ON FRACTAL DIMENSION OF INVARIANT SETS

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We give an upper bound for the box dimension of an invariant set of a differentiable function $f : U \to M$. Here U is an open subset of a Riemannian manifold M.

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1. INTRODUCTION

Let $f : M \to M$ be a C^1-diffeomorphism defined on a Riemannian manifold M. A compact subset K of M is called invariant set of f if $f(K) \subset K$. K is usually a fractal set and it is interesting to compute or estimate the fractal dimension of K. Invariant set theories have many applications in dynamical systems and chaos. Put $f^m = f \circ f \circ \cdots \circ f$ (m-times) and define

$$b = \lim_{m\to\infty} \frac{1}{m} \log(\min\{|\det(D_x f^m)|;\, x \in K\}),$$

$$s = \lim_{m\to\infty} \frac{1}{m} \log(\max\{\|D_x f^m\|;\, x \in K\}).$$

C. Wolf proved (in [6]) that, *if $M = R^n$ and $b > 0$, then $s > 0$ and*

$$\overline{\dim}_B K \le n - \frac{b}{s} < n.$$

Here, $\overline{\dim}_B K$ denotes the upper box dimension of K. Wolf's theorem has been generalized to complete Riemannian manifolds with non-negative Ricci curvature (see [2]). In this paper, we generalize Wolf's theorem, to complete Riemannian manifolds (without conditions on curvature), as follows:

THEOREM 1.1. *Let U be an open subset of a Riemannian manifold M and $f : U \to M$ a C^1-diffeomorphism on its image. Let $K \subset U$ be a compact f-invariant set. Define*

$$b = \lim_{m\to\infty} \frac{1}{m} \log(\min\{|\det D_x f^m|,\, x \in K\}),$$

$$s = \lim_{m\to\infty} \frac{1}{m} \log(\max\{\|D_x f^m\|,\, x \in K\}).$$

If $b > 0$*, then* $s > 0$ *and*

$$\overline{\dim}_B K \leq n - \frac{b}{s} < n.$$

2. PRELIMINARIES

Let M and N be Riemannian manifolds and $f : M \to N$ be a differentiable function. We denote the tangent map of f at the point $x \in M$ by $D_x f$ and the norm of f at that point, is defined by

$$\|D_x f\| = \sup\{|D_x f(v)| : v \in T_x M;\ |v| = 1\}.$$

If $A \subset M$ then the following set is called the ϵ-neighborhood of A

$$B_\epsilon(A) = \{x \in M : d(x, a) < \epsilon \text{ for some } a \in A\}.$$

If A is bounded then the upper box dimension of A is defined (see [3]) by

$$\overline{\dim}_B A = \limsup_{\delta \to 0} \frac{\log(m_\delta A)}{-\log \delta}.$$

Here, $m_\delta A$ is the maximum number of disjoint balls of radius δ with the centers contained in A.

Theorem 2.1 (see [4, p. 143]). *Let* c_n *be the volume of the unit ball* $\{x : |x| \leq 1\}$ *in* R^n*,* M *be a Riemannian manifold of dimension* n *and* $S(x)$ *be the scalar curvature of* M *at the point* $x \in M$*. Then*

$$\mathrm{vol}(B_r(x)) = c_n r^n \left(1 - \frac{r^2}{6(n+2)} S(x) + o(r^2)\right).$$

Lemma 2.2. *If* K *is a compact subset of* M *then*

$$\overline{\dim}_B K \leq n + \limsup_{\rho \to 0} \frac{\log(\mathrm{vol}(B_\rho K))}{-\log(\rho)}.$$

Proof. Let $m_\rho(K) = m$ be the maximum number of the points $\{x_1, \ldots, x_m\}$ in K such that the balls $\{B_\rho(x_1), \ldots, B_\rho(x_m)\}$ are disjoint. We have

$$\mathrm{vol}(B_\rho K) \geq \mathrm{vol}(B_\rho(x_1)) + \cdots + \mathrm{vol}(B_\rho(x_m)).$$

Choose the number $r > 0$ small enough, such that the set $E = \overline{B_r K}$ be compact. By Theorem 2.1,

$$\mathrm{vol}(B_\rho(x)) = c_n \rho^n \left(1 - \frac{S(x)\rho^2}{6(n+2)} + o(\rho^2)\right).$$

We can choose the positive number ρ so small that $B_\rho(K) \subset E$. The continuous function $S : M \to R$, has maximum and minimum on the compact set E. Thus there is a constant number α such that for each $x \in K$

$$\mathrm{vol}(B_\rho(x)) \geq c_n \rho^n \left(1 - \frac{\alpha\rho^2}{6(n+2)} + o(\rho^2)\right).$$

Therefore,

$$\mathrm{vol}(B_\rho K) \geq m_\rho(K) c_n \rho^n \left(1 - \frac{\alpha\rho^2}{6(n+2)} + o(\rho^2)\right).$$

Then

$$\limsup_{\rho\to 0} \frac{\log(\mathrm{vol}(B_\rho K))}{-\log\rho} \geq \limsup_{\rho\to 0} \frac{\log m_\rho(K)}{-\log\rho} - n.$$

This gives the result. $\square$

We recall that for each point x in a Riemannian manifold M, there are normal open balls around x (see [4, p. 89, Theorem 2.92]). If a point y belongs to a normal ball $B_r(x)$, there is a minimizing geodesic in $B_r(x)$, joining x to y (a geodesic $\lambda : [0,1] \to B_r(x)$ such that the length of λ is equal to $d(x,y)$).

Remark 2.3 (see [2]). Let $B \subset U$ be an open subset of a Riemannian manifold M and $\varphi : U \to M$ a C^1-map. If B is bounded then

$$\mathrm{vol}(\varphi(B)) \geq \inf_{x\in B} |\det D_x\varphi| \,\mathrm{vol}(B)$$

which is called the transformation formula.

3. PROOF OF THE THEOREM

Consider a number $\delta > 0$. Since K is compact, by definition of b, s and continuity arguments, there is a number $k = k_\delta \in N$ and a positive real number ϵ, such that for each $x \in B_\epsilon(K)$

$$1 < \exp(k(b-\delta)) < |\det D_x f^k| \tag{1}$$

and

$$\|D_x f^k\| < \exp(k(s+\delta)). \tag{2}$$

Also, we can choose ϵ so small that for each $x \in K$ and each positive number $\mu \leq \epsilon$, $B_\mu(x)$ be a normal open ball around x. From now on consider the map $g = f^k$ and put

$$r_m = \frac{\epsilon}{(\exp k(s+\delta))^m}, \quad m \in N.$$

Let $x \in K$, $d(x,y) < r_1$ and consider a minimizing geodesic $\lambda : [0,1] \to B_{r_1}(x)$ joining x to y. We have

$$d(g(x),g(y)) \leq \int_0^1 |(g(\lambda(t)))'|\mathrm{d}t = \int_0^1 |(D_{\lambda(t)}g)(\lambda'(t))|\mathrm{d}t \leq$$
$$\leq \int_0^1 \|D_{\lambda(t)}g\| \cdot |\lambda'(t)|\mathrm{d}t \leq \exp(k(s+\delta)) \int_0^1 |\lambda'(t)|\mathrm{d}t =$$
$$= \exp(k(s+\delta))d(x,y) < \exp(k(s+\delta))r_1 = \epsilon.$$

This means that $g(y) \in B_\epsilon(g(x))$. Since $g(K) \subset K$ then $g(y) \in B_\epsilon K$. Therefore,

$$g(B_{r_1}K) \subset B_\epsilon K. \tag{3}$$

Now, let $x \in K$ and $d(x,y) < r_2$. We have

$$d(g^2(x),g^2(y)) \leq \int_0^1 |(g^2(\lambda(t)))'|\mathrm{d}t \leq \int_0^1 \|D_{\lambda(t)}g\| \cdot \|D_{g(\lambda(t))}g\| \cdot |\lambda'(t)|\mathrm{d}t.$$

Since $\lambda(t) \in B_{r_1}K$, then by (3) we have $g(\lambda(t)) \in B_\epsilon K$, so

$$\|D_{g(\lambda(t))}g\| \leq \exp(k(s+\delta)).$$

Thus

$$d(g^2(x),g^2(y)) \leq (\exp(k(s+\delta)))^2 \int_0^1 |\lambda'(t)|\mathrm{d}t = (\exp(k(s+\delta)))^2 r_2 = \epsilon. \tag{4}$$

Since $g^2(K) \subset K$, by using (4), we get

$$g^2(B_{r_2}K) \subset B_\epsilon K.$$

In a similar way and by induction, we can show that

$$g^m(B_{r_m}(K)) \subset B_\epsilon(K). \tag{5}$$

By Remark 2.3, we have

$$\mathrm{vol}(g^m(B_{r_m}(K))) \geq \inf_{x \in (B_{r_m}K)} |\det D_x g^m| \,\mathrm{vol}(B_{r_m}(K)). \tag{6}$$

By (1), we have $|\det D_x g| > \exp(k(b-\delta))$, so $|\det D_x g^m| > (\exp(k(b-\delta)))^m$. Thus by (6)

$$\mathrm{vol}(B_{r_m}(K)) \leq (\exp(k(b-\delta)))^{-m}\,\mathrm{vol}(g^m(B_{r_m}(K)). \tag{7}$$

Let $\mathrm{vol}\,B_\epsilon(K) = C$. Using (5) and (7) we get

$$\mathrm{vol}(B_{r_m}(K)) \leq (\exp(k(b-\delta)))^{-m}C. \tag{8}$$

Therefore,

$$\limsup_{\rho\to 0}\frac{\log(\mathrm{vol}(B_\rho(K)))}{-\log(\rho)}=\limsup_{r_m\to 0}\frac{\log(\mathrm{vol}\,B_{r_m}(K))}{-\log(r_m)}\le$$
$$\le\lim_{m\to\infty}\frac{\log(\exp(k(b-\delta))^{-m}C)}{-\log(\frac{\epsilon}{\exp(k(s+\delta))^m})}=-\frac{b-\delta}{s+\delta}.$$

Since δ is arbitrary small, then

$$\limsup_{\rho\to 0}\frac{\log(\mathrm{vol}(B_\rho(K)))}{-\log(\rho)}\le-\frac{b}{s}. \tag{9}$$

Now by (9) and Lemma 2.2, we get

$$\overline{\dim}_B K\le n-\frac{b}{s}.\quad\square$$

4. SOME APPLICATIONS

Let M be a smooth n-dimensional Riemannian manifold and $V:U\to TM$ be a C^1-vector field and let us consider the corresponding differential equation $\dot{x}=V(x)$. The differential equation generates a flow $\varphi^t:U\to M$. Consider the covariant derivative ∇V and let $S=\frac{1}{2}(\nabla V+\nabla V^t)$. Denote the eigenvalues of S by $\lambda_1(x)\ge\lambda_2(x)\ge\cdots\ge\lambda_n(x)$ ordered with respect to size and multiplicity. Now for $k=1,2,\dots n$, put $\mathrm{div}_k\,V(x)=\lambda_{n-k+1}(x)+\cdots+\lambda_n(x)$ and let

$$\Lambda_{m,K}(V)=\min_{x\in K}\mathrm{div}_m\,V(x),\Theta_K=\max_{x\in K}\mathrm{div}\,V(x).$$

The following theorem gives an upper bound estimate for box dimension of φ^t-invariant sets.

THEOREM 4.1. *If M is a Riemannian manifold and K is a compact φ^t-invariant set for some $t>0$, and*

$$\Theta_K(V)-\Lambda_{n-1,K}(V)\ge\frac{1}{n}\Lambda_{n,K}(V)>0.$$

Then

$$\overline{\dim}_B K\le n-\frac{\Lambda_{n,K}(V)}{\Theta_K(V)-\Lambda_{n-1,K}(V)}.$$

Proof. This theorem is proved in [2] for Riemannian manifolds of non-negative Ricci curvature. The proof is based on techniques used in [5] (which are not dependent on curvature) and the relation $\overline{\dim}_B K\le n-\frac{b}{s}$, which we have proved for complete Riemannian manifolds. Thus the proof is valid also for all complete Riemannian manifolds. $\square$

As another application, we give an upper bound for the box dimension of an invariant set of a conformal map. Recall that a differentiable map $f : M \to M$ on a Riemannian manifold M is called conformal if there is a differentiable positive function $\lambda : M \to R$ such that for each $x \in M$ and $V, W \in T_xM$ we have $\langle D_xf(V), D_xf(W)\rangle = (\lambda(x))^2\langle V, W\rangle$ ($\langle\ ,\ \rangle$ is the inner product of vectors). The function λ is called the conformal coefficient of f.

COROLLARY 4.2. *Let U be an open subset of a complete Riemannian manifold M and $f : U \to M$ a conformal diffeomoprphism on its image with the conformal coefficient λ, and let K be a compact f-invariant subset of U. Then we have*

$$\overline{\dim}_B K \le n\left(1 - \frac{\min_{x\in K}\lambda(x)}{\max_{x\in K}\lambda(x)}\right).$$

Proof. Since $|D_xf(V)| = \lambda(x)|V|$, then

$$|D_xf^m(V)| = \lambda(x)\lambda(f(x))\dots\lambda(f^{m-1}(x))|V|.$$

Thus

$$\begin{aligned}\|D_xf^m\| &= \sup\{|D_xf^m(V)| : |V| = 1\} = \\ &= \lambda(x)\lambda(f(x))\dots\lambda(f^{m-1}(x)) \le (\max\{\lambda(x) : x \in K\})^m.\end{aligned}$$

Then, we have

$$s \le \log(\max\{\lambda(x) : x \in K\}). \tag{10}$$

The linear map $\frac{1}{\lambda(x)}D_xf : T_xM \to T_{f(x)}M$ is an isometry, so

$$\left|\det\left(\frac{1}{\lambda(x)}D_xf\right)\right| = 1 \Rightarrow |\det D_xf| = (\lambda(x))^n.$$

Then

$$\begin{aligned}|\det D_xf^m| &= |(\det D_xf)(\det D_{f(x)}f)\dots(\det D_{f^{m-1}(x)}f)| \\ &= (\lambda(x)\lambda(f(x))\dots\lambda(f^{m-1}(x)))^n \\ &\Rightarrow \min\{|\det D_xf^m| : x \in K\} \ge (\min\{\lambda(x) : x \in K\})^{mn}.\end{aligned}$$

Therefore,

$$b \ge n\log(\min\{\lambda(x) : x \in K\}). \tag{11}$$

Now, by (10) and (11) and our main theorem, we get the result. □

4.1. **The Lorenz model and Henon maps**

The Lorenz model is a well known model in physics (see [3, p. 186]). E. Lorenz derived this model from equations arising in atmosphere physics. The Lorenz model has important implications for climate and weather prediction. To study the attractor and periodic orbits of the Lorenz system, the Poincare section-maps play an important role. We recall that a Poincare section is a two dimensional manifold S which is transversal to the flow of the Lorenz system. Let a Poincare section S be a plane and $r(t)$ be an orbit intersecting S. If x is a point in S, contained in $r(t)$, then we have $x = r(t_x)$, for some real number t_x. Let $t'_x = \inf\{t : t > t_x \text{ and } r(t) \in S\}$. A Poincare section- map is defined by $P : S(\simeq R^2) \to S(\simeq R^2)$, $P(x) = r(t'_x)$. Simplified models of the Poincare-section maps are Henon maps. We recall that a Henon map is a polynomial diffeomorphisms of the form $f : R^2 \to R^2$, $f(x, y) = (y, g(y) + ax)$ where $g(y)$ is a real polynomial of degree at least two and a is a non-zero real number. Let K be the union of periodic orbits of f. K is a compact invariant set of f. If $|a| < 1$ then by using Theorem 1.1, one can show that

$$(*) \qquad \overline{\dim}_B K \leq 2 - \frac{\log(|a|^{-1})}{s}.$$

If f has an attractor then $(*)$ gives also an upper bound for the box dimension of the attractor.

5. **CONCLUSION**

1. The inequality $\overline{\dim}_B K \leq n - \frac{b}{s}$ theoretically gives a good upper bound estimate. But calculation of b and s is not easy in general, because max and min in the formula of b and s are taken on invariant set K, which is not characterized in general. When M is compact then the continuous functions $|\det Df|, \|Df\| : M \to R$ have maximum and minimum which may be useful, for estimating b and s. Also the mean values $e = \frac{1}{\mathrm{Vol}(M)} \int_M |\det Df|$ and $g = \frac{1}{\mathrm{Vol}(M)} \int_M \|Df\|$ may give good bounds for b and s.

2. Because the box dimension of a set is always greater or equal to its Hausdorff dimension, the inequality of our theorem, also gives an upper bound for the Hausdorff dimension.

3. If the f-invariant set K admits an equivariant splitting of the tangent bundle, one can see other estimates for Hausdorff dimension of K in [5].

4. Our theorem is about forward f-invariant sets. A compact set is called backward f-invariant if $f^{-1}(K) \subset K$. By similar way to this paper, we can find upper bound for box dimension of backward invariant sets, related to the values and behaviour of $\det D_x f$ and $\|D_x f\|$, $x \in K$.

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