ON THE CONNECTIVITY OF THE ATTRACTORS OF RECURRENT ITERATED FUNCTION SYSTEMS

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The aim of the paper is to give necessary and sufficient conditions for the attractor of a recurrent iterated function system to be arcwise connected. Recurrent iterated function systems are a generalization of iterated function systems. Instead of taking contractions from a metric space (X, d) to itself in the definition of an iterated function system we take contractions from $X \times X$ to X.

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1. INTRODUCTION

Iterated function systems (IFSs) were conceived in the present form by John Hutchinson [4] and popularized by Michael Barnsley [1] and are one of the most common and most general ways to generate fractals. Many of the important examples of sets and functions with special and unusual properties in analysis turn out to be fractal sets or functions whose graph are a fractal sets and a great part of them are attractors of IFSs. There is a current effort to extend the classical Hutchinson's framework to more general spaces and infinite iterated function systems (IIFSs) or more generally to multifunction systems and to study them. A recent such extension of the IFS theory can be found in [7], where the Lipscomb's space-which was an important example in dimension theory – can be obtained as an attractor of an IIFS defined in a very general setting. In this setting the attractor can be a closed and bounded set in contrast with the classical theory where only compact sets are considered. Although the fractal sets are defined with measure theory – being sets with noninteger Hausdorff dimension [2], [3] – it turns out that they have interesting topological properties as we can see from the above example [7]. One of the most important result in this direction is given in Theorem 1.2 below (see [11] for a proof) which states when the attractor of an IFS is a connected set. We intend to extend this result to recurrent iterated function system (see [8], [9]; see [5], [6] for a generalization of results from [8]).

The paper is divided in four parts. The first part is the introduction. In the second part is given the description of the shift space of a recurrent iterated function system. The main result, Theorem 3.1, is contained in the third part. The last part contains some examples.

We start by a short presentation of recurrent iterated function systems, RIFS for short. We will also fix the notations.

Let (X, d) be a metric space and $A \subset X$. By $\delta(A)$ we understand the diameter of A, i.e., $\delta(A) = \sup_{x,y \in A} d(x, y)$.

Let (X, d) be a metric space and $K(X)$ be the set of nonvoid compact subsets of X. $K(X)$ with the distance Hausdorff-Pompeiu $h : K(X) \times K(X) \to [0, +\infty)$ defined by

$$h(A, B) = \max(D(A, B), D(B, A)),$$

where

$$D(A, B) = \sup_{x \in A} d(x, B) = \sup_{x \in A} \Big(\inf_{y \in B} d(x, y) \Big).$$

is a metric space.

$(K(X), h)$ is a complete metric space provided that (X, d) is a complete metric space, compact provided that (X, d) is compact and separable provided that (X, d) is separable (see [1], [2] or [10]).

For a function $f : X \to X$ let us denote by $\mathbf{Lip}(f) \in [0, +\infty]$ the Lipschitz constant associated to f,

$$\mathbf{Lip}(f) = \sup_{x,y \in X;\, x \neq y} \frac{d(f(x), f(y))}{d(x, y)}.$$

f is a Lipschitz function if $\mathbf{Lip}(f) < +\infty$ and a contraction if $\mathbf{Lip}(f) < 1$.

For a function $f : X \times X \to X$ the number

$$\mathbf{Lip}(f) = \sup_{\substack{x,y,x_1,y_1 \in X;\\ x \neq y \text{ or } x_1 \neq y_1}} \frac{d(f(x, x_1), f(y, y_1))}{\max\{d(x, y), d(x_1, y_1)\}}$$

is named the Lipschitz constant of f.

The function $f : X \times X \to X$ is a Lipschitz function if $\mathbf{Lip}(f) < +\infty$ and a contraction if $\mathbf{Lip}(f) < 1$.

An iterated function systems on X consists of a finite family of contractions $(f_k)_{k=\overline{1,n}}$ on X and it is denoted by $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$.

A recurrent iterated function systems on X consists of a finite family of contractions $(f_k)_{k=\overline{1,n}}$, $f_k : X \times X \to X$ and is denoted by $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$.

For an IFS (respectively for a RIFS) $F_{\mathcal{S}} : K(X) \to K(X)$ (respectively $F_{\mathcal{S}} : K(X) \times K(X) \to K(X)$ for a RIFS) is the function defined by $F_{\mathcal{S}}(B) =$

$\bigcup_{k=1}^{n} f_k(B)$ (respectively by $F_{\mathcal{S}}(K,H) = \bigcup_{k=1}^{n} f_k(K,H)$ for a RIFS, where for a function $f : X \times X \to X$, $f(K,H) = f(K \times H) = \{f(x,y) \mid x \in K,\, y \in H\}$).

The function $F_{\mathcal{S}}$ is in both cases a contraction with

$$\mathbf{Lip}(F_{\mathcal{S}}) \leq \max_{k=\overline{1,n}} \mathbf{Lip}(f_k).$$

We remark that every IFS is a particular case of a RIFS.

Using the Banach contraction theorem there exists, for an IFS or a RIFS, a unique set $A(\mathcal{S})$ such that $F_{\mathcal{S}}(A(\mathcal{S})) = A(\mathcal{S})$, respectively $F_{\mathcal{S}}(A(\mathcal{S}), A(\mathcal{S})) = A(\mathcal{S})$. We state the results for RIFS (see [8], [9] or [5], [6] for a general case).

Theorem 1.1. *Let (X,d) be a complete metric space and $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ be a* RIFS *with $c = \max_{k=\overline{1,n}} \mathbf{Lip}(f_k) < 1$. Then there exists a unique set $A(\mathcal{S}) \in K(X)$ such that $F_{\mathcal{S}}(A(\mathcal{S}), A(\mathcal{S})) = A(\mathcal{S})$. Moreover, for any $H_0, H_1 \in K(X)$ the sequence $(H_n)_{n\geq 1}$ defined by $H_{n+1} = F_{\mathcal{S}}(H_n, H_{n-1})$ is convergent to $A(\mathcal{S})$. For the speed of the convergence we have the following estimation*

$$h(H_n, A(\mathcal{S})) \leq \frac{2c^{\left[\frac{n}{2}\right]}}{1-c} \max\left\{h(H_0,H_1), h(H_1,H_2)\right\}.$$

Definition 1.1. Let (X,d) be a metric space and $(A_i)_{i\in I}$ a family of nonvoid subsets of X. The family $(A_i)_{i\in I}$ is said to be connected if for every $i,j \in I$ there exists $(i_k)_{k=\overline{1,n}} \subset I$ such that $i_1 = i$, $i_n = j$ and $A_{i_k} \cap A_{i_{k+1}} \neq \emptyset$ for every $k \in \{1,2,\ldots,n-1\}$.

Definition 1.2. A metric space (X,d) is arcwise connected if for every $x,y \in X$ there exists a continuous function $\varphi : [0,1] \to X$ such that $\varphi(0) = x$ and $\varphi(1) = y$.

Concerning the connectivity of the attractor of an IFS we have the following theorem (see [11]).

Theorem 1.2. *Let (X,d) be a complete metric space, $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ be an* IFS *with $c = \max_{k=\overline{1,n}} \mathbf{Lip}(f_k) < 1$ and $A(\mathcal{S})$ be the attractor of $\mathcal{S}$. The following three statements are equivalent:*

(1) *the family $(A_i)_{i=\overline{1,n}}$ is connected, where $A_i = f_i(A(\mathcal{S}))$;*

(2) *$A(\mathcal{S})$ is arcwise connected;*

(3) *$A(\mathcal{S})$ is connected.*

We want to prove a similar result for a RIFS.

2. THE SHIFT SPACE FOR A RIFS

In this section we describe the construction of the shift space of a RIFS and we present the main properties concerning the relation between the attractor of a RIFS and the shift space. The shift space for a RIFS will be used in the proof of Theorem 3.1. The proofs can be found in [9].

Through this section (X, d) will be a fixed complete metric space and $\mathcal{S} = \left(X, (f_k)_{k=\overline{1,n}}\right)$ a fixed RIFS with n functions.

We will start with some notations: k, l, m, i, j, m', i_j denote natural numbers, if we do not say otherwise, $\mathbb{N}$ denotes the set of natural number, $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$ and $\mathbb{N}^*_n = \{1, 2, \ldots, n\}$.

For a nonvoid set I and a family of functions $f_i : X_i \to Y_i$ where $i \in I$, $\underset{i\in I}{\times} f_i$ denotes the function $\underset{i\in I}{\times} f_i : \underset{i\in I}{\times} X_i \to \underset{i\in I}{\times} Y_i$ defined by $\underset{i\in I}{\times} f_i((x_i)_{i\in I}) = (f_i(x_i))_{i\in I}$.

Let $\Omega_k = \mathbb{N}^*_{n^{2^{k-1}}}$ for $k \in \mathbb{N}^*$. On Ω_k we consider the discrete metric $d_k : \Omega_k \times \Omega_k \to \mathbb{R}_+$ given by $d_k(x, y) = 1 - \delta^y_x$ where $\delta^y_x = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$.

Let us fix a bijection ϕ_k between $\Omega_k \times \Omega_k = \mathbb{N}^*_{n^{2^{k-1}}} \times \mathbb{N}^*_{n^{2^{k-1}}}$ and $\Omega_{k+1} = \mathbb{N}^*_{n^{2^k}}$ given by $\phi_k(a, b) = n^{2^{k-1}}(a - 1) + b$.

Let $p^k_1 : \Omega_k \times \Omega_k \to \Omega_k$ and $p^k_2 : \Omega_k \times \Omega_k \to \Omega_k$ be defined by $p^k_1(x, y) = x$ and $p^k_2(x, y) = y$.

Let $\psi^k_1 : \Omega_{k+1} \to \Omega_k$ and $\psi^k_2 : \Omega_{k+1} \to \Omega_k$ be defined by $\psi^k_1(x) = p^k_1 \circ (\phi_k)^{-1}(x)$ and $\psi^k_2(x) = p^k_2 \circ (\phi_k)^{-1}(x)$.

More general we consider $\phi_{lk} : (\Omega_l)^{2^{k-l}} \to \Omega_k$, the function defined by

$$\phi_{lk} = \phi_{k-1} \circ (\phi_{k-2} \times \phi_{k-2}) \circ \cdots \circ \left(\underset{i=1}{\overset{2^{k-l-1}}{\times}} \phi_l\right) \text{ for } 0 < l < k.$$

In particular, $\phi_{(k-1)k} = \phi_{k-1}$.

Also, let $\theta_k : \Omega_k \to \Omega_{k-1} \times \Omega_{k-1}$ be the inverse of ϕ_{k-1} and $\theta_{kl} : \Omega_k \to (\Omega_l)^{2^{k-l}}$ be the inverse of ϕ_{lk}. That is $\theta_k = \phi^{-1}_{k-1}$ and

$$\theta_{kl} = \left(\underset{i=1}{\overset{2^{k-l-1}}{\times}} \phi^{-1}_l\right) \circ \left(\underset{i=1}{\overset{2^{k-l-2}}{\times}} \phi^{-1}_{l-1}\right) \circ \cdots \circ \left(\phi^{-1}_{k-2} \times \phi^{-1}_{k-2}\right) \circ \phi^{-1}_{k-1} =$$

$$= \left(\underset{i=1}{\overset{2^{k-l-1}}{\times}} \theta_{l+1}\right) \circ \left(\underset{i=1}{\overset{2^{k-l-2}}{\times}} \theta_l\right) \circ \cdots \circ (\theta_{k-1} \times \theta_{k-1}) \circ \theta_k \text{ for } k > l > 0.$$

Set $p^n_j = p^{n,X}_j : X^{2^n} \to X$, $p^{n,l}_j(x_1, x_2, \ldots, x_{2^n}) = x_j$ where X is a nonvoid set. In particular, if $X = \Omega_l$ set $p^{n,l}_j = p^{n,\Omega_l}_j$. Then $p^{n,l}_j : (\Omega_l)^{2^n} \to \Omega_l$, $p^{n,l}_j(x_1, x_2, \ldots, x_{2^n}) = x_j$.

Set also $\tau_j^{k,l} = p_j^{k-l,l} \circ \theta_{kl} : \Omega_k \to \Omega_l$ for $0 < l < k$ and $j \in \{1, 2, \ldots, 2^{k-l}\}$.

Remark 2.1. With the above notations we have
(1) $\tau_j^{k+1,k} = \psi_j^k$, where $j \in \{1, 2\}$.
(2) $\left(\underset{i=1}{\overset{2^{k-l}}{\times}} \theta_{lm}\right) \circ \theta_{kl} = \theta_{km}$, where $0 < m < l < k$.
(3) $p_{i_2}^{m,l} \circ p_{i_1}^{m',X} = p_{i_2+2^m(i_1-1)}^{m'+m,l}$, where $m', m, l \in \mathbb{N}^*$, $i_1 \in \{1, 2, \ldots, 2^{m'}\}$, $i_2 \in \{1, 2, \ldots, 2^m\}$ and $X = (\Omega_l)^{2^m}$.
(4) $\theta_{mk} \circ p_i^{u,m} = p_i^{u,X} \circ \left(\underset{i=1}{\overset{2^u}{\times}} \theta_{mk}\right)$, where $0 < k < m$, $u \in \mathbb{N}^*$ and $X = (\Omega_k)^{2^{m-k}}$. In particular if $u = l - m$ we obtain $\theta_{mk} \circ p_i^{l-m,m} = p_i^{l-m,X} \circ \left(\underset{i=1}{\overset{2^{l-m}}{\times}} \theta_{mk}\right)$, where $k < m < l$ and $X = (\Omega_k)^{2^{m-k}}$.
(5) $\tau_j^{m,k} \circ \tau_i^{l,m} = \tau_{j+2^{m-k}(i-1)}^{l,k}$, where $k < m < l$.
(6) $\tau_j^{k,l} = \psi_{i_1}^l \circ \psi_{i_2}^{l+1} \circ \cdots \circ \psi_{i_{k-l}}^k$, where $k < l$, $i_1, i_2, \ldots, i_{k-l} \in \{1, 2\}$ and $j = i_1 + (i_2 - 1)2 + \cdots + (i_{k-l} - 1)2^{k-l-1}$.

Definition 2.1. The space (Ω, d_Ω), where $\Omega = \underset{k\geq 1}{\times} \Omega_k$, and d_Ω is the distance $d_\Omega : \Omega \times \Omega \to \mathbb{R}_+$ given by $d_\Omega(\alpha, \beta) = \sum\limits_{k\geq 1} \frac{d_k(\alpha_k, \beta_k)}{3^k} = \sum\limits_{k\geq 1} \frac{1-\delta_{\alpha_k}^{\beta_k}}{3^k}$ is named the shift space or the code space for a RIFS with n components. An element $\omega \in \Omega$ will be written as an infinite word $\omega = \omega_1\omega_2 \ldots \omega_m\omega_{m+1} \ldots$ where $\omega_m \in \Omega_m$. In other words, $\Omega = \{f : \mathbb{N}^* \to \mathbb{N}^* \mid$ such that $f(k) \leq n^{2^{k-1}}\}$.

We remark that the convergence in (Ω, d_Ω) is in fact the convergence on components.

LEMMA 2.1. *(Ω, d_Ω) is a compact metric space.*

Definition 2.2. Let $F_k : \Omega \times \Omega \to \Omega$ be defined by $F_k(\alpha, \beta) = \omega = k\phi_1(\alpha_1, \beta_1)\phi_2(\alpha_2, \beta_2) \ldots \phi_m(\alpha_m, \beta_m) \ldots$ that is $\omega_1 = k$ and $\omega_m = \phi_{m-1}(\alpha_{m-1}, \beta_{m-1})$ for $k \in \{1, 2, \ldots, n\}$.

Definition 2.3. (1) Let $R_1 : \Omega \to \Omega$ be the function defined by $R_1(\omega) = \psi_1^1(\omega_2)\psi_1^2(\omega_3) \ldots \psi_1^m(\omega_{m+1}) \ldots$ that is $(R_1(\omega))_n = \psi_1^n(\omega_{n+1})$.
(2) Let $R_2 : \Omega \to \Omega$ be defined by $R_2(\omega) = \psi_2^1(\omega_2)\psi_2^2(\omega_3) \ldots \psi_2^m(\omega_{m+1}) \ldots$ that is $(R_2(\omega))_n = \psi_2^n(\omega_{n+1})$.
(3) Let $R : \Omega \to \Omega \times \Omega$ be defined by $R(\alpha) = (R_1(\alpha), R_2(\alpha))$.

Remark 2.2. R is a continuous function.

LEMMA 2.2. *The functions* $F_k : \Omega \times \Omega \to \Omega$ *defined as above are contractions with Lipschitz constant less than* $2/3$ *and* $\Omega = \bigcup_{k=1}^{n} F_k(\Omega, \Omega)$. *In other words,* Ω *is the attractor of the* RIFS $(\Omega, (F_k)_{k=\overline{1,n}})$.

Let $[\Omega]_m = \mathop{\times}\limits_{k=1}^{m} \Omega_k = \{f : \mathbb{N}^*_m \to \mathbb{N}^* \mid \text{such that } f(k) \leq n^{2^{k-1}}\}$. The elements of $[\Omega]_m$ will be represented by words of length m, $\omega = \omega_1\omega_2 \ldots \omega_m$, where $\omega_k \in \Omega_k$. $\Omega^* = \bigcup_{m \geq 1} [\Omega]_m$ is the set of all finite words. For $\omega \in \Omega^*$, $|\omega|$ denotes the length of ω. If $\omega \in \Omega$ then $|\omega| = +\infty$.

As above, if $\omega = \omega_1\omega_2 \ldots \omega_m\omega_{m+1} \ldots$ then $[\omega]_m = \omega_1\omega_2 \ldots \omega_m$ and $[\omega]_m \in [\Omega]_m$. For $\alpha \in \Omega^*$ and $\beta \in \Omega^*$ or $\beta \in \Omega$ we denote $\alpha \prec \beta$ if $|\alpha| \leq |\beta|$ and $[\beta]_{|\alpha|} = \alpha$.

The functions F_k, for $k \in \{1, 2, \ldots, n\}$, and R, R_1, R_2 as above can be defined in a similar way on finite words on which they have similar properties.

Let $\omega \in [\Omega]_m$. We are going to define the function $f_\omega : \mathop{\times}\limits_{k=1}^{2^m} X \to X$ and its extension $f_\omega : \mathop{\times}\limits_{k=1}^{2^m} P(X) \to P(X)$. We will use the same notations for f_ω and for its extension. Since f_ω is a continuous function, one has $f_\omega\Big(\mathop{\times}\limits_{k=1}^{2^m} K(X)\Big) \subset K(X)$ and so we can consider $f_\omega : \mathop{\times}\limits_{k=1}^{2^m} K(X) \to K(X)$.

The construction will be made by induction with respect to the length of the word ω.

For $\omega = \omega_1\omega_2\omega_3 \ldots \in \Omega$ we have:

$f_{[\omega]_1} = f_{\omega_1}$, where $f_{\omega_1} : X \times X \to X$ is the ω_1 function from the definitions of the RIFS $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$;

$f_{[\omega]_2} = f_{\omega_1\omega_2}$ is the function $f_{[\omega]_2} : X \times X \times X \times X \to X$ given by

$$f_{[\omega]_2}(x_1, x_2, x_3, x_4) = f_{[\omega]_1}(f_{\psi^1_1(\omega_2)}(x_1, x_2), f_{\psi^1_2(\omega_2)}(x_3, x_4)) = \\ = f_{[\omega]_1}(f_{\tau^{2,1}_1(\omega_2)}(x_1, x_2), f_{\tau^{2,1}_2(\omega_2)}(x_3, x_4)).$$

In general,

$$f_{[\omega]_m}(x_1, x_2, \ldots, x_{2^m}) = \\ = f_{[\omega]_1}(f_{R_1([\omega]_m)}(x_1, x_2, \ldots, x_{2^{m-1}}), f_{R_2([\omega]_m)}(x_{2^{m-1}+1}, x_{2^{m-1}+2}, \ldots, x_{2^m})).$$

Let (X, d) be a complete metric space and $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ be a RIFS. Let $H, H_k \subset X$ be sets. Then

$$f_{[\omega]_m}(H_1, H_2, \ldots, H_{2^m}) = f_{[\omega]_m}(H_1 \times H_2 \times \cdots \times H_{2^m}) = \\ = \{f_{[\omega]_m}(x_1, x_2, \ldots, x_{2^m}) \mid x_k \in H_k\}$$

and

$$H_{[\omega]_m} = f_{[\omega]_m}(H, H, \ldots, H).$$

Notation 2.1. Let (X, d) be a complete metric space, $m \in \mathbb{N}^*$ and let $f : \overset{m}{\underset{k=1}{\times}} X \to X$ be a contraction. We denote by e_f the fixed point of f. If $f = f_\omega$ then we denote by e_{f_ω} or by e_ω the fixed point of $f = f_\omega$.

The main properties of the shift space and its relation with the attractor of a RIFS is contained in the following theorem.

THEOREM 2.1. *If* $A = A(\mathcal{S})$ *is the attractor of the* RIFS $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ *then:*

(1) *For* $\omega \in \Omega$, *one has* $A_{[\omega]_{m+1}} \subset A_{[\omega]_m}$. *In other words, if* $\alpha \in \Omega^*$ *and* $\beta \in \Omega^*$ *such that* $\alpha \prec \beta$ *then* $A_\beta \subset A_\alpha$.

(2) $\delta(A_{[\omega]_m}) \to 0$ *when* $m \to \infty$, *more precisely* $\delta(A_{[\omega]_m}) \leq c^m \delta(A)$.

(3) *For every* $\omega \in \Omega$ *there exists a unique* a_ω *such that* $\{a_\omega\} = \bigcap\limits_{m \geq 1} A_{[\omega]_m}$.

(4) *We have* $A = \bigcup\limits_{\omega \in [\Omega]_1} f_\omega(A, A) = \bigcup\limits_{\omega \in [\Omega]_1} A_\omega$, $A_\omega = \bigcup\limits_{\alpha \in \Omega_{|\omega|+1}} A_{\omega\alpha}$ *for* $\omega \in \Omega^*$, $A = \bigcup\limits_{\omega \in [\Omega]_m} f_\omega(A, A, \ldots, A) = \bigcup\limits_{\omega \in [\Omega]_m} A_\omega$ *and* $A_\omega = \bigcup\limits_{\beta \in [\Omega]_{|\omega|+n}, \omega \prec \beta} A_\beta$.

(5) *For* $\omega \in \Omega$, $e_{[\omega]_m} \in A_{[\omega]_m} \subset A$ *and if* a_ω *is defined by* $\{a_\omega\} = \bigcap\limits_{m \geq 1} A_{[\omega]_m}$, *then* $d(e_{[\omega]_m}, a_\omega) \to 0$ *when* $m \to \infty$.

(6) $A = \bigcup\limits_{\omega \in \Omega} \{a_\omega\}$ *and the set* $\{e_{[\omega]_m} \mid \omega \in \Omega$ *and* $m \in \mathbb{N}^*\}$ *is dense in* A. *Similarly,* $A_\alpha = \bigcup\limits_{\omega \in \Omega, \alpha \prec \omega} \{a_\omega\}$ *for every* $\alpha \in \Omega^*$ *and the set* $\{e_{[\omega]_m} \mid \omega \in \Omega$, $\alpha \prec \omega$ *and* $m \in \mathbb{N}^*\}$ *is dense in* A_α.

(7) *The function* $\pi : \Omega \to A$ *defined by* $\pi(\omega) = a_\omega$ *is continuous and surjective.*

(8) $f_k(A_{[\alpha]_m}, A_{[\beta]_m}) = A_{[F_k(\alpha,\beta)]_{m+1}}$ *and* $\pi(F_k(\alpha, \beta)) = f_k(\pi(\alpha), \pi(\beta))$ *for every* $\alpha, \beta \in \Omega$ *and* $k \in \{1, 2, \ldots, n\}$.

3. THE MAIN RESULT

For the proof of the main result (Theorem 3.1) we need the following lemma.

LEMMA 3.1. *Let* (X, d) *be a complete metric space and* $(a_n)_{n \geq 1}$ *be a sequence of positive numbers convergent to* 0. *Let* $(\Delta_l)_{l \geq 0}$ *be a sequence of divisions of the unit interval* $[0, 1]$ (*i.e.,* $\Delta_l = (y_0^l = 0 < y_1^l < \cdots < y_{n_l}^l = 1)$) *such that* $\Delta_l \subset \Delta_{l+1}$, $\lim\limits_{l \to +\infty} \|\Delta_l\| = 0$, *where* $\|\Delta_l\| = \max\limits_{i=1}^{n_l} (y_i^l - y_{i-1}^l)$. *Let* $(g_l)_{l \geq 0}$ *be* *a sequence of functions* $g_l : \Delta_l \to X$ *such that* $g_{l+1}|_{\Delta_l} = g_l$ *and for every* $m \geq n$ *and every* $y_i^m \in \Delta_m$, $\max\{d(g_m(y_i^m), g_n(y_j^n)), d(g_m(y_i^m), g_n(y_{j+1}^n))\} \leq a_n$ *whenever* $y_i^m \in [y_j^n, y_{j+1}^n]$. *Then there exists a continuous function* $g : [0,1] \to X$ *such that* $g|_{\Delta_l} = g_l$.

Proof. Let $A = \bigcup\limits_{n \geq 1} \Delta_n$ and $\tilde{g} : A \to X$ be the function defined by $\tilde{g}(x) = g_l(x)$ if $x \in \Delta_l$. The function $\tilde{g}$ is well defined because $g_m|_{\Delta_l} = g_l$ for every $m \geq l$. We intend to prove that $\tilde{g}$ is uniformly continuous. Let $\varepsilon > 0$ be fixed. Since the sequence $(a_n)_{n\geq 1}$ is convergent to 0, there exists n_ε such that for every $n \geq n_\varepsilon$ $a_n < \varepsilon/2$. Set $\delta_l = \min\limits_{i=1}^{n_l}\big(y_i^l - y_{i-1}^l\big)$ and $\delta = \delta_{n_\varepsilon}$. We have $0 < \delta_{l+1} \leq \delta_l \leq \|\Delta_l\|$.

Let $c, d \in [0,1] \cap A$ be such that $c < d$ and $d - c < \delta$. There exists $m_0 \geq n_\varepsilon$ such that $c, d \in \Delta_{m_0}$. The set $(c,d) \cap \Delta_{n_\varepsilon}$ has at most one element.

If $(c,d) \cap \Delta_{n_\varepsilon} = \emptyset$, then there exists a $j \in \{0, 1, \ldots, n_{n_\varepsilon} - 1\}$ such that $y_j^{n_\varepsilon} \leq c < d \leq y_{j+1}^{n_\varepsilon}$. In this case we have

$$d(\tilde{g}(c), \tilde{g}(d)) = d(g_{m_0}(c), g_{m_0}(d)) \leq$$
$$\leq d(g_{m_0}(c), g_{n_\varepsilon}(y_j^{n_\varepsilon})) + d(g_{n_\varepsilon}(y_j^{n_\varepsilon}), g_{m_0}(d)) \leq 2a_{n_\varepsilon} < \varepsilon.$$

If $(c,d) \cap \Delta_{n_\varepsilon} = \{y_j^{n_\varepsilon}\}$ we have $y_{j-1}^{n_\varepsilon} < c < y_j^{n_\varepsilon} < d < y_{j+1}^{n_\varepsilon}$ and

$$d(\tilde{g}(c), \tilde{g}(d)) = d(g_{m_0}(c), g_{m_0}(d)) \leq$$
$$\leq d(g_{m_0}(c), g_{n_\varepsilon}(y_j^{n_\varepsilon})) + d(g_{n_\varepsilon}(y_j^{n_\varepsilon}), g_{m_0}(d)) \leq 2a_{n_\varepsilon} < \varepsilon.$$

It follows that $\tilde{g}$ is an uniformly continuous function. Then there exists a unique continuous function $g : [0,1] \to X$ such that $g|_A = \tilde{g}$. We also have $g|_{\Delta_l} = g_l$.

THEOREM 3.1. *Let* (X, d) *be a complete metric space,* $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ *be a* RIFS, $c = \max\limits_{k=\overline{1,n}} \mathbf{Lip}(f_k) < 1$ *and* $A = A(\mathcal{S})$ *be the attractor of* $\mathcal{S}$. *The following three statements are equivalent:*

(1) *The family* $(A_i)_{i=\overline{1,n}}$ *is connected, where* $A_i = f_i(A(\mathcal{S}), A(\mathcal{S}))$;

(2) $A(\mathcal{S})$ *is arcwise connected;*

(3) $A(\mathcal{S})$ *is connected.*

Proof. Firstly we remark that, according to Theorem 1.1, we have $A(\mathcal{S}) \in K(X)$ and so $A_\omega = A(\mathcal{S})_\omega = f_\omega(A(\mathcal{S}), A(\mathcal{S}), \ldots, A(\mathcal{S})) \in K(X)$ for every $\omega \in \Omega^*$. We can suppose that $\delta(A(\mathcal{S})) \neq 0$. The case $\delta(A(\mathcal{S})) = 0$ is obvious. In this case the set A contains one point and $A = A_\omega$ for every $\omega \in \Omega^*$.

(2) $\Rightarrow$ (3) is obvious, since every arcwise connected set is connected.

(3) $\Rightarrow$ (1) Let M be $\{j \in \{1, 2, \ldots, n\} |$ there exist $(i_k)_{k=\overline{1,m}}$ such that $i_1 = 1$, $i_m = j$ and $A_{i_k} \cap A_{i_{k+1}} \neq \emptyset$ for every $k \in \{1, 2, \ldots, m-1\}\}$.

Set $V_1 = \bigcup_{j\in M} A_j$ and $V_2 = \bigcup_{j\notin M} A_j$. Then $V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 = A(\mathcal{S})$ and V_1 and V_2 are compact sets. Since $A(\mathcal{S})$ is connected and $V_1 \neq \emptyset$ (because $A_1 \subset V_1$) it follows that $A(\mathcal{S}) = V_1$.

(1) $\Rightarrow$ (2) For every indexes i and j such that $A_i \cap A_j \neq \emptyset$ let us fix $x_{i,j} \in A_i \cap A_j$. Also, for every indexes i and j let us fix $(i_k)_{k=\overline{1,m(i,j)}}$ such that $i_1 = i$, $i_{m(i,j)} = j$ and $A_{i_k} \cap A_{i_{k+1}} \neq \emptyset$ for every $k \in \{1,2,\dots,m(i,j)-1\}$. The family of indexes $(i_k)_{k=\overline{1,m(i,j)}}$ can be taken without repetition. Then $m(i,j) \leq n$. We can suppose that $m(i,j) = n$ (by taking $A_{i_n} = \dots = A_{i_{m(i,j)+1}} = A_{i_{m(i,j)}}$).

Since $A(\mathcal{S}) = \bigcup_{i=1}^{n} f_i(A(\mathcal{S}), A(\mathcal{S}))$, it follows that for every z there exists $i(z)$ such that $z \in A_{i(z)} = f_{i(z)}(A(\mathcal{S}), A(\mathcal{S}))$. Let $i : A \to \{1,2,\dots,n\}$ be a fixed function such that $z \in i(z)$.

Then, for every two elements z_0 and z_1, we have fixed $A_{i(z_0)}$ and $A_{i(z_1)}$, a family of indexes $(i_k)_{k=\overline{1,n}}$ such that $i_1 = i(z_0)$, $i_n = i(z_1)$ and $A_{i_k} \cap A_{i_{k+1}} \neq \emptyset$ for every $k \in \{1,2,\dots,n-1\}$ and elements $x_{i_k,i_{k+1}} \in A_{i_k} \cap A_{i_{k+1}}$ for $k \in \{1,2,\dots,n-1\}$. Set $w_0(z_0,z_1) = z_0$, $w_n(z_0,z_1) = z_1$, $i_k(z_0,z_1) = i_k$ for every $k \in \{1,2,\dots,n\}$ and $w_k(z_0,z_1) = x_{i_k,i_{k+1}}$ for every $k \in \{1,2,\dots,n-1\}$. We remark that $w_k(z_0,z_1), w_{k+1}(z_0,z_1) \in A_{i_k(z_0,z_1)}$ for every $k \in \{0,1,\dots,n-1\}$.

Let x_0 and x_1 be two fixed different elements from $A(\mathcal{S})$. We will define by induction after l $\Delta_l = (y_0^l = 0 < y_1^l < \dots < y_{n^l}^l = 1)$, divisions of the unit interval $[0,1]$, the functions $g_l : \Delta_l \to A(\mathcal{S})$ and the elements $\omega_k^l \in [\Omega]_l$, where $k \in \{0,1,\dots,n^l-1\}$ such that $\Delta_l \subset \Delta_{l+1}$, $g_{l+1}|_{\Delta_l} = g_l$, $g_l(y_k^l), g_l(y_{k+1}^l) \in A_{\omega_k^l}$ for $k \in \{0,1,\dots,n^l-1\}$ and if $l' \geq l$ and $y_{k'}^{l'} \in [y_k^l, y_{k+1}^l]$ then $\omega_k^l \prec \omega_{k'}^{l'}$.

Set $\Delta_0 = (y_0^0 = 0 < y_1^0 = 1)$, $g_0(0) = z_0$ and $g_0(1) = z_1$.

Set $\Delta_1 = (y_0^1 = 0 < y_1^1 < \dots < y_n^1 = 1)$, where $y_k^1 = \frac{k}{n}$, and $g_1(y_k^1) = w_k(x_0,x_1)$ for $k \in \{0,1,\dots,n\}$.

In general we will take $y_k^l = \frac{k}{n^l}$. Then $y_k^l = y_{kn}^{l+1}$ and $\Delta_l \subset \Delta_{l+1}$.

Let us suppose that g_l and ω_k^l for $k \in \{0,1,\dots,n^l-1\}$ are defined. Let $k \in \{0,1,\dots,n^l-1\}$ be fixed. Then $g_l(y_k^l), g_l(y_{k+1}^l) \in A_{\omega_k^l}$. It follows that $g(y_k^l) = f_{\omega_k^l}(z_1,z_2,\dots,z_{2^l})$ and $g(y_{k+1}^l) = f_{\omega_k^l}(z_1',z_2',\dots,z_{2^l}')$ for some $z_1,z_2,\dots,z_{2^l}$, $z_1',z_2',\dots,z_{2^l}' \in A(\mathcal{S})$. Set

$$g_{l+1}(y_{kn+j}^{l+1}) = f_{\omega_k^l}(w_j(z_1,z_1'), w_j(z_2,z_2'),\dots,w_j(z_{2^l},z_{2^l}')) \text{ for } j \in \{0,1,\dots,n\}.$$

We have

$$g_{l+1}(y_{kn}^{l+1}) = f_{\omega_k^l}(w_0(z_1,z_1'), w_0(z_2,z_2'),\dots,w_0(z_{2^l},z_{2^l}')) =$$
$$= f_{\omega_k^l}(z_1,z_2,\dots,z_{2^l}) = y_k^l$$

and

$$g_{l+1}(y^{l+1}_{kn+n}) = f_{\omega^l_k}(w_n(z_1,z'_1), w_n(z_2,z'_2), \ldots, w_n(z_{2^l},z'_{2^l})) =$$
$$= f_{\omega^l_k}(z'_1, z'_2, \ldots, z'_{2^l}) = y^l_{k+1}.$$

This means that g_{l+1} is well defined and $g_{l+1}|_{\Delta_l} = g_l$.

Let $j \in \{0,1,\ldots,n\}$ be fix. Then $w_j(z_i,z'_i), w_{j+1}(z_i,z'_i) \in A_{i_j(z_i,z'_i)}$ for every $i \in \{1,2,\ldots,2^l\}$. Let $\omega^{l+1}_{kn+j} = \omega^l_k \phi_{1l}(i_j(z_1,z'_1), i_j(z_2,z'_2), \ldots, i_j(z_{2^l},z'_{2^l}))$. Then

$$g_{l+1}(y^{l+1}_{kn+j}) = f_{\omega^l_k}(w_j(z_1,z'_1), w_j(z_2,z'_2), \ldots, w_j(z_{2^l},z'_{2^l})) \in$$
$$\in f_{\omega^l_k}(A_{i_j(z_1,z'_1)}, A_{i_j(z_2,z'_2)}, \ldots, A_{i_j(z_{2^l},z'_{2^l})}) =$$
$$= A_{\omega^l_k \phi_{1l}(i_j(z_1,z'_1), i_j(z_2,z'_2), \ldots, i_j(z_{2^l},z'_{2^l}))} = A_{\omega^{l+1}_{kn+j}}$$

and

$$g_{l+1}(y^{l+1}_{kn+j+1}) = f_{\omega^l_k}(w_{j+1}(z_1,z'_1), w_{j+1}(z_2,z'_2), \ldots, w_{j+1}(z_{2^l},z'_{2^l})) \in$$
$$\in f_{\omega^l_k}(A_{i_j(z_1,z'_1)}, A_{i_j(z_2,z'_2)}, \ldots, A_{i_j(z_{2^l},z'_{2^l})}) =$$
$$= A_{\omega^l_k \phi_{1l}(i_j(z_1,z'_1), i_j(z_2,z'_2), \ldots, i_j(z_{2^l},z'_{2^l}))} = A_{\omega^{l+1}_{kn+j}}.$$

We also have that $\omega^l_k \prec \omega^{l+1}_{kn+j}$. This implies that if $l' \geq l$ then $\omega^l_{[\frac{k}{n^{l'-l}}]} \prec \omega^{l'}_k$.

The induction hypothesis are now cheeked.

Since, for every l and $k \in \{1,2,\ldots,n^l\}$, $g_l(y^l_k), g_l(y^l_{k+1}) \in A_{\omega^l_k}$ and $\omega^l_k \in [\Omega]_l$, we have $d(g_l(y^l_k), g_l(y^l_{k+1})) \leq \delta(A_{\omega^l_k}) \leq c^l \delta(A)$.

We will apply Lemma 3.1 to the functions g_l and divisions Δ_l defined above. We have seen that $\Delta_l \subset \Delta_{l+1}$ and $g_{l+1}|_{\Delta_l} = g_l$. Since $\|\Delta_l\| = \frac{1}{n^l}$, $\lim\limits_{l\to+\infty} \|\Delta_l\| = 0$. Set $a_l = c^l\delta(A)$. Let $l' \geq l$ and $y^{l'}_{k'} \in \Delta_{l'}$ be fix. We have two cases, $y^{l'}_{k'} \in \Delta_l$ and $y^{l'}_{k'} \notin \Delta_l$.

In the first case, $y^{l'}_{k'} \in \Delta_l$, $k' = [\frac{k'}{n^{l'-l}}]n^{l'-l}$ and $y^{l'}_{k'} = y^l_k$, where $k = [\frac{k'}{n^{l'-l}}]$. Then $g_l(y^l_k) = g_l(y^{l'}_{k'}), g_l(y^l_{k+1}) \in A_{\omega^l_k}$,where $\omega^l_k \in [\Omega]_l$, and so

$$d(g_l(y^{l'}_{k'}), g_l(y^l_{k+1})) = d(g_l(y^l_k), g_l(y^l_{k+1})) \leq \delta(A_{\omega^l_{k-1}}) \leq c^l\delta(A) = a_l.$$

Also, we have

$$d(g_l(y^{l'}_{k'}), g_l(y^l_{k-1})) = d(g_l(y^l_k), g_l(y^l_{k-1})) \leq \delta(A_\omega) \leq c^l\delta(A) = a_l.$$

In the second case we have $y^{l'}_{k'} \in (y^l_k, y^l_{k+1})$, where $k = [\frac{k'}{n^{l'-l}}]$. Then $y^l_k, y^l_{k+1} \in A_{\omega^l_k}$, $\omega^l_k \prec \omega^{l'}_{k'}$ and $y^{l'}_{k'} \in A_{\omega^{l'}_{k'}} \subset A_{\omega^l_k}$. It follows that

$$\max\{d(g_{l'}(y^{l'}_{k'}), g_l(y^l_k)), d(g_{l'}(y^{l'}_{k'}), g_l(y^l_{k+1}))\} \leq \delta(A_{\omega^l_k}) \leq c^l\delta(A) = a_l.$$

In view of Lemma 3.1, there exists a continuous function $g : [0,1] \to X$ such that $g|_{\Delta_l} = g_l$. Since $g(\Delta_l) = g_l(\Delta_l) \subset A$, g is a continuous function and A is a compact set we have, $g([0,1]) \subset A$. This proves that A is arcwise connected.

4. EXAMPLES

Example 4.1. Every IFS can be seen as a RIFS. Indeed, let $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ be an IFS. Let $\overline{\mathcal{S}} = (X, (\overline{f_k})_{k=\overline{1,n}})$ be the RIFS defined by $\overline{f_k}(x,y) = f_k(x)$. Then $A(\mathcal{S}) = A(\overline{\mathcal{S}})$. In this way Theorem 1.1 is a particular case of Theorem 3.1.

Example 4.2. Let $X = \mathbb{R}$ and $g, f : \mathbb{R}\times R \to R$ be defined by $f(x,y) = \frac{x}{3} + \frac{y}{3}$ and $g(x,y) = \frac{x}{3} + \frac{2}{3}$. Let $\mathcal{S} = (\mathbb{R}, (f,g))$ be an RIFS. Let $F : K(\mathbb{R})\times K(\mathbb{R}) \to K(\mathbb{R})$ be defined by $F(K,H) = f(K,H)\cup g(K,H)$. The attractor of $\mathcal{S}$ is $[0,1]$. Indeed $f([0,1],[0,1]) = [0,2/3] \subset [0,1]$ and $g([0,1],[0,1]) = [2/3,1] \subset [0,1]$ and so $F([0,1],[0,1]) = [0,1]$. It follows that $A(\mathcal{S}) = [0,1]$.

We remark that $A(\mathcal{S})$ is connected. We also remark that the family of sets $\{A_f(\mathcal{S}) = f([0,1],[0,1]) = [0,2/3], A_g(\mathcal{S}) = g([0,1],[0,1]) = [2/3,1]\}$ is connected.

Example 4.3. Let $X = \mathbb{R}$ and $g, f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be defined by $f(x,y) = \frac{x}{5} + \frac{y}{5}$ and $g(x,y) = \frac{x}{5} + \frac{y}{5} + \frac{3}{5}$. Let $\mathcal{S} = (\mathbb{R}, (f,g))$ be a RIFS. Let $F : K(\mathbb{R})\times K(\mathbb{R}) \to K(\mathbb{R})$ be defined by $F(K,H) = f(K,H)\cup g(K,H)$. The attractor of $\mathcal{S}$ is $[0,2/5]\cup[3/5,1]$. Indeed

$$f([0,2/5]\cup[3/5,1],[0,2/5]\cup[3/5,1]) = f([0,2/5],[0,2/5]) \cup f([0,2/5],[3/5,1])\cup$$
$$\cup f([3/5,1],[3/5,1]) = [0,4/25]\cup[3/25,7/25]\cup[6/25,2/5] = [0,2/5].$$

In a similar way, $g([0,2/5]\cup[3/5,1],[0,2/5]\cup[3/5,1]) = [3/5,1]$.

We remark that $A(\mathcal{S}) = [0,2/5]\cup 3/5,1]$ is not connected. Moreover, the family of sets $\{A_f(\mathcal{S}) = [0,2/5],\ A_g(\mathcal{S}) = [3/5,1]\}$ is not connected.

Example 4.4. Let $\mathbb{X}$ be one of the spaces l_p, l_∞ or c_0 where $p \ge 1$. The elements of these spaces will be sequences of real numbers $(x_n)_{n\ge1}$.

Let $j : \mathbb{X} \to \mathbb{X}$, $i_m : \mathbb{R}^m \to \mathbb{X}$ and $\pi_1 : \mathbb{X} \to \mathbb{R}$ be given by

$$j(x_1,x_2,\ldots,x_m,\ldots) = (0,x_1,x_2,\ldots,x_m,\ldots),$$
$$i_m(x_1,x_2,\ldots,x_m) = (x_1,x_2,\ldots,x_m,0,0,0,\ldots)$$

and $\pi_1((x_n)_{n\ge1}) = x_1$.

We consider the RIFS $\mathcal{S} = (\mathbb{X}, (f_0, f_1))$ where $f_0 : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ and $f_1 : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ are given by

$$f_0(x, y) = i_1\left(\frac{\pi_1(x)}{2}\right) + \frac{j(y)}{2},$$
$$f_1(x, y) = i_1\left(\frac{\pi_1(x)}{2} + \frac{1}{2}\right) + \frac{j(y)}{2}.$$

Then $A(\mathcal{S}) = \mathop{\times}\limits_{k=0}^{\infty}[0, \frac{1}{2^k}]$.

Proof. We put $A = \mathop{\times}\limits_{k=0}^{\infty}[0, \frac{1}{2^k}]$. Then $j(A) = \{0\} \times \left(\mathop{\times}\limits_{k=0}^{\infty}\left[0, \frac{1}{2^k}\right]\right)$ and $\pi_1(A) = [0, 1]$.

We also have

$$f_0(A, A) = i_1\left(\frac{\pi_1(A)}{2}\right) + \frac{j(A)}{2} =$$
$$= [0, \tfrac{1}{2}] \times \{(0, 0, 0, \ldots)\} + \{0\} \times \left(\mathop{\times}\limits_{k=0}^{\infty}\left[0, \frac{1}{2^{k+1}}\right]\right) = [0, \tfrac{1}{2}] \times \left(\mathop{\times}\limits_{k=0}^{\infty}\left[0, \tfrac{1}{2^{k+1}}\right]\right)$$

and

$$f_1(A, A) = i_1\left(\frac{\pi_1(A)}{2} + \frac{1}{2}\right) + \frac{j(A)}{2} =$$
$$= \left[\tfrac{1}{2}, 1\right] \times \{(0, 0, 0, \ldots)\} + \{0\} \times \left(\mathop{\times}\limits_{k=0}^{\infty}\left[0, \frac{1}{2^{k+1}}\right]\right) = \left[\tfrac{1}{2}, 1\right] \times \left(\mathop{\times}\limits_{k=0}^{\infty}[0, \frac{1}{2^{k+1}}]\right).$$

Then $A = f_0(A, A) \cup f_1(A, A)$.

This proves that $A(\mathcal{S}) = A = \mathop{\times}\limits_{k=0}^{\infty}[0, \frac{1}{2^k}]$.

We remark that $A(\mathcal{S})$ is connected and that

$$A_f(\mathcal{S}) \cap A_g(\mathcal{S}) = f(A(\mathcal{S}), A(\mathcal{S})) \cap g(A(\mathcal{S}), A(\mathcal{S})) = \{1/2\} \times \left(\mathop{\times}\limits_{k=0}^{\infty}\left[0, \frac{1}{2^{k+1}}\right]\right).$$

It follows that the family of sets $\{A_f(\mathcal{S}), A_g(\mathcal{S})\}$ is connected.

Example 4.5 (a Sierpinsky like RIFS). Let $X = \mathbb{R}^2$ and $f, g, h : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f((x_1, x_2), (y_1, y_2)) = (\frac{x_1}{4} + \frac{y_1}{4}, \frac{x_2}{4} + \frac{y_2}{4})$, $g((x_1, x_2), (y_1, y_2)) = (\frac{x_1}{2} + \frac{1}{2}, \frac{x_2}{2})$ and $h((x_1, x_2), (y_1, y_2)) = (\frac{x_1}{2}, \frac{x_2}{2} + \frac{1}{2})$. Let $\mathcal{S} = (\mathbb{R}^2, (f, g, h))$ be an RIFS. Let $F_{\mathcal{S}} : K(\mathbb{R}) \times K(\mathbb{R}) \to K(\mathbb{R})$ be defined by $F_{\mathcal{S}}(K, H) = f(K, H) \cup g(K, H) \cup h(K, H)$. Let $A(\mathcal{S})$ be such that $A(\mathcal{S}) = F_{\mathcal{S}}(A(\mathcal{S}), A(\mathcal{S}))$.

Let us denote $T = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq 1\}$, $T_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq 1/2\}$, $T_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 1/2,\ x_2 \geq 0,\ x_1 + x_2 \leq 1\}$ and $T_3 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 \geq 1/2,\ x_1 + x_2 \leq 1\}$. Then $T_1 = f(T, T)$, $T_2 = g(T, T)$, $T_3 = h(T, T)$ and $F_{\mathcal{S}}(T, T) \subset T$. It follows that $A(\mathcal{S}) \subset T$ and $f(A(\mathcal{S}), A(\mathcal{S})) \subset f(T, T) = T_1$.

We give now a direct proof that $A(\mathcal{S})$ is connected.

Let $\overline{f}, \overline{g}, \overline{h} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\overline{f}(x_1, x_2) = (\frac{x_1}{2}, \frac{x_2}{2})$, $\overline{g}(x_1, x_2) = (\frac{x_1}{2} + \frac{1}{2}, \frac{x_2}{2})$ and $\overline{h}(x_1, x_2) = (\frac{x_1}{2}, \frac{x_2}{2} + \frac{1}{2})$. We remark that $\overline{g}(x_1, x_2) = g((x_1, x_2), (y_1, y_2))$ and $\overline{h}(x_1, x_2) = h((x_1, x_2), (y_1, y_2))$, for every $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$. Also, $\overline{f}(x_1, x_2) = f((x_1, x_2), (x_1, x_2))$. So, $\overline{f}(A(\mathcal{S})) \subset f(A(\mathcal{S}), A(\mathcal{S})) \subset A(\mathcal{S})$. Let $\mathbb{T}$ be the attractor of the IFS $\mathcal{S}_1 = (\mathbb{R}^2, (\overline{f}, \overline{g}, \overline{h}))$. Then $\mathbb{T}$ is the Sierpinsky triangle. It is well-known that $\mathbb{T}$ is a connected set. We have $F_{\mathcal{S}_1}(A(\mathcal{S})) = \overline{f}(A(\mathcal{S})) \cup \overline{g}(A(\mathcal{S})) \cup \overline{h}(A(\mathcal{S})) \subset A(\mathcal{S})$. This implies that $\mathbb{T} \subset A(\mathcal{S})$. We also have $f(\mathbb{T}, \mathbb{T}) = T_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq 1/2\}$. It follows that $T_1 = f(A(\mathcal{S}), A(\mathcal{S})) \subset A(\mathcal{S})$.

Let $G : K(\mathbb{R}) \to K(\mathbb{R})$ be defined by $G(K) = F_{\mathcal{S}_1}(K) \cup T_1$. G is a contraction. We have $G(A(\mathcal{S})) = A(\mathcal{S})$. Indeed,

$$G(A(\mathcal{S})) = F_{\mathcal{S}_1}(A(\mathcal{S})) \cup T_1 \subset A(\mathcal{S}) = F_{\mathcal{S}}(A(\mathcal{S}), A(\mathcal{S})) =$$
$$= f(A(\mathcal{S}), A(\mathcal{S})) \cup g(A(\mathcal{S}), A(\mathcal{S})) \cup h(A(\mathcal{S}), A(\mathcal{S})) =$$
$$= f(A(\mathcal{S}), A(\mathcal{S})) \cup \overline{g}(A(\mathcal{S})) \cup \overline{h}(A(\mathcal{S})) \subset T_1 \cup \overline{g}(A(\mathcal{S})) \cup \overline{h}(A(\mathcal{S})) = G(A(\mathcal{S})).$$

In fact, G is the set function associated to the IIFS $\mathcal{S}_2 = (\mathbb{R}^2, (\overline{f}, \overline{g}, \overline{h}, (t_{(a,b)})_{(a,b) \in T_1}))$. Let $K_0 = \mathbb{T}$ and $K_{n+1} = G(K_n)$. Then $K_0 \subset K_1 = \mathbb{T} \cup T_1$ and so $K_n \subset K_{n+1}$. It results that $A(\mathcal{S}_2) = G(A(\mathcal{S}_2)) = \overline{\bigcup_{n \geq 0} K_n} = A(\mathcal{S})$.

We remark that the sets K_n are connected. This can be proved by induction with respect to $n = 1, 2, \ldots$. $K_0 = \mathbb{T}$ is a connected set. Let us suppose that K_n is connected. Then $K_{n+1} = K_n \cup \overline{f}(K_n) \cup \overline{g}(K_n) \cup \overline{h}(K_n)$. The sets $K_n, \overline{f}(K_n), \overline{g}(K_n), \overline{h}(K_n)$ are connected and $K_n \cap \overline{f}(K_n) \supset \mathbb{T} \cap \overline{f}(\mathbb{T}) \supset \{(0,0)\}$, $K_n \cap \overline{g}(K_n) \supset \mathbb{T} \cap \overline{g}(\mathbb{T}) \supset \{(1,0)\}$ and $K_n \cap \overline{h}(K_n) \supset \mathbb{T} \cap \overline{h}(\mathbb{T}) \supset \{(0,1)\}$. This proves that K_{n+1} is connected. Since $A(\mathcal{S}) = \overline{\bigcup_{n \geq 0} K_n}$ it follows that $A(\mathcal{S})$ is connected.

Next, $A(\mathcal{S})_f = f(A(\mathcal{S}), A(\mathcal{S})) = T_1$, $A(\mathcal{S})_g = g(A(\mathcal{S}), A(\mathcal{S})) = A(\mathcal{S}) \cap T_2$ and $A(\mathcal{S})_h = h(A(\mathcal{S}), A(\mathcal{S})) = A(\mathcal{S}) \cap T_3$. $A(\mathcal{S})_f \cap A(\mathcal{S})_g = \{(1/2, 0)\}$, $A(\mathcal{S})_f \cap A(\mathcal{S})_h = \{(0, 1/2)\}$ and $A(\mathcal{S})_g \cap A(\mathcal{S})_h = \{(1/2, 1/2)\}$. It follows that the family of sets $\{A(\mathcal{S})_f, A(\mathcal{S})_g, A(\mathcal{S})_h\}$ is connected.

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