{\partial c_i^0}{\partial t} = \overline{\delta}\delta_i \Delta c_i^0 - \overline{\beta_1}\beta_{1i} c_i^0 b_i^0 \text{ in } Q_i, \tag{39}$$

$$b_i^0(0,\xi) = b_{i,0}(\xi) \text{ in } \Omega_i, \tag{40}$$

$$c_i^0(0,\xi) = c_{i,0}(\xi) \text{ in } \Omega_i, \tag{41}$$

for $i=\overline{1,n}$,

$$D_i \frac{\partial b_i^0}{\partial x} = D_{i+1} \frac{\partial b_{i+1}^0}{\partial x} \text{ on } \Sigma_i, \tag{42}$$

$$b_i^0 = b_{i+1}^0 \text{ on } \Sigma_i, \tag{43}$$

$$c_i^0 = c_{i+1}^0 \text{ on } \Sigma_i, \tag{44}$$

$$\delta_i \frac{\partial c_i^0}{\partial x} = \delta_{i+1} \frac{\partial c_{i+1}^0}{\partial x} \text{ on } \Sigma_i, \tag{45}$$

for $i=\overline{1,n-1}$,

$$\frac{\partial b_1^0}{\partial x}(t,\xi)=0, \quad (t,\xi)\in\Sigma_0, \tag{46}$$

$$\frac{\partial b_n^0}{\partial x}(t,\xi)=0, \quad (t,\xi)\in\Sigma_n, \tag{47}$$

$$\nabla b_i^0\cdot\nu=0, \quad (t,\xi)\in\Sigma_i^{lat},\ i=\overline{1,n}, \tag{48}$$

$$c_i^0(t,\xi)=0, \quad (t,\xi)\in\Sigma=\Sigma_0\cup\Sigma_n\cup\Sigma_i^{lat},\ i=\overline{1,n}. \tag{49}$$

Next, identifying the coefficients of ε^1 we obtain the following system for the ε^1-order approximation

$$\frac{\partial b_i^1}{\partial t}-\overline{D}D_i\Delta b_i^1+\overline{f}k_ib_i^1=F_i(t,\xi) \text{ in } Q_i, \tag{50}$$

$$\frac{\partial c_i^1}{\partial t}-\overline{\delta}\delta_i\Delta c_i^1+\overline{\beta_1}\beta_{1i}b_i^0c_i^1=H_i(t,\xi) \text{ in } Q_i, \tag{51}$$

$$c_i^1(0,\xi)=0 \text{ in } \Omega_i, \tag{52}$$

$$b_i^1(0,\xi)=0 \text{ in } \Omega_i, \tag{53}$$

for $i=\overline{1,n}$,

$$\left(-\overline{D}D_i\frac{\partial b_i^1}{\partial x}+\overline{D}D_{i+1}\frac{\partial b_{i+1}^1}{\partial x}\right)\bigg|_{x=x_i}=G_i\left(t,x_i,\xi'\right) \text{ on } \Sigma_i, \tag{54}$$

$$b_i^1=b_{i+1}^1 \text{ on } \Sigma_i, \tag{55}$$

$$c_i^1=c_{i+1}^1 \text{ on } \Sigma_i, \tag{56}$$

$$\delta_i\frac{\partial c_i^1}{\partial x}=\delta_{i+1}\frac{\partial c_{i+1}^1}{\partial x} \text{ on } \Sigma_i, \tag{57}$$

for $i=\overline{1,n-1}$,

$$-\overline{D}D_1\frac{\partial b_1^1}{\partial x}=G_0\left(t,x_0,\xi'\right) \text{ on } \Sigma_0, \tag{58}$$

$$\overline{D}D_n\frac{\partial b_n^1}{\partial x}=G_n\left(t,x_n,\xi'\right) \text{ on } \Sigma_n, \tag{59}$$

$$\nabla b_i^1\cdot\nu=0 \text{ on } \Sigma_i^{lat}, \tag{60}$$

$$c_i^1(t,\xi)=0, \quad (t,\xi)\in\Sigma=\Sigma_0\cup\Sigma_n\cup\Sigma_i^{lat}, \tag{61}$$

for $i=\overline{1,n}$, where we have denoted by $\xi':=(y,z)\in\Omega_2$,

$$F_i(t,\xi)=\overline{f}\widetilde{f}_i\left(b_i^0,c_i^0\right)-\nabla\cdot\left[b_i^0\left(t,\xi\right)K_i\left(b_i^0,c_i^0\right)\nabla c_i^0\left(t,\xi\right)\right], \tag{62}$$

$$H_i(t,\xi)=-\beta_{2i}c_i^0(t,\xi)\frac{\partial c_i^0}{\partial t}+\overline{\delta}\delta_i\beta_{2i}c_i^0(t,\xi)\Delta c_i^0-\overline{\beta_1}\beta_{1i}c_i^0(t,\xi)b_i^1(t,\xi), \tag{63}$$

for all $i = \overline{1, n}$,

$$G_i(t, x_i, \xi') = \left(-b_i^0(t, \xi) K_i \left(b_i^0, c_i^0\right) \frac{\partial c_i^0}{\partial x} + \right. \\ \left. \left. + b_{i+1}^0(t, \xi) K_{i+1} \left(b_{i+1}^0, c_{i+1}^0\right) \frac{\partial c_{i+1}^0}{\partial x} \right) \right|_{x = x_i}, \tag{64}$$

for all $i = \overline{1, n-1}$, and by

$$G_0\left(t, x_0, \xi'\right) = \left. \left(-b_1^0\left(t, x\right) K_1 \left(b_1^0, c_1^0\right) \frac{\partial c_1^0}{\partial x} \right) \right|_{x = x_0}, \tag{65}$$

$$G_n\left(t, x_n, \xi'\right) = \left. \left(b_n^0\left(t, x\right) K_n \left(b_n^0, c_n^0\right) \frac{\partial c_n^0}{\partial x} \right) \right|_{x = x_n}. \tag{66}$$

Once solved the system for the ε^0-order approximation the functions $F_i(t, \xi)$, $H_i(t, \xi)$, $G_i(t, x_i, \xi')$, $G_0\left(t, x_0, \xi'\right)$, $G_n\left(t, x_n, \xi'\right)$ become known.

We retain only the first two approximations because the equations for the next approximations raise the same mathematical treatment as those corresponding to the ε^1-order system. Further approximations can be used for numerical purposes.

We shall resume in detail the definition of the solutions b_i^0, c_i^0 and b_i^1, c_i^1 in the next sections.

3. THE ε^0-ORDER APPROXIMATION

We resume the system (38)–(49) for the ε^0-order approximation. For the moment we recall only the equations for b_i^0. In order to simplify the writing we shall no longer write the “0” superscript symbol. We get the system

$$\frac{\partial b_i}{\partial t} - \overline{D} D_i \Delta b_i + \overline{f} k_i b_i = 0 \text{ in } Q_i, \quad i = \overline{1, n}, \tag{67}$$

$$b_i(0, \xi) = b_{i,0}(\xi) \text{ in } \Omega_i, \quad i = \overline{1, n}, \tag{68}$$

$$D_i \frac{\partial b_i}{\partial x} = D_{i+1} \frac{\partial b_{i+1}}{\partial x} \text{ on } \Sigma_i, \quad i = \overline{1, n-1}, \tag{69}$$

$$b_i = b_{i+1} \text{ on } \Sigma_i, \quad i = \overline{1, n-1}, \tag{70}$$

$$\frac{\partial b_1}{\partial x} = 0 \text{ on } \Sigma_0, \tag{71}$$

$$\frac{\partial b_n}{\partial x} = 0 \text{ on } \Sigma_n, \tag{72}$$

$$\nabla b_i \cdot \nu = 0 \text{ on } \Sigma_i^{lat}, \quad i = \overline{1, n}. \tag{73}$$

Functional framework. We introduce the functions

$$b(t,\xi),\ c(t,\xi) = \begin{cases} b_1(t,\xi),\ c_1(t,\xi), & x \in (x_0,x_1) \\ \cdots \\ b_n(t,\xi),\ c_n(t,\xi), & x \in (x_{n-1},x_n) \end{cases}, \tag{74}$$

$$b_0(\xi),\ c_0(\xi) = \begin{cases} b_{1,0}(\xi),\ c_{1,0}(\xi), & x \in (x_0,x_1) \\ \cdots \\ b_{n,0}(\xi),\ c_{n,0}(\xi), & x \in (x_{n-1},x_n) \end{cases},$$

$$D(x),\ \delta(x) = \begin{cases} \overline{D}D_1,\ \bar{\delta}\delta_1, & x \in (x_0,x_1) \\ \cdots \\ \overline{D}D_n,\ \bar{\delta}\delta_n, & x \in (x_{n-1},x_n) \end{cases},$$

$$k(x) = \begin{cases} \overline{f}k_1, & x \in (x_0,x_1) \\ \cdots \\ \overline{f}k_n, & x \in (x_{n-1},x_n) \end{cases},$$

$$\beta_1(x) = \begin{cases} \overline{\beta_1}\beta_{11}, & x \in (x_0,x_1) \\ \cdots \\ \overline{\beta_1}\beta_{1n}, & x \in (x_{n-1},x_n) \end{cases}. \tag{75}$$

Similarly, we define $f(b,c,x)$ and $K(b,c,x)$.

We notice that assumption (7) and hypotheses i_1)–i_4) imply similar properties for the functions defined before. Namely, we have

$$c_0(\xi) \geq 0, \tag{76}$$

$$b_0(\xi) \geq 0, \tag{77}$$

$$D(x) \geq D_0 > 0, \tag{78}$$

$$\delta(x) \geq \delta_0 > 0, \tag{79}$$

$$\beta_1(x) \geq 0. \tag{80}$$

Recall that $\Omega = (x_0,x_n)$. We consider the Sobolev space $V = H^1(\Omega)$ endowed with the standard norm

$$\|\psi\|_V = \left(\|\psi\|^2 + \|\nabla\psi\|^2\right)^{1/2}.$$

We denote V' the dual of V and $H = L^2(\Omega)$ with $V \subset H \subset V'$. We also specify that by $(\cdot\,,\cdot)$ and $\|\cdot\|$ we shall denote the scalar product and the norm in $L^2(\Omega)$. The value of $g \in V'$ at $\psi \in V$ is

$$g(\psi) = \langle g,\psi\rangle_{V',V}\,, \tag{81}$$

where $\langle\cdot\,,\cdot\rangle_{V',V}$ represents the duality between V' and V.

Now, we define the operator $A : V \to V'$ by

$$\langle Ab, \psi\rangle_{V',V} = \sum_{i=1}^{n} \int_{\Omega_i} \left[\overline{D} D_i \nabla b_i \cdot \nabla\psi + \overline{f} k_i b_i \psi\right] \mathrm{d}\xi = \int_{\Omega} \left[D(x)\nabla b \cdot \nabla\psi + k(x) b\psi\right] \mathrm{d}\xi, \quad \forall \psi \in V. \tag{82}$$

So, we are led to the Cauchy problem

$$\frac{\mathrm{d}b}{\mathrm{d}t}(t) + Ab(t) = 0 \text{ a.e. } t \in (0,T), \tag{83}$$

$$b(0) = b_0. \tag{84}$$

We shall prove that (83)–(84) has a strong solution implying that (67)–(73) has a solution in the sense of distributions and that this solution is unique.

3.1. Main results

Theorem 3.1. *Let* $b_0 \in L^2(\Omega)$*. Then problem* (83)–(84) *has a unique strong solution*

$$b \in W^{1,2}([0,T]; V') \cap L^2(0,T;V) \cap C([0,T]; L^2(\Omega)) \tag{85}$$

which satisfies the estimates

$$\|b(t)\|^2 + \alpha_0 \int_0^t \|b(\tau)\|_V^2 \, \mathrm{d}\tau \le \|b_0\|^2, \tag{86}$$

$$\left\|b(t) - \overline{b}(t)\right\| \le \left\|b_0 - \overline{b_0}\right\|, \tag{87}$$

where $\alpha_0 = 2\min\{D_0, k_0\}$ *and* $\overline{b}(t)$ *is another solution of* (83) *with* $\overline{b}(0) = \overline{b_0}$*.*

In addition, if $b_0 \in V$*, we have the regularity*

$$b \in W^{1,2}\left([0,T]; L^2(\Omega)\right) \cap L^\infty(0,T;V). \tag{88}$$

Proof. We prove the existence of the strong solution using the Lions' theorem for the time independent case (see [15]). To this end we show that the operator A is positively defined, bounded and coercive. We have

$$\begin{aligned}
\|Ab(t)\|_{V'} &= \sup_{\psi\in V;\ \|\psi\|_V \le 1} \left|\langle Ab(t), \psi\rangle_{V',V}\right| \\
&\le \sup_{\psi\in V;\ \|\psi\|_V \le 1} \left|\int_\Omega \left(D(x)\nabla b\cdot\nabla\psi + k(x) b\psi\right)\mathrm{d}\xi\right| \\
&\le \sup_{\psi\in V;\ \|\psi\|_V \le 1} \left(D_\infty \|\nabla b\| \, \|\psi\|_V + k_\infty \|b\| \, \|\psi\|_V\right) \\
&\le \max\{D_\infty, k_\infty\} \sup_{\psi\in V;\ \|\psi\|_V \le 1} \|b\|_V \, \|\psi\|_V \le \max\{D_\infty, k_\infty\} \|b\|_V,
\end{aligned}$$

where $D_\infty = \max\limits_{i=\overline{1,n}} D_i$ and $k_\infty = \max\limits_{i=\overline{1,n}} k_i$.

Then we compute

$$\langle Ab(t), b(t)\rangle_{V',V} = \int_\Omega D(x)\,(\nabla b)^2\,\mathrm{d}\xi + \int_\Omega k(x) b^2 \mathrm{d}\xi$$
$$\geq D_0\,\|\nabla b\|^2 + k_0\|b\|^2 \geq \min\{D_0, k_0\}\,\|b\|_V^2 \geq 0$$

and we conclude that the operator A is coercive.

It follows that the operator A satisfies the hypotheses of Lions' theorem and we conclude that the problem (83)–(84) has a unique strong solution as claimed in (85).

To obtain (86) we multiply (83) by b, integrate over $(0,t)$ and we get

$$\|b(t)\|^2 - \|b_0\|^2 + 2\int_0^t \langle Ab(\tau), b(\tau)\rangle_{V',V}\,\mathrm{d}\tau = 0.$$

Using the fact that the operator A is positively defined we have

$$\|b(t)\|^2 - \|b_0\|^2 + 2D_0\int_0^t \|\nabla b(\tau)\|^2\,\mathrm{d}\tau + 2k_0\int_0^t \|b(\tau)\|^2\,\mathrm{d}\tau \leq 0$$

which implies (86) with $\alpha_0 = 2\min\{D_0, k_0\}$.

In order to obtain (87), let us consider two solutions b and $\overline{b}$ corresponding to the initial data b_0 and $\overline{b_0}$. Writing (86) for $\left(b(t) - \overline{b}\,(t)\right)$ we get

$$\left\|b(t) - \bar{b}(t)\right\|^2 + \alpha_0\int_0^t \left\|b(\tau) - \bar{b}(\tau)\right\|_V^2\,\mathrm{d}\tau \leq \left\|b_0 - \overline{b_0}\right\|^2,$$

which implies (87) since the second term in the left hand side is positive.

Next, we multiply (83) by $\frac{\mathrm{d}b}{\mathrm{d}t}$ and integrate over $(0,t)$. We get

$$\int_0^t\int_\Omega \left(\frac{\mathrm{d}b}{\mathrm{d}\tau}\right)^2 \mathrm{d}\xi\mathrm{d}\tau + \int_0^t\int_\Omega D(x)\frac{\mathrm{d}}{\mathrm{d}\tau}\left(|\nabla b|^2\right)\mathrm{d}\xi\mathrm{d}\tau +$$
$$+ \int_0^t\int_\Omega k(x) b\frac{\mathrm{d}b}{\mathrm{d}\tau}\mathrm{d}\xi\mathrm{d}\tau = 0.$$

Using (78) we can write

$$\int_0^t \left\|\frac{\mathrm{d}b}{\mathrm{d}\tau}(\tau)\right\|^2 \mathrm{d}\tau + D_0\,\|\nabla b(t)\|^2 \leq D_\infty\,\|\nabla b_0\|^2 - k_\infty\int_0^t \|b(\tau)\|\left\|\frac{\mathrm{d}b}{\mathrm{d}\tau}(\tau)\right\|\mathrm{d}\tau.$$

Further,

$$\int_0^t \left\|\frac{\mathrm{d}b}{\mathrm{d}\tau}(\tau)\right\|^2 \mathrm{d}\tau + D_0\,\|\nabla b(t)\|^2 \leq$$
$$\leq D_\infty\,\|\nabla b_0\|^2 + \frac{k_\infty^2}{2}\int_0^t \|b(\tau)\|^2\,\mathrm{d}\tau + \frac{1}{2}\int_0^t \left\|\frac{\mathrm{d}b}{\mathrm{d}\tau}(\tau)\right\|^2 \mathrm{d}\tau.$$

Then we can write

$$\frac{1}{2}\int_0^t \left\|\frac{\mathrm{d}b}{\mathrm{d}\tau}(\tau)\right\|^2 \mathrm{d}\tau + D_0\left\|\nabla b(t)\right\|^2 \le D_\infty \left\|\nabla b_0\right\|^2 + \frac{k_\infty^2}{2}\int_0^t \left\|b(\tau)\right\|^2 \mathrm{d}\tau.$$

We deduce that

$$\int_0^t \left\|\frac{\mathrm{d}b}{\mathrm{d}\tau}(\tau)\right\|^2 \mathrm{d}\tau + 2D_0\left\|\nabla b(t)\right\|^2 \le 2D_\infty \left\|\nabla b_0\right\|^2 + k_\infty^2 T\left\|b_0\right\|^2 = C'. \tag{89}$$

We use (86) and finally we get

$$\left\|b(t)\right\|_V \le C_V, \tag{90}$$

where $C_V = \sqrt{\left\|b_0\right\|^2 + \frac{C'}{2D_0}}$.

Because $b_0 \in V$ the relation above implies (88) and it yields that

$$b_i \in W^{1,2}\left([0,T]\,; L^2(\Omega_i)\right) \cap L^\infty\left(0,T; H^1(\Omega_i)\right). \quad \square$$

We return in the equation (67) and we deduce that

$$\overline{D}D_i \triangle b_i = \frac{\partial b_i}{\partial t} + \overline{f}k_i b_i \in L^2\left(0,T; L^2(\Omega_i)\right)$$

which together with the boundary conditions on the externals boundaries implies that $b_i \in L^2\left(0,T; H^2(\Omega_i)\right)$, $i = \overline{1,n}$. This assertion is proved in a similar way with the regularity of the weak solutions (see [3]) modifying the proof correspondingly to the Neumann boundary conditions on Γ_i.

In conclusion, in each layer the functions b_i^0 have the regularity

$$\begin{aligned} b_i^0 \in\, & W^{1,2}([0,T]; (H^1(\Omega_i)') \cap L^2\left(0,T; H^1(\Omega_i)\right) \cap \\ & \cap\, C([0,T]; L^2(\Omega_i)) \cap L^2\left(0,T; H^2(\Omega_i)\right). \end{aligned} \tag{91}$$

In order to show the connection with the physical model we verify if the solution falls within an accepted physical domain. This being a concentration we shall check its positiveness.

Proposition 3.2. *Assume $b_0 \in L^2(\Omega)$, $b_0 \ge 0$ a.e. in Ω and let b_M be a positive constant such that $0 \le b_0 \le b_M$. Then the solution b to problem* (83)–(84) *satisfies*

$$0 \le b(t) \le b_M \ \textit{a.e.} \text{ in } \Omega\,, \quad \forall t \in [0,T]. \tag{92}$$

Proof. Recalling that the positive and negative parts of b are $b^+ = \max\{b, 0\}$ and $b^- = -\min\{b, 0\}$, we have to prove that $b^-(t) = 0$ for each $t \in [0,T]$. We multiply (83) scalarly by $b^-(t)$ and get

$$\int_\Omega \frac{\mathrm{d}b}{\mathrm{d}t}(t)b^-(t)\mathrm{d}\xi + \int_\Omega D(x)\nabla b(t)\cdot\nabla b^-(t)\mathrm{d}\xi + \int_\Omega k(x)b(t)b^-(t)\mathrm{d}\xi = 0.$$

We use the Stampacchia's lemma

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|b^{-}(t)\right\|^{2}-\int_{\Omega}D(x)\left|\nabla b^{-}(t)\right|^{2}\mathrm{d}\xi-\int_{\Omega}k(x)\left|b^{-}(t)\right|^{2}\mathrm{d}\xi=0 \tag{93}$$

and integrating over $(0,T)$ we have

$$\frac{1}{2}\left\|b^{-}(t)\right\|^{2}-\frac{1}{2}\left\|b^{-}(0)\right\|^{2}+D_0\int_0^t\left\|\nabla b^{-}(\tau)\right\|^{2}\mathrm{d}\tau+k_0\int_0^t\left\|b^{-}(\tau)\right\|^{2}\mathrm{d}\tau\leq 0.$$

But $b_0^{-}=0$ since $b_0(t)\geq 0$ and the two last terms are positive since $D_0,\ k_0\geq 0$. It follows that $\|b^{-}(t)\|=0$ whence $b(t)\geq 0$ for each $t\in[0,T]$.

Now, we consider the equation (83) written equivalently

$$\int_0^T\int_{\Omega}\frac{\mathrm{d}b}{\mathrm{d}t}\psi\mathrm{d}\xi\mathrm{d}t+\int_0^T\int_{\Omega}D(x)\nabla b\cdot\nabla\psi\mathrm{d}\xi\mathrm{d}t+\int_0^T\int_{\Omega}k(x)b\psi\mathrm{d}\xi\mathrm{d}t=0$$

for any $\psi\in L^2(0,T;V)$.

Next, we still have

$$\int_0^T\int_{\Omega}\frac{\mathrm{d}}{\mathrm{d}t}(b-b_M)\psi\mathrm{d}\xi\mathrm{d}t+\int_0^T\int_{\Omega}D(x)\nabla(b-b_M)\nabla\psi\mathrm{d}\xi\mathrm{d}t$$
$$+\int_0^T\int_{\Omega}\left[k(x)(b-b_M)+k(x)b_M\right]\psi\mathrm{d}\xi\mathrm{d}t=0$$

and make $\psi=(b-b_M)^{+}$. We obtain

$$\frac{1}{2}\left\|(b-b_M)^{+}(t)\right\|^{2}-\frac{1}{2}\left\|(b_0-b_M)^{+}\right\|^{2}+\int_0^T\int_{\Omega}D(x)\left|\nabla(b-b_M)^{+}\right|^{2}\mathrm{d}\xi\mathrm{d}t+$$
$$+\int_0^T\int_{\Omega}k(x)\left|(b-b_M)^{+}\right|^{2}\mathrm{d}\xi\mathrm{d}t=-\int_0^T\int_{\Omega}k(x)b_M(b-b_M)^{+}\mathrm{d}\xi\mathrm{d}t.$$

Since $b_0\leq b_M$ it follows that $(b_0-b_M)^{+}=0$ and we have

$$\frac{1}{2}\left\|(b-b_M)^{+}(t)\right\|^{2}+D_0\int_0^T\int_{\Omega}\left|\nabla(b-b_M)^{+}\right|^{2}\mathrm{d}\xi\mathrm{d}t+$$
$$+k_0\int_0^T\int_{\Omega}\left|(b-b_M)^{+}\right|^{2}\mathrm{d}\xi\mathrm{d}t\leq 0.$$

Finally, we get

$$\left\|(b-b_M)^{+}(t)\right\|^{2}=0\ .$$

This means that $b(t,\xi)\leq b_M$ a.e. in Ω for any $t\in[0,T]$. □

Now we resume the system of the ε^0-order approximation for c_i^0 written without "0" superscript symbol

$$\frac{\partial c_i}{\partial t} - \overline{\delta}\delta_i \Delta c_i + \overline{\beta_1}\beta_{1i} b_i c_i = 0 \text{ in } Q_i, \quad i = \overline{1,n}, \tag{94}$$

$$c_i(0,\xi) = c_{i,0}(\xi) \text{ in } \Omega_i, \quad i = \overline{1,n}, \tag{95}$$

$$\delta_i \frac{\partial c_i}{\partial x} = \delta_{i+1}\frac{\partial c_{i+1}}{\partial x} \text{ on } \Sigma_i, \quad i = \overline{1,n-1}, \tag{96}$$

$$c_i = c_{i+1} \text{ on } \Sigma_i, \quad i = \overline{1,n-1}, \tag{97}$$

$$c_i(t,\xi) = 0, \quad (t,\xi) \in \Sigma = \Sigma_0 \cup \Sigma_n \cup \Sigma_i^{lat}, \quad i = \overline{1,n}. \tag{98}$$

The problem (94)–(98) is similar to the problem (67)–(73) treated in the space $V_0 = H_0^1(\Omega)$ endowed with the standard norm

$$\|\psi\|_{V_0} = \|\nabla\psi\|$$

with $V_0' = H^{-1}(\Omega)$ its dual.

Here we define the time dependent operator $B(t) : V_0 \to V_0'$ by

$$\begin{aligned} \langle B(t)c, \psi\rangle_{V_0', V_0} &= \sum_{i=1}^{n} \int_{\Omega_i} \left[\overline{\delta}\delta_i \nabla c_i \cdot \nabla\psi + \overline{\beta_1}\beta_{1i} b_i(t) c_i \psi\right] \mathrm{d}\xi \\ &= \int_{\Omega} \left[\delta(x)\nabla c \cdot \nabla\psi + \beta_1(x) b(t) c\psi\right] \mathrm{d}\xi, \quad \forall \psi \in V_0. \end{aligned} \tag{99}$$

So, we are led to the Cauchy problem

$$\frac{\mathrm{d}c}{\mathrm{d}t}(t) + B(t)c(t) = 0 \text{ a.e. } t \in (0,T)\,, \tag{100}$$

$$c(0) = c_0 \tag{101}$$

and we can give the next result.

Theorem 3.3. *Let* $c_0 \in L^2(\Omega)$ *. Then problem* (100)–(101) *has a unique strong solution*

$$c \in W^{1,2}([0,T]; V_0') \cap L^2\left(0,T; V_0\right) \cap C([0,T]; L^2(\Omega)) \tag{102}$$

which satisfies the estimates

$$\|c(t)\|^2 + 2\delta_0 \int_0^t \|c(\tau)\|_{V_0}^2 \,\mathrm{d}\tau \le \|c_0\|^2\,, \tag{103}$$

$$\|c(t) - \overline{c}(t)\|^2 + 2\delta_0 \int_0^t \|c(\tau) - \overline{c}(\tau)\|_{V_0}^2 \,\mathrm{d}\tau \le \|c_0 - \overline{c_0}\|^2\,, \tag{104}$$

$$\|c(t)\|_{V_0} \le C_{V_0}, \tag{105}$$

where C_{V_0} *is a constant and* $\overline{c}\,(t)$ *is another solution of* (100) *with* $\overline{c}(0) = \overline{c_0}$.

In addition, if $c_0 \in V_0$ *we have the regularity*

(106) $$c \in W^{1,2}\left([0,T]\,; L^2(\Omega)\right) \cap L^{\infty}\left(0,T; V_0\right).$$

Proof. The proof is similar to the one of Theorem 3.1, just that the space $H^1(\Omega)$ is replaced by $H_0^1(\Omega)$.

In order to obtain (103) we multiply (100) by $c\,(t)$, we integrate over $(0,t)$ and we get

$$\frac{1}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\left\|c(\tau)\right\|^2 \mathrm{d}\tau + \int_0^t \int_\Omega \delta\,(x)\,(\nabla c(\tau))^2\,\mathrm{d}\xi\mathrm{d}\tau + \\ + \int_0^t \int_\Omega \beta_1(x) b(\tau) c^2(\tau)\mathrm{d}\xi\mathrm{d}\tau = 0.$$

For the last term in the right hand side we take into account the relations (80) and (92) and we can write that

$$\frac{1}{2}\left\|c(t)\right\|^2 + \delta_0 \int_0^t \left\|\nabla c(\tau)\right\|^2 \mathrm{d}\tau \le \frac{1}{2}\left\|c_0\right\|^2$$

which implies (103).

To obtain (104) we consider two solutions c and $\overline{c}$ corresponding to the initial data c_0 and $\overline{c_0}$ and we write (103) for $(c(t) - \overline{c}(t))$.

Next, we multiply (100) by $\frac{\mathrm{d}c}{\mathrm{d}t}(t)$ and integrate over $(0,t)$. We have

$$\int_0^t \left\|\frac{\mathrm{d}c}{\mathrm{d}\tau}(\tau)\right\|^2 \mathrm{d}\tau + \int_0^t \int_\Omega \delta(x)\,(\nabla c(\tau))^2\,\mathrm{d}\xi\mathrm{d}\tau + \\ + \int_0^t \int_\Omega \beta_1(x) b(\tau) c(\tau) \frac{\mathrm{d}c}{\mathrm{d}\tau}(\tau)\mathrm{d}\xi\mathrm{d}\tau = 0.$$

Using (79) we can write

$$\int_0^t \left\|\frac{\mathrm{d}c}{\mathrm{d}\tau}(\tau)\right\|^2 \mathrm{d}\tau + \delta_0\left\|\nabla c(t)\right\|^2 \le \delta_\infty \left\|\nabla c_0\right\|^2 - \int_0^t \int_\Omega \beta_1(x) b(\tau) c(\tau) \frac{\mathrm{d}c}{\mathrm{d}\tau}(\tau)\mathrm{d}\xi\mathrm{d}\tau$$

and further

$$\int_0^t \left\|\frac{\mathrm{d}c}{\mathrm{d}\tau}(\tau)\right\|^2 \mathrm{d}\tau + \delta_0\left\|\nabla c(t)\right\|^2 \le \\ \le \delta_\infty \left\|\nabla c_0\right\|^2 + \beta_{1\infty} b_M \int_0^t \int_\Omega |c(\tau)| \left|\frac{\mathrm{d}c}{\mathrm{d}\tau}(\tau)\right| \mathrm{d}\xi\mathrm{d}\tau \\ \le \delta_\infty \left\|\nabla c_0\right\|^2 + \beta_{1\infty} b_M \int_0^t \left\|c(\tau)\right\| \left\|\frac{\mathrm{d}c}{\mathrm{d}\tau}(\tau)\right\| \mathrm{d}\tau \\ \le \delta_\infty \left\|\nabla c_0\right\|^2 + \frac{\beta_{1\infty}^2 b_M^2}{2} \int_0^t \left\|c(\tau)\right\|^2 \mathrm{d}\tau + \frac{1}{2}\int_0^t \left\|\frac{\mathrm{d}c}{\mathrm{d}\tau}(\tau)\right\|^2 \mathrm{d}\tau.$$

So, we get

$$\int_0^t \left\| \frac{\mathrm{d}c}{\mathrm{d}\tau}(\tau) \right\|^2 \mathrm{d}\tau + 2\delta_0 \left\| \nabla c(t) \right\|^2 \leq 2\delta_\infty \left\| \nabla c_0 \right\|^2 + \beta_{1\infty}^2 b_M^2 T \left\| c_0 \right\|^2 = C_1$$

and from here we deduce that $\|c(t)\|_{V_0} \leq C_{V_0}$ with $C_{V_0} = \sqrt{\frac{C_1}{2\delta_0}}$. $\square$

It is obvious that in the same way as before we get in each layer

$$c_i^0 \in W^{1,2}\left([0,T]; L^2(\Omega_i)\right) \cap L^\infty\left(0,T; H_0^1(\Omega_i)\right) \cap L^2\left(0,T; H^2(\Omega_i)\right). \tag{107}$$

Proposition 3.4. *Assume* $c_0 \in L^2(\Omega)$, $c_0 \geq 0$ *a.e. in* Ω *and let* c_M *be a positive constant such that* $0 \leq c_0 \leq c_M$. *Then the solution* c *to problem* (100)–(101) *satisfies*

$$0 \leq c(t) \leq c_M \text{ a.e. in } \Omega, \quad \forall t \in [0,T]. \tag{108}$$

This result ends the proof of the existence and uniqueness of the solution for the system (38)–(49) of ε^0-order approximation.

4. THE ε^1-ORDER APPROXIMATION

We resume the system (50)–(61) for the ε^1-order approximation. We have again two systems, one for b_i^1 and one for c_i^1. In order to simplify the writing we shall no longer write the "1" superscript symbol. First we write the system

$$\frac{\partial b_i}{\partial t} - \overline{D}D_i \Delta b_i + \overline{f} k_i b_i = F_i(t,\xi) \text{ in } Q_i, \quad i = \overline{1,n}, \tag{109}$$

$$b_i(0,\xi) = 0 \text{ in } \Omega_i, \quad i = \overline{1,n}, \tag{110}$$

$$\left. \left(-\overline{D}D_i \frac{\partial b_i}{\partial x} + \overline{D}D_{i+1} \frac{\partial b_{i+1}}{\partial x} \right) \right|_{x=x_i} = G_i\left(t, x_i, \xi'\right) \text{ on } \Sigma_i, \tag{111}$$

$$b_i = b_{i+1} \text{ on } \Sigma_i, \quad i = \overline{1, n-1}, \tag{112}$$

$$-\overline{D}D_1 \frac{\partial b_1}{\partial x} = G_0\left(t, x_0, \xi'\right) \text{ on } \Sigma_0, \tag{113}$$

$$-\overline{D}D_n \frac{\partial b_n}{\partial x} = G_n\left(t, x_n, \xi'\right) \text{ on } \Sigma_n, \tag{114}$$

$$\nabla b_i \cdot \nu = 0 \text{ on } \Sigma_i^{lat}, \quad i = \overline{1,n}, \tag{115}$$

where F_i, G_i are given by the relations (62) and (64)–(66).

We recall that $0 \leq b_i^0(t,\xi) \leq b_M$ for any $t \in [0,T]$ by Proposition 3.2 and $K_i\left(b_i^0, c_i^0\right)$ are bounded in absolute value by hypothesis i_5).

We calculate

$$\left\|b_i^0(t)K_i\left(b_i^0(t),c_i^0(t)\right)\frac{\partial c_i^0}{\partial x}(t)\right\|^2_{L^2(\Omega_i)}=\int_\Omega\left|b_i^0(t)K_i\left(b_i^0(t),c_i^0(t)\right)\frac{\partial c_i^0}{\partial x}(t)\right|^2\mathrm{d}\xi\le$$

$$\le K_M^2b_M^2\int_\Omega\left|\frac{\partial c_i^0}{\partial x}(t)\right|^2\mathrm{d}\xi\le K_M^2b_M^2\left\|c_i^0(t)\right\|^2_{H_0^1(\Omega_i)}\le C\text{ a.e. }t\in(0,T)\,. \tag{116}$$

Here we used the relation (105).

Now, we recall the next result (see [9]): if $\eta\in H^1(\Omega)$, $\theta\in H^1(\Omega)$ then $\eta\theta\in L^2(\Omega)$ and

$$\|\eta\theta\|\le C\,\|\eta\|_{H^1(\Omega)}\,\|\theta\|_{H^1(\Omega)} \tag{117}$$

and we calculate

$$\left\|\frac{\partial}{\partial x}\left(b_i^0(t)K_i\big(b_i^0(t),c_i^0(t)\big)\frac{\partial c_i^0}{\partial x}(t)\right)\right\|_{L^2(\Omega_i)}\le \tag{118}$$

$$\le K_M\left\|\frac{\partial b_i^0}{\partial x}(t)\frac{\partial c_i^0}{\partial x}(t)+b_i^0(t)\frac{\partial^2c_i^0}{\partial x^2}(t)\right\|_{L^2(\Omega_i)}\le$$

$$\le K_M\left\|\frac{\partial b_i^0}{\partial x}(t)\frac{\partial c_i^0}{\partial x}(t)\right\|_{L^2(\Omega_i)}+K_M\left\|b_i^0(t)\frac{\partial^2c_i^0}{\partial x^2}(t)\right\|_{L^2(\Omega_i)}.$$

But $b_i^0\in L^2\left(0,T;H^2(\Omega_i)\right)$ and we get that $\frac{\partial b_i^0}{\partial x}(t)\in H^1(\Omega_i)$ for $i=\overline{1,n}$. So, for the first norm we use (117). For the second norm we have $0\le b_i^0(t)\le b_M$ for any $t\in[0,T]$ and we use $\left\|b_i^0(t)\frac{\partial^2c_i^0}{\partial x^2}(t)\right\|_{L^2(\Omega_i)}\le b_M\left\|\frac{\partial^2c_i^0}{\partial x^2}(t)\right\|_{L^2(\Omega_i)}$. We return in (118) and we obtain

$$\left\|\frac{\partial}{\partial x}\left(b_i^0(t)K_i\left(b_i^0(t),c_i^0(t)\right)\frac{\partial c_i^0}{\partial x}(t)\right)\right\|_{L^2(\Omega_i)}\le C\text{ a.e. }t\in(0,T). \tag{119}$$

We mention that C represents several constants.

We deduce that

$$b_i^0\,(t,\cdot)\,K_i\left(b_i^0,c_i^0\right)\frac{\partial c_i^0}{\partial x}\,(t,\cdot)\in H^1\left(\Omega_i\right)\text{ a.e. }t\in(0,T) \tag{120}$$

and so its trace on Γ_i exists on $L^2\left(\Gamma_i\right)$ implying that

$$G_i\in L^2\left(0,T;L^2\left(\Gamma_i\right)\right),\quad i=\overline{1,n}. \tag{121}$$

Next, we know by (62) that

$$F_i(t,\xi)=\overline{f}\widetilde{f}_i\left(b_i^0,c_i^0\right)-\nabla\cdot\left[b_i^0K_i\left(b_i^0,c_i^0\right)\nabla c_i^0\right].$$

By hypothesis i$_6$), $\widetilde{f}_i\left(b_i^0,c_i^0\right)\in L^\infty\left(Q_i\right)$ and by (120) it yields that

$$\nabla\cdot\left[b_i^0K_i\left(b_i^0,c_i^0\right)\nabla c_i^0\right]\in L^2\left(Q_i\right).$$

So, we obtain that $F_i \in L^2(Q_i)$.

We recall the definition of operator $A : V \to V'$

$$\langle Ab, \psi \rangle_{V',V} = \int_\Omega [D(x)\nabla b \cdot \nabla \psi + k(x) b\psi] \, d\xi, \quad \forall \psi \in V.$$

Then for a.e. $t \in (0,T)$ we define $E(t) : V \to V'$ by

$$\langle E(t), \psi \rangle_{V',V} = \sum_{i=1}^n \int_{\Omega_i} F_i(t)\psi d\xi + \sum_{i=1}^n \int_{\Gamma_i} G_i\left(t, x, \xi'\right) \psi d\sigma \tag{122}$$

for any $\psi \in V$ and so we are led to the Cauchy problem

$$\frac{db}{dt}(t) + Ab(t) = E(t) \text{ a.e. } t \in (0,T), \tag{123}$$

$$b(0) = 0. \tag{124}$$

Equivalently, it can be written

$$\int_0^T \left\langle \frac{db}{dt}(t), \psi(t) \right\rangle_{V',V} dt + \int_Q D(x)\nabla b \cdot \nabla \psi d\xi dt + \int_Q k(x) b\psi d\xi dt$$
$$= \sum_{i=1}^n \int_{Q_i} F_i \psi d\xi dt + \sum_{i=1}^n \int_{\Sigma_i} G_i(t, x_i, \xi')\psi d\sigma dt$$

for any $\psi \in V$.

THEOREM 4.1. *The problem* (123)–(124) *has a unique strong solution*

$$b \in W^{1,2}([0,T]; V') \cap L^2(0,T; V) \cap C([0,T]; L^2(\Omega)) \tag{125}$$

which satisfies the estimate

$$\|b(t)\|^2 + \alpha_0 \int_0^t \|b(\tau)\|_V^2 \, d\tau \le \frac{1}{\alpha_0} \int_0^t \|E(\tau)\|_{V'}^2 \, d\tau, \tag{126}$$

where $\alpha_0 = \min\{D_0, k_0\}$.

Proof. We know that the operator A satisfies the hypotheses of Lions' theorem and that $E(t) \in L^2(0,T;V')$. We conclude that the system (123)–(124) has a unique strong solution as claimed in (125).

To obtain (126) we multiply (123) by b, integrate over $(0,t)$ and we have

$$\frac{1}{2}\|b(t)\|^2 - \frac{1}{2}\|b_0\|^2 + \int_0^t D(x)\|\nabla b(\tau)\|^2 \, d\tau +$$
$$+ \int_0^t k(x)\|b(\tau)\|^2 \, d\tau \le \int_0^t \|E(\tau)\|_{V'} \|b(\tau)\|_V \, d\tau.$$

Using the hypotheses for $D(x)$ and $k(x)$ we can write

$$\frac{1}{2}\|b(t)\|^2 + D_0\int_0^t \|\nabla b(\tau)\|^2\,\mathrm{d}\tau + k_0\int_0^t \|b(\tau)\|^2\,\mathrm{d}\tau \le$$

$$\le \frac{1}{2\alpha_0}\int_0^t \|E(\tau)\|_{V'}^2\,\mathrm{d}\tau + \frac{\alpha_0}{2}\int_0^t \|b(\tau)\|_V^2\,\mathrm{d}\tau$$

and so we get

$$\frac{1}{2}\|b(t)\|^2 + \alpha_0\int_0^t \|b(\tau)\|_V^2\,\mathrm{d}\tau \le \frac{1}{2\alpha_0}\int_0^t \|E(\tau)\|_{V'}^2\,\mathrm{d}\tau + \frac{\alpha_0}{2}\int_0^t \|b(\tau)\|_V^2\,\mathrm{d}\tau,$$

where $\alpha_0 = \min\{D_0, k_0\}$. From here we get (126) as we claimed. □

Now, we resume the system (50)–(61) for the ε^1-order approximation for c_i^1 written without the "1" superscript symbol

$$\frac{\partial c_i}{\partial t} - \overline{\delta}\delta_i\Delta c_i + \overline{\beta_1}\beta_{1i}b_i^0(t)c_i = H_i(t,\xi) \text{ in } Q_i, \quad i=\overline{1,n}, \tag{127}$$

$$c_i(0,\xi) = 0 \text{ in } \Omega_i, \quad i=\overline{1,n}, \tag{128}$$

$$\delta_i\frac{\partial c_i}{\partial x} = \delta_{i+1}\frac{\partial c_{i+1}}{\partial x} \text{ on } \Sigma_i, \quad i=\overline{1,n-1}\ , \tag{129}$$

$$c_i = c_{i+1} \text{ on } \Sigma_i, \quad i=\overline{1,n-1}, \tag{130}$$

$$c_i(t,\xi) = 0, \quad (t,\xi)\in\Sigma = \Sigma_0\cup\Sigma_n\cup\Sigma_i^{lat}, \quad i=\overline{1,n}. \tag{131}$$

We recall that $H_i(t,\xi)$ is given by relation (63). We know by Proposition 3.4 that c_i^0 is bounded and by (107) we deduce that $\frac{\partial c_i^0}{\partial t}(t)\in L^2(\Omega_i)$ a.e. $t\in(0,T)$. Since $\triangle c_i^0(t)\in L^2(\Omega_i)$ a.e. $t\in(0,T)$ and $b_i^1(t)\in L^2(\Omega_i)$, with these arguments we obtain that $H_i\in L^2\left(0,T;L^2(\Omega_i)\right)$.

We define the operator $B_1(t): V_0\to V_0'$ by

$$\langle B_1(t)c,\psi\rangle_{V_0',V_0} = \int_\Omega \left[\delta(x)\nabla c\cdot\nabla\psi + \beta_1(x)b^0(t)c\psi\right]\mathrm{d}\xi$$

for any $\psi\in V_0$ and $H(t): V_0\to V_0'$ by

$$\langle H(t),\psi\rangle_{V_0',V_0} = \sum_{i=1}^n\int_{\Omega_i} H_i(t)\psi\,\mathrm{d}\xi \text{ a.e. } t\in(0,T).$$

So, we have the Cauchy problem

$$\frac{\mathrm{d}c}{\mathrm{d}t}(t) + B_1(t)c(t) = H(t) \text{ a.e. } t\in(0,T), \tag{132}$$

$$c(0) = 0. \tag{133}$$

THEOREM 4.2. *The problem* (132)–(133) *has a unique strong solution*

$$c\in W^{1,2}([0,T];V_0')\cap L^2(0,T;V_0)\cap C([0,T];L^2(\Omega)) \tag{134}$$

which satisfies the estimate

$$\|c(t)\|^2 + \delta_0 \int_0^t \|c(\tau)\|_{V_0}^2 \,\mathrm{d}\tau \le \frac{1}{\delta_0} \int_0^t \|H(\tau)\|_{V_0'}^2 \,\mathrm{d}\tau. \tag{135}$$

Proof. The proof is the same like in Theorem 4.1. To obtain (135) we multiply (132) by $c(t)$ and integrate over $(0,t)$

$$\frac{1}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau} \|c(\tau)\|^2 \,\mathrm{d}\tau + \int_0^t \langle B(\tau)c(\tau), c(\tau)\rangle_{V_0',V_0} \,\mathrm{d}\tau = \int_0^t \langle H(\tau), c(\tau)\rangle_{V_0',V_0} \,\mathrm{d}\tau.$$

Further, we can write

$$\frac{1}{2}\|c(t)\|^2 + \int_0^t \int_\Omega \delta(x) \left(\nabla c(\tau)\right)^2 \mathrm{d}\xi \mathrm{d}\tau + \int_0^t \int_\Omega \beta_1(x) b^0(\tau) c^2(\tau) \mathrm{d}\xi \mathrm{d}\tau$$
$$= \int_0^t \langle H(\tau), c(\tau)\rangle_{V_0',V_0} \,\mathrm{d}\tau$$

and using the hypotheses and the result from Proposition 3.2 we get

$$\frac{1}{2}\|c(t)\|^2 + \delta_0 \int_0^t \|\nabla c(\tau)\|^2 \,\mathrm{d}\tau \le \int_0^t \|H(\tau)\|_{V_0'} \|c(\tau)\|_{V_0} \,\mathrm{d}\tau.$$

Next, we have that

$$\frac{1}{2}\|c(t)\|^2 + \delta_0 \int_0^t \|c(\tau)\|_{V_0}^2 \,\mathrm{d}\tau \le \frac{1}{2\delta_0} \int_0^t \|H(\tau)\|_{V_0'}^2 \,\mathrm{d}\tau + \frac{\delta_0}{2} \int_0^t \|c(\tau)\|_{V_0}^2 \,\mathrm{d}\tau$$

and from here we obtain (135). □

It is obvious that by Theorem 4.1 and Theorem 4.2 we can write

$$b_i^1 \in W^{1,2}([0,T]; \left(H^1(\Omega_i)\right)') \cap L^2\left(0,T; H^1(\Omega_i)\right) \cap C([0,T]; L^2(\Omega_i)),$$
$$c_i^1 \in W^{1,2}([0,T]; H^{-1}(\Omega_i)) \cap L^2\left(0,T; H_0^1(\Omega_i)\right) \cap C([0,T]; L^2(\Omega_i))$$

for each layer i, $i = \overline{1,n}$.

Finally, with the results for the approximations ε^0 and ε^1 it can be characterized the solution for the chemotaxis model.

Corollary 4.3. *Problem* (22)–(35) *admits a unique solution up the order of approximation* ε,

$$b \in W^{1,2}\left([0,T]; V'\right) \cap L^2\left(0,T; V\right) \cap C\left([0,T]; L^2(\Omega)\right),$$
$$c \in W^{1,2}([0,T]; V_0') \cap L^2\left(0,T; V_0\right) \cap C([0,T]; L^2(\Omega)), \tag{136}$$

given by

$$b(t,\xi) = b^0(t,\xi) + \varepsilon b^1(t,\xi) + 0(\varepsilon^2),$$
$$c(t,\xi) = c^0(t,\xi) + \varepsilon c^1(t,\xi) + 0(\varepsilon^2) \to 0. \tag{137}$$

Remark. This implies that problem (22)–(35) has a unique solution up to the order ε in each layer i, $i = \overline{1,n}$ given by

$$b_i^1 \in W^{1,2}([0,T]; (H^1(\Omega_i))') \cap L^2\left(0,T; H^1(\Omega_i)\right) \cap C([0,T]; L^2(\Omega_i)),$$
$$c_i^1 \in W^{1,2}([0,T]; H^{-1}(\Omega_i)) \cap L^2\left(0,T; H_0^1(\Omega_i)\right) \cap C([0,T]; L^2(\Omega_i)),$$

with

$$b_i(t,\xi) = b_i^0(t,\xi) + \varepsilon b_i^1(t,\xi) + 0(\varepsilon^2),$$
$$c_i(t,\xi) = c_i^0(t,\xi) + \varepsilon c_i^1(t,\xi) + 0(\varepsilon^2) \to 0.$$

The further order of approximations corresponding to ε^n, $n = 2, 3, \dots$ involve equations which are similar with those for the ε^1-order of approximation, so that we no longer study them. However, for the next order of approximations it might be necessary to complete the set of hypotheses with additional assumptions regarding the properties of the functions K_i and $\widetilde{f}_i$.

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