PARABOLIC FLOW ASSOCIATED TO BLOW-UP BOUNDARY SOLUTIONS

DUMITRU FELICIAN PREDA

We prove the existence of a minimal boundary blow-up flow associated with the parabolic problem $u_t - \Delta u + f(u) = 0$ in $\Omega \times (0,\infty)$, $u(t,x) = +\infty$ in $\partial\Omega \times (0,\infty)$ and $u(0,\cdot) = u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$. Here Ω is a smooth and bounded domain in $\mathbb{R}^D$, $D \geq 1$ and f is a non-negative C^1 function satisfying $\int_1^\infty [\int_0^s f(t)\mathrm{d}t]^{-1/2}\mathrm{d}s < \infty$.

AMS 2010 Subject Classification: Primary 35K58; Secondary 35K67, 35J75, 35B50.

Key words: blow-up solution, parabolic problem, Keller-Osserman condition.

1. INTRODUCTION

Consider the following problem

$$\begin{cases} u_t - \Delta u + f(u) = 0 & \text{in } \Omega \times (0,\infty), \\ u(t,x) = +\infty & \text{in } \partial\Omega \times (0,\infty), \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^D$, $D \geq 1$, $u_0 \in L^\infty(\Omega)$ satisfies $u_0 \geq 0$. The second condition in (1.1) means in fact that $u(t,\cdot)$ blows-up on the boundary of Ω in the sense that

$$\lim_{\Omega \ni x \to x_0} u(t,x) = +\infty \quad \text{for all } x_0 \in \partial\Omega.$$

Throughout this paper we assume that $f \in C^1([0,\infty))$ is an increasing function such that $f(0) = 0$ and such that f satisfies the so-called the Keller-Osserman condition (at infinity), that is,

$$\int_1^\infty \frac{\mathrm{d}s}{\sqrt{F(s)}} < +\infty \quad \text{where } F(s) = \int_0^s f(t)\mathrm{d}t,\ s \geq 0. \tag{1.2}$$

We are interested in the boundary blow-up flows associated with (1.1) in the sense given by the definition bellow.

Definition 1.1. We say that a smooth function $u : (0,\infty) \to (0,\infty)$ is a *boundary blow up flow* associated with (1.1) starting from u_0 if u satisfies $(1.1)_1$–$(1.1)_2$ and $u(t) \to u_0$ as $t \to 0$.

The boundary blow-up solutions for elliptic problems has been largely investigated in the past decades. The study of such singular boundary value problems goes back to the pioneering work of Bieberbach in [4] on the equation $\Delta u = e^u$ in the plane. He showed that in this case there exists a unique solution u such that $u(x) - \log(d(x)^{-2})$ is bounded as $x \to \partial\Omega$. Problems of this type arise in Riemannian geometry with constant Gaussian curvature. Motivated by a problem in mathematical physics, Rademacher [14] continued the study of Bieberbach on smooth bounded domains in the space. Condition (1.2) was independently introduced by Keller [11] and Osserman [13].

To stress the importance of the above Keller-Osserman condition let us remind some interesting results from [9]. Consider first $\Phi : (0,\infty) \to (0,\infty)$ defined by

$$\Phi(\alpha) = \frac{1}{\sqrt{2}} \int_\alpha^\infty \frac{\mathrm{d}s}{\sqrt{F(s) - F(\alpha)}},$$

where we assume $\Phi(\alpha) = \infty$ if the above integral is divergent or $F(s) = F(\alpha)$ in a set having positive Lebesgue measure. We say that f satisfies the Sharpened Keller-Osserman condition if

$$\liminf_{\alpha\to\infty} \Phi(\alpha) = 0.$$

As observed in [9, Theorem 1.3], if f is increasing then the standard and the Sharpened Keller-Osserman conditions are equivalent.

Furthermore, from [9, Theorem 1.1, 1.4] we have the following equivalence between the Keller-Osserman conditions and the existence of boundary blow up solutions for stationary problems associated to (1.1):

(a) f satisfies the Keller-Osserman condition if and only if the problem

$$\begin{cases} \Delta u = f(u), & \text{in } \Omega, \\ u = +\infty, & \text{as } x \in \partial\Omega, \end{cases} \tag{1.3}$$

has a solution for a certain ball $\Omega = B_R(0) \subset \mathbb{R}^D$, $D \geq 1$;

(b) f satisfies the Sharpened Keller-Osserman condition if and only if the problem (1.3) has a solutions for any smooth and bounded domain $\Omega \subset \mathbb{R}^D$, $D \geq 1$.

Recently, problems of type (1.3) under the hypothesis f is increasing and satisfies the Keller-Osserman condition (1.2) have been studied in [1, 2, 3, 5, 6, 7, 8, 12]. We also refer the reader to the book [10] and to the survey [15] for an extensive account on this topic.

The main result in ths paper establishes that for any initial data $u_0 \in L^\infty(\Omega)$ there exists a minimal boundary blow-up flow $u(t,x)$ staring from u_0 which is also stable. Furthermore, we are also able to derive its behavior as $t \to \infty$. More precisely, we have:

THEOREM 1.2. *Assume f satisfies the Keller-Osserman condition and let $u_0 \in L^\infty(\Omega)$.*

(i) *There exists a minimal boundary blow-up flow $u(t,x)$ starting from u_0 which is stable.*

(ii) *For any $x \in \Omega$ we have*

$$\lim_{t\to\infty} u(t,\cdot) = u^* \quad \textit{pointwise in } \Omega, \tag{1.4}$$

where u^* is the unique stationary boundary blow-up solution associated with (1.1)*, that is, u^* satisfies*

$$\begin{cases} \Delta u^* = f(u^*) & \textit{in } \Omega, \\ u^* = +\infty & \textit{on } \partial\Omega. \end{cases} \tag{1.5}$$

2. PROOF OF THEOREM 1.2

(i) We shall divide the proof of Theorem 1.2(i) into two steps.

Step 1 : *An approximated problem*

Let $N \geq 1$ be a positive integer such that $N \geq \|u_0\|_{L^\infty(\Omega)}$. We prove that there exists u^N a solution of the following approximated problem

$$\begin{cases} u^N_t - \Delta u^N + f(u^N) = 0 & \text{in } \Omega \times (0,\infty), \\ u^N(t,x) = N & \text{in } \partial\Omega \times (0,\infty), \\ u^N(0,\cdot) = u_0 & \text{in } \Omega. \end{cases} \tag{2.1}$$

In order to prove the above claim, we shall use the monotone iterations combined with the fact that f is a C^1 increasing function. We first let $\Lambda_N > 0$ be such that $u \to \Lambda_N u - f(u)$ is increasing on $[0,N]$. Next, starting from $v^0 = 0$, let $\{v^k\}$ be defined as

$$\begin{cases} v^{k+1}_t - \Delta v^{k+1} + \Lambda_N v^{k+1} = \Lambda_N v^k - f(v^k) & \text{in } \Omega \times (0,\infty), \\ v^{k+1} = N & \text{in } \partial\Omega \times (0,\infty), \\ v^{k+1}(0,\cdot) = u_0 & \text{in } \Omega. \end{cases} \tag{2.2}$$

For $k = 1$ we have that v^1 satisfies

$$\begin{cases} v^1_t - \Delta v^1 + \Lambda_N v^1 = 0 & \text{in } \Omega \times (0,\infty), \\ v^1 = N & \text{in } \partial\Omega \times (0,\infty), \\ v^1(0,\cdot) = u_0 \geq 0 & \text{in } \Omega. \end{cases}$$

Since 0 is a subsolution and N is a supersolution, by the standard Maximum Principle for parabolic equations we find $0 \leq v^1(x) \leq N$ in $\Omega \times (0,\infty)$. An induction argument yields

$$0 \leq v^k(t,x) \leq v^{k+1}(t,x) \leq N \quad \text{in } \Omega \times (0,\infty).$$

Thus, the sequence $\{v^k\}$ converges increasingly to a certain u^N which is a solution of (2.1). Furthermore, using the boundary condition satisfied by u^N we find that $\{u^N\}$ is increasing with respect to N.

Step 2 : *Interior a-priori estimates*

Fix $x_0 \in \Omega$ and let $r > 0$ be such that $B(x_0,\rho) \subset \Omega$. Since f satisfies the Keller-Osserman condition, by [9, Theorem 1.3] there exists a solution u_ρ of the problem

$$\begin{cases} \Delta(u_\rho) = f(u_\rho) & \text{in } B(x_0,\rho), \\ u_\rho = +\infty & \text{on } \partial B(x_0,\rho). \end{cases}$$

Furthermore, from the results in [9], u_ρ can be chosen to be radially symmetric with respect to x_0, that is, u_ρ satisfies

$$\begin{cases} (r^{N-1}u_\rho')' = r^{N-1}f(u_\rho) & \text{for all } 0 \leq r < \rho, \\ u_\rho(\rho) = +\infty. \end{cases}$$

We next multiply the first equality in the above system by $r^{N-1}u_\rho'$ and integrate over $[0,r]$. We obtain

$$\begin{aligned} [r^{N-1}u_\rho'(r)]^2 &= 2\int_0^r s^{2N-2} f(u_\rho(s))u_\rho'(s)\mathrm{d}s \\ &\leq 2r^{2N-2}\int_0^r s^{2N-2} f(u_\rho(s))u_\rho'(s)\mathrm{d}s \\ &= 2r^{2N-2}F(u_\rho(r)), \end{aligned}$$

for all $0 \leq r < \rho$. This yields

$$\frac{u_\rho'(r)}{\sqrt{2F(u_\rho(r))}} \leq 1 \quad \text{for all } 0 \leq r < \rho.$$

Integrating in the above inequality and changing the variable we find

$$\int_{u_\rho(0)}^\infty \frac{\mathrm{d}s}{\sqrt{2F(s)}} \leq \rho.$$

Now, by taking ρ small enough, we may assume that

$$\inf u_\rho = u_\rho(0) \geq \| u_0 \|_{L^\infty(\Omega)}.$$

Also remark that ρ is independent of the point x_0. Hence, for such a value of ρ, u_ρ is a supersolution of

$$\begin{cases} u_t - \Delta u + f(u) = 0 & \text{in } B(x_0,\rho) \times (0,\infty), \\ u(0,\cdot) = u_0 & \text{in } B(x_0,\rho). \end{cases}$$

By Maximum Principle we now obtain

$$u^N(t,x) \le u_\rho(x) \quad \text{in } B(x_0,\rho).$$

If K is a compact subset of Ω, we cover K by a finite number of balls of radius ρ and deduce an uniform a-priori estimate on K for u^N. Hence, the sequence $\{u^N\}$ is bounded in $L^\infty(\mathbb{R}_t^+, L^\infty(K))$ for any compact subset $K \subset \Omega$. We can now pass to the limit in (2.1) to deduce the existence of a boundary blow up flow u associated to (1.1) that starts from u_0. By virtue of our construction and Maximum Principle, if v is another boundary blow-up flow associated with (1.1) and u_0, we easily find $v \le u^N$ for all large values of N, so $v \ge u$, that is, u is the minimal boundary blow-up flow associated (1.1) that starts from u_0. This completes the proof of (i) in our Theorem.

(ii) The stability follows from the Proposition below.

Proposition 2.1. *Let u, v be two minimal boundary blow-up flows associated with* (1.1) *starting from u_0 and v_0 respectively, with $u_0, v_0 \in L^\infty(\Omega)$. Then, for any compact set $K \subset \Omega$ we have*

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^\infty(K)} \le \|u_0 - v_0\|_{L^\infty(\Omega)} \quad \textit{for all } t > 0.$$

Proof. Let u^N and v^N be the approximations obtained at Step 1 in the proof of (i) above and let $\omega^N = v^N - u^N$. Then

$$\begin{cases} \omega_t^N - \Delta\omega^N + \frac{f(v^N)-f(u^N)}{v^N-u^N}\omega^N = 0 & \text{in } \Omega, \\ \omega^N = 0 & \text{on } \partial\Omega, \\ \omega(0) = v(0) - u(0). \end{cases}$$

Therefore, using that fact that f is increasing and the Maximum Principle, we obtain

$$\|\omega^N(t)\|_{L^\infty(\Omega} \le \|v(0) - u(0)\|_{L^\infty(\Omega)}$$

and so

$$\|\omega^N(t)\|_{L^\infty(K)} \le \|v(0) - u(0)\|_{L^\infty(\Omega)}.$$

Letting now $N \to +\infty$ we obtain the conclusion. This ends the proof of Proposition. □

(iii) Let $u_0 \in L^\infty(\Omega)$ and u be the minimal boundary blow-up flow associated with (1.1) and starting from u_0. Define $v^N(t,x) = u^N(t+h,x) - u^N(t,x)$, where $h > 0$. Then, v^N satisfies

$$\begin{cases} v_t^N - \Delta v^N + \frac{f(u^N(t+h))-f(u^N(t))}{u^N(t+h)-u^N(t)}v^N = 0 & \text{in} \quad \Omega, \\ v^N(t) = 0 & \text{on } \partial\Omega, \\ v^N = u^N(h) \ge 0 & \text{at } t = 0. \end{cases}$$

By Maximum Principle we deduce that v^N is positive, so u^N is increasing in t. Hence, u is also increasing in the time variable. Let

$$\overline{u}(x) := \lim_{t\to+\infty} u(t,x) = \sup_{t>0} u(t,x).$$

Since the blow-up solution $u^*(x)$ is a super-solution of the parabolic problem, it follows that $\overline{u}(x) \le u^*(x)$ in Ω.

To prove the converse inequality, let now $\varphi \in C_0^\infty(\Omega)$. From (1.1) we find

$$\lim_{T\to\infty} \frac{1}{T}\int_\Omega \int_0^T \left[u_t(t,x)\varphi(x) - \Delta u(t,x)\varphi(x) + f(u(t,x))\varphi(x)\right] \mathrm{d}t\mathrm{d}x = 0.$$

We next analyze separately the limits of the three quantities arising under the above integrals. We have

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T}\int_\Omega \int_0^T u_t(t,x)\varphi(x)\mathrm{d}t\mathrm{d}x &= \int_\Omega \varphi(x)\left[\lim_{T\to+\infty}\frac{1}{T}\int_0^T u_t(t,x)\mathrm{d}t\right]\mathrm{d}x\\ &= \int_\Omega \varphi(x)\left[\lim_{T\to\infty}\frac{1}{T}(u(t,x)-u_0)\right]\mathrm{d}x\\ &= 0,\end{aligned}$$

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T}\int_\Omega \int_0^T \Delta u(t,x)\varphi(x)\mathrm{d}t\mathrm{d}x &= \lim_{T\to+\infty}\frac{1}{T}\int_0^T\int_\Omega u(t,x)\Delta\varphi(x)\mathrm{d}x\mathrm{d}t\\ &= \int_\Omega \Delta\varphi(x)\left[\lim_{T\to+\infty}\frac{1}{T}\int_0^T u(t,x)\mathrm{d}t\right]\mathrm{d}x\\ &= \int_\Omega \Delta\varphi(x)\overline{u}(x)\mathrm{d}x,\end{aligned}$$

and finally,

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T}\int_\Omega \int_0^T f(u(t,x))\varphi(x)\mathrm{d}t\mathrm{d}x &= \int_\Omega \varphi(x)\left[\lim_{T\to+\infty}\frac{1}{T}\int_0^T f(u(t,x))\mathrm{d}t\right]\mathrm{d}x\\ &= \int_\Omega \varphi(x)f(\overline{u}(x))\mathrm{d}x.\end{aligned}$$

Combining the last three limits we find

$$\int_\Omega \overline{u}(x)\Delta\varphi(x)\mathrm{d}x = \int_\Omega f(\overline{u}(x))\varphi(x)\mathrm{d}x,$$

that is, $\overline{u}(x)$ is a weak solution of the problem (1.5). Hence $\overline{u}(x) = u^*(x)$ which concludes the proof of our Theorem.

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Received 6 April 2011

Nicolae Titulescu High School
Street Brazda lui Novac no. 115
200690, Craiova, Romania
felician.preda@yahoo.com