# A SELECTION PORTFOLIO PROBLEM WITH FUZZY LINEAR EXPONENTIAL DISTRIBUTION

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We consider the optimal selection problem of the rebalancing portfolio with chance constraints, where some of the parameters are uncertain. We model these uncertainties using fuzzy numbers. The random variable of the chance constraints follows fuzzy linear exponential distribution.

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## 1. INTRODUCTION

The selection problem of the rebalancing portfolio, under nonconvex transactional costs, was defined in Konno *et al.* [7], [8], which proposed a branch and bound algorithm for calculating an optimal solution of the minimum cost rebalancing problem under concave transactional costs.

In order to purchase (invest) and/or sell (disinvest) assets, sometimes the investor has to pay certain fees.

In this paper, the fees associated with $x = (x_1, x_2, \dots, x_n)$ are named transaction cost, where $x_j$ represents the amount of investment (or disinvestment) of the asset $j$ $(j = 1, \dots, n)$. The transaction cost of the entire investment is $\sum_{j=1}^{n} c_j(x_j)$, where $c_j(x_j)$ is a non-decreasing nonconvex function up to certain point $x_j$ [7].

Let us now consider a time horizon, composed from $T$ moments of time; we denote by $(r_{1t}, r_{2t}, \dots, r_{nt})$ the vector of the rates of return of the $n$ assets at the $t$ moment, and $p_t = Pr\{(R_1, R_2, \dots, R_n) = (r_{1t}, r_{2t}, \dots, r_{nt})\}$, $t = 1, \dots, T$ is known and where $R_j$ is a random variable, which represents the rate of the return of the $j$ asset (Pr stands for probability) ([5], [13], [14]).

Let $r_j = \sum_{t=1}^{T} p_t r_{jt}$ be the expected rate of the return of the asset $j$ without transaction costs, and $\sum_{j=1}^{n} r_j x_j$ the expected rate of the return of the portfolio $x = (x_1, x_2, \dots, x_n)$.

Because the current portfolio may deviate from the present efficient frontier, the investors are inclined to “rebalance” the portfolio due to the change of investment environment.

Let $x^0 = (x_1^0, x_2^0, \dots, x_n^0)$ [7], [8] be the portfolio at time 0. The investor wants to change the portfolio at a certain later point, say one year later. Let $x = (x_1, x_2, \dots, x_n)$ the new portfolio, with the condition that its expected rate of return $\sum_{j=1}^{n} r_j x_j$ is greater than a constant $g_2$ and smaller than a constant $g_1$, because the investor wants to obtain an minimum guaranteed expected rate of return, and also a maximum expected rate of return, if it is possible.

Let $W[r(x)] = p_t \big| \sum_{j=1}^{n} (r_{jt} - r_j) x_j \big|$ be the risk of the problem, assumed to be bounded, which means that the investor does not want a very big risk $W_0$, but neither a small risk $w_0$, which gives a less expected rate of return.

The investor is limited at a minimum $M_2$ (maximum $M_1$) capital which he wishes to invest in the rebalancing problem.

Let us introduce the new portfolio at a certain later point $x = y + x^0$, with $x^0 = (x_1^0, x_2^0, \dots, x_n^0)$ being the portfolio at time point 0, made up by all the operations resulted from the rebalancing, a portfolio that has the following meanings:

– if $y_j > 0$, $j = 1, \dots, n$, then $c_j(y_j)$ is the associated cost with purchasing $y_j$ units of the asset $j$;

– if $y_j < 0$, $j = 1, \dots, n$, then $c_j(y_j)$ is the associated cost with selling $|y_j|$ units of the asset $j$.

In this paper, we model the chance constraints from the minimal cost rebalancing problem using fuzzy theory approach. Then, we solve the deterministic equivalent of the fuzzy chance constrained minimal cost rebalancing problem.

This paper is organized as follows. In Section 2, we review the theoretical background concerning the triangular fuzzy numbers. In Section 3, we state the minimum cost rebalancing problem under the mean-absolute deviation (MAD) model. In Section 4, we model the programming problem under fuzzy linear exponential distribution with different choices of the distribution parameters. In the last section, we solve the deterministic problem with modified subgradient algorithm.

## 2. FUZZY NUMBERS THEORY. SOME PRELIMINARIES

The purpose of this section is to recall some concepts which will be needed in the sequel. The programming problem with chance constraints has some uncertain informations which can be modeled through different methods involving randomness and fuzziness in different scenarios. Buckley [1], [2], [3] has defined fuzzy probability using fuzzy numbers as parameters in probability density function. The fuzzy numbers are obtained from the set of confidence interval.

According to Dash *et al.* [4], the fuzzy chance constrained programming problem is a chance constrained programming problem in the presence of ambiguous information, where the random variable follows different fuzzy distributions. In this paper, fuzzy random variables for chance constrained rebalancing problem follow a linear exponential distribution, which is very efficient from the practical point of view.

We place a " $\sim$ " over a symbol to denote a fuzzy set. All our fuzzy sets will be fuzzy subsets of the real numbers. So, $\widetilde{a_i}, \widetilde{A}, \widetilde{x}$ all represent fuzzy subsets of the real numbers.

Given the fuzzy number $\widetilde{F}$ with membership function $\mu_{\widetilde{F}} : \mathbf{R} \to [0,1]$, the set $\{b \mid \mu_{\tilde{F}}(b) \geq \alpha, \forall\, 0 \leq \alpha \leq 1\} = \tilde{F}[\alpha]$ is $\alpha$-*cut* of $\widetilde{F}$. $\tilde{F}[0]$ is separately defined as the closure of the union of all the $\tilde{F}[\alpha]$, $0 < \alpha \leq 1$. So, a fuzzy number $\widetilde{F}$ is a fuzzy subset of the real numbers satisfying the following conditions:

– $\mu_{\tilde{F}}(b) = 1$ for some $b$ (normalized);

– $\tilde{F}[\alpha]$ is a closed and bounded interval for $0 \leq \alpha \leq 1$.

A triangular fuzzy number $\widetilde{F}$ is a triplet $(y^1, y^2, y^3) \in \mathbf{R}^3$. The membership function of $\widetilde{F}$ is defined in connection with the real numbers $y^1, y^2, y^3$ as follows:

$$\widetilde{F}(x) = \begin{cases} 0 & x \in (-\infty, y^1), \\ \frac{x-y^1}{y^2-y^1} & x \in [y^1, y^2], \\ \frac{x-y^3}{y^2-y^3} & x \in (y^2, y^3], \\ 0 & x \in (y^3, \infty). \end{cases}$$

$\widetilde{F}(x)$ represents a number in [0,1], which is the membership function of $\widetilde{F}$ evaluated in $x$. For any fuzzy number $\widetilde{F}$, the $\alpha$-*cut* of $\widetilde{F}$ is a closed and bounded interval for $0 \leq \alpha \leq 1$, i.e., $\widetilde{F}([\alpha]) = [q_*(\alpha), q^*(\alpha)]$. We define a partial order relation $\succeq$ between two fuzzy numbers $\widetilde{F}_1$ and $\widetilde{F}_2$ using $\alpha$-*cuts* $\widetilde{F}_1[\alpha]$ and $\widetilde{F}_2[\alpha]$. Let $\widetilde{F}_1[\alpha] = \left[q^1_*(\alpha), q^{1*}(\alpha)\right]$ and $\widetilde{F}_2[\alpha] = \left[q^2_*(\alpha), q^{2*}(\alpha)\right]$. Then, $\widetilde{F}_1 \succeq \widetilde{F}_2$ iff $q^1_*(\alpha) \geq q^{2*}(\alpha)$, for each $\alpha \in [0,1]$.

## 3. PROBLEM STATEMENT

We assume that for the expected rate of return to have chance constraints with the probabilities $p'_t = Pr\{(R_1, R_2, \ldots, R_n) = (r'_{1t}, r'_{2t}, \ldots, r'_{nt})\}$, $t = 1, \ldots, T$ and $p''_t = Pr\{(R_1, R_2, \ldots, R_n) = (r''_{1t}, r''_{2t}, \ldots, r''_{nt})\}$, $t = 1, \ldots, T$ are known. The minimum cost rebalancing problem under the mean-absolute deviation (MAD) model can be formulated as follows

$$(P) \qquad \min \sum_{j=1}^{n} c_j(y_j)$$

*subject to*

$$Pr\Big(\sum_{j=1}^{n} r_j(y_j + x_j^0) \le g_1 \le \sum_{j=1}^{n} r'_j(y_j + x_j^0)\Big) \ge q_1,$$

$$Pr\Big(\sum_{j=1}^{n} r''_j(y_j + x_j^0) \le g_2 \le \sum_{j=1}^{n} r_j(y_j + x_j^0)\Big) \ge q_2,$$

$$M_2 \le \sum_{j=1}^{n}(y_j + x_j^0) \le M_1,$$

$$\gamma'_j \le y_j + x_j^0 \le \gamma_j, \quad j = \overline{1, n},$$

$$w_0 \le \sum_{t=1}^{T} p_t \Big| \sum_{j=1}^{n}(r_{jt} - r_j)y_j + (r_{jt} - r_j)x_j^0 \Big| \le W_0.$$

Let us consider a set of nonnegative variables $z_t$, $z'_t$, $t = 1, \ldots, T$, satisfying the following conditions

$$z_t - z'_t = p_t \sum_{j=1}^{n} \big[(r_{jt} - r_j)y_j + (r_{jt} - r_j)x_j^0\big],$$

$$z_t z'_t = 0, \ z_t \ge 0, \ z'_t \ \ge 0, \ t = 1, \ldots, T.$$

Then we have

$$z_t + z'_t = \Big| p_t \sum_{j=1}^{n} \big[(r_{jt} - r_j)y_j + (r_{jt} - r_j)x_j^0\big] \Big|, \quad t = 1, \ldots, T.$$

Therefore, instead of Problem $(P)$ we consider the following problem

$$(P_1) \qquad \min \sum_{j=1}^{n} c_j(y_j)$$

*subject to*

$$Pr\Big(\sum_{j=1}^{n} r_j(y_j + x_j^0) \le g_1 \le \sum_{j=1}^{n} r'_j(y_j + x_j^0)\Big) \ge q_1,$$

$$Pr\Big(\sum_{j=1}^{n} r''_j(y_j + x_j^0) \le g_2 \le \sum_{j=1}^{n} r_j(y_j + x_j^0)\Big) \ge q_2,$$

$$
\begin{aligned}
& M_2 \le \sum_{j=1}^{n} (y_j + x_j^0) \le M_1, \\
& \gamma_j' \le y_j + x_j^0 \le \gamma_j, \quad j = 1, \ldots, n, \\
& w_0 \le \sum_{t=1}^{T} (z_t + z_t') \le W_0, \\
& z_t - z_t' = p_t \sum_{j=1}^{n} \left[ (r_{jt} - r_j) y_j + (r_{jt} - r_j) x_j^0 \right], \\
& z_t z_t' = 0,\ z_t \ge 0,\ z_t' \ \ge 0,\ t = 1, \ldots, T.
\end{aligned}
$$

As in [7] we can prove a similar result for problem $(P_1)$.

THEOREM 1. *The complementarity constraint* $z_t z_t' = 0$, $z_t \ge 0$, $z_t' \ge 0$, $t = 1, 2, \ldots, T$, *can be eliminated from problem* $(P_1)$. *Moreover,* $\sum_{t=1}^{T}(z_t - z_t') = 0$.

*Proof.* Let $(y_1^*, \ldots, y_n^*, z_1^*, \ldots, z_T^*, {z'}_1^*, \ldots, {z'}_T^*)$ be an optimal solution of the problem $(P_1)$ without complementarity constraint and let us assume that $z_t^* {z'}_t^* > 0$, $z_t^* \ge 0$, ${z'}_t^* \ge 0$, $\forall t \in I \subset \{1, \ldots, T\}$.

We see that

– if $z_t^* \ge {z'}_t^* \ge 0$, $\forall t \in I$ then consider a solution of problem $(P_1)$ $(\widetilde{z}_t, \widetilde{z'_t}) = (z_t^* - {z'}_t^*, 0), \forall t \in I$;

– if $0 \le z_t^* \le {z'}_t^*$, $\forall t \in I$ then consider a solution of problem $(P_1)$ $(\widetilde{z}_t, \widetilde{z'_t}) = (0, {z'}_t^* - z_t^*)$, $\forall t \in I$.

Therefore, $(y_1^*, \ldots, y_n^*, \widetilde{z}_1, \ldots, \widetilde{z_T}, \widetilde{z'_1}, \ldots, \widetilde{z'_T})$ satisfies all the constraint of $(P_1)$, while the objective function value in $(y_1^*, \ldots, y_n^*, \widetilde{z}_1, \ldots, \widetilde{z_T}, \widetilde{z'_1}, \ldots, \widetilde{z'_T})$ is equal with the objective function value calculated in $(y_1^*, \ldots, y_n^*, z_1^*, \ldots, z_T^*, {z'}_1^*, \ldots, {z'}_T^*)$. Moreover, it checks that

$$
\begin{aligned}
\sum_{t=1}^{T} \left( z_t - {z'}_t \right) &= \sum_{t=1}^{T} p_t \sum_{j=1}^{n} \left[ (r_{jt} - r_j) y_j + (r_{jt} - r_j) x_j^0 \right] \\
&= \sum_{t=1}^{T} \sum_{j=1}^{n} \left[ p_t (r_{jt} - r_j) y_j + p_t (r_{jt} - r_j) x_j^0 \right] = 0.
\end{aligned}
$$

The theorem is proved. □

Thus, the mean-absolute deviation (MAD) model from Marinescu [10], can be represented as a crisp chance constrained programming problem for the minimal cost of the portfolio transaction of the form

$$(P_2) \qquad \min \sum_{j=1}^{n} c_j(y_j)$$

*subject to*

$$Pr\left(\sum_{j=1}^{n} r_j(y_j + x_j^0) \le g_1 \le \sum_{j=1}^{n} r'_j(y_j + x_j^0)\right) \ge q_1,$$

$$Pr\left(\sum_{j=1}^{n} r''_j(y_j + x_j^0) \le g_2 \le \sum_{j=1}^{n} r_j(y_j + x_j^0)\right) \ge q_2,$$

$$\gamma'_j \le y_j + x_j^0 \le \gamma_j, \quad j = 1, \ldots, n,$$

$$w_0 \le 2\sum_{t=1}^{T} z_t \le W_0,$$

$$p_t \sum_{j=1}^{n}(r_{jt} - r_j)y_j - z_t \le -p_t \sum_{j=1}^{n}(r_{jt} - r_j)x_j^0, \quad t = 1, \ldots, T,$$

where $g_i$, $i = 1, 2$, is uncertain and $\sum_{j=1}^{n} r_j(y_j + x_j^0)$ represents the expected rate of return for rebalancing portfolio.

We suppose that $g_1$ and $g_2$ are fuzzy random variables and the others real parameters $M_1,\ M_2,\ \gamma_t,\ \gamma'_t,\ W_0,\ w_0$ are given. Their significations and definitions are given in [9], [10], [11].

## 4. THE PROGRAMMING PROBLEM UNDER FUZZY LINEAR EXPONENTIAL DISTRIBUTION

Let $g_1$ and $g_2$ be continuous random variables with probability density function linear exponential

$$f(g_i; \lambda, \mu) = \begin{cases} e^{-\lambda g_i - \frac{\mu g_i^2}{2}}(\lambda + \mu g_i) & \text{if } g_i > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda$ and $\mu$ are uncertain parameters, describing the probability density function. We will study both the case when the parameter $\lambda$ is a fuzzy number and the case when the parameter $\mu$ is a fuzzy number.

### 4.1. The parameter $\lambda$ is a fuzzy number

Let $\widetilde{g}_1$ and $\widetilde{g}_2$ be fuzzy random variables with fuzzy density functions linear exponential, $\widetilde{\lambda}$ a fuzzy parameter and $\mu$ is given parameter. So, the fuzzy chance constrained programming problem for the minimal cost of the portfolio transaction is of the form

$$
(P_3) \qquad \begin{aligned}
&\min \sum_{j=1}^{n} c_j(y_j)\\
&\textit{subject to}\\
&\widetilde{Pr}\left(\sum_{j=1}^{n} r_j(y_j+x_j^0) \le \widetilde{g}_1 \le \sum_{j=1}^{n} r'_j(y_j+x_j^0)\right) \succeq \widetilde{q}_1,\\
&\widetilde{Pr}\left(\sum_{j=1}^{n} r''_j(y_j+x_j^0)) \le \widetilde{g}_2 \le \sum_{j=1}^{n} r_j(y_j+x_j^0)\right) \succeq \widetilde{q}_2,\\
&M_2 \le \sum_{j=1}^{n}(y_j+x_j^0) \le M_1,\\
&\gamma'_j \le y_j + x_j^0 \le \gamma_j, \quad j=1,\dots,n,\\
&w_0 \le 2\sum_{t=1}^{T} z_t \le W_0,\\
&p_t\sum_{j=1}^{n}(r_{jt}-r_j)y_j - z_t \le -p_t\sum_{j=1}^{n}(r_{jt}-r_j)x_j^0, \quad t=1,\dots,T.
\end{aligned}
$$

Let $u=\sum_{j=1}^{n} r_j(y_j+x_j^0)$, $u'=\sum_{j=1}^{n} r'_j(y_j+x_j^0)$ and $u''=\sum_{j=1}^{n} r''_j(y_j+x_j^0)$. Then $u \le \widetilde{g}_i \le u'$ is a fuzzy event. $\widetilde{Pr}\,(u \le \widetilde{g}_i \le u')$ is the probability of this event which is a fuzzy number. Its $\alpha$-*cut* is the set

$$
\widetilde{Pr}(u\le\widetilde{g}_i\le u')[\alpha]=\left\{\int_u^{u'} e^{-\lambda g_i-\frac{\mu g_i^2}{2}}\,(\lambda+\mu g_i)\,\mathrm{d}g_i \mid \lambda\in\widetilde{\lambda}[\alpha]\right\}=\left[q_*^i(\alpha),q^{i*}(\alpha)\right],
$$

for $0\le\alpha\le 1$. Denote $\widetilde{q}_i[\alpha]=\left[q_{i*}(\alpha),q_i^*(\alpha)\right]$, the $\alpha$-cut of the fuzzy number $\widetilde{q}_i$.

LEMMA 1. *If $\widetilde{g}_1$ is a fuzzy random variable linear exponential distributed with $\mu$ real parameter given, and $\widetilde{\lambda}_1$ is a fuzzy number, then the minimum $\alpha$-cut of the fuzzy probability of the event $(u\le\widetilde{g}_1\le u')$ is*

$$
q_*^1(\alpha)=\min\left\{e^{-\lambda_{1*}(\alpha)u-\frac{\mu u^2}{2}}-e^{-\lambda_{1*}(\alpha)u'-\frac{\mu u'^2}{2}},\ e^{-\lambda_1^*(\alpha)u-\frac{\mu u^2}{2}}-e^{-\lambda_1^*(\alpha)u'-\frac{\mu u'^2}{2}}\right\}
$$

*for $\alpha\in[0,1]$, where $\widetilde{\lambda}_1[\alpha]=[\lambda_{1*}(\alpha),\lambda_1^*(\alpha)]$.*

*Proof.*

$$\widetilde{Pr}\left(u \le \widetilde{g}_1 \le u'\right)[\alpha] = \left\{ \int_u^{u'} e^{-\lambda_1 g_1 - \frac{\mu g_1^2}{2}} (\lambda_1 + \mu g_1)\, \mathtt{d} g_1 \middle| \lambda_1 \in \widetilde{\lambda}_1[\alpha] \right\} =$$

$$= \left\{ e^{-\lambda_1 u - \frac{\mu u^2}{2}} - e^{-\lambda_1 u' - \frac{\mu u'^2}{2}} \middle| \widetilde{\lambda}_{1*}(\alpha) \le \lambda_1 \le \widetilde{\lambda}_1^*(\alpha) \right\} = \left[q_*^1(\alpha), q^{1*}(\alpha)\right].$$

The function

$$f_1(u, u'; \lambda_1) = e^{-\lambda_1 u - \frac{\mu u^2}{2}} - e^{-\lambda_1 u' - \frac{\mu u'^2}{2}}$$

is an increasing function of in $\lambda_1$ until a maximum and it is decreasing at infinite. So, it results that the minimum of the function $f_1$ is attained in $\lambda_{1*}(\alpha)$ for $f_1$ increasing, or if $f_1$ is decreasing then the minimum of $f_1$ is attained in $\lambda_1^*(\alpha)$. Thus, the minimum $\alpha$-*cut* of the fuzzy probability of the event $(u \le \widetilde{g}_1 \le u')$ is

$$q_*^1(\alpha) = \min\left\{ e^{-\lambda_{1*}(\alpha) u - \frac{\mu u^2}{2}} - e^{-\lambda_{1*}(\alpha) u' - \frac{\mu u'^2}{2}},\ e^{-\lambda_1^*(\alpha) u - \frac{\mu u^2}{2}} - e^{-\lambda_1^*(\alpha) u' - \frac{\mu u'^2}{2}} \right\}. \quad \square$$

Lemma 2. *If $\widetilde{g}_2$ is a fuzzy random variable linear exponential distributed with $\mu$ real parameter given, and $\widetilde{\lambda}_2$ is a fuzzy number, then we have the following relation for the minimum $\alpha$-cut of the fuzzy probability of the event $(u'' \le \widetilde{g}_2 \le u)$ is*

$$q_*^2(\alpha) = \min\left\{ e^{-\lambda_{2*}(\alpha) u'' - \frac{\mu u''^2}{2}} - e^{-\lambda_{2*}(\alpha) u - \frac{\mu u^2}{2}},\ e^{-\lambda_2^*(\alpha) u'' - \frac{\mu u''^2}{2}} - e^{-\lambda_2^*(\alpha) u - \frac{\mu u^2}{2}} \right\}$$

*for $\alpha \in [0,1]$, where $\widetilde{\lambda}_2[\alpha] = [\lambda_{2*}(\alpha), \lambda_2^*(\alpha)]$.*

*Proof.*

$$\widetilde{Pr}\left(u'' \le \widetilde{g}_2 \le u\right)[\alpha] = \left\{ \int_{u''}^{u} e^{-\lambda_2 g_2 - \frac{\mu g_2^2}{2}} (\lambda_2 + \mu g_2) \mathtt{d} g_2 \middle| \lambda_2 \in \widetilde{\lambda}_2[\alpha] \right\} =$$

$$= \left\{ e^{-\lambda_2 u'' - \frac{\mu u''^2}{2}} - e^{-\lambda_2 u - \frac{\mu u^2}{2}} \middle| \widetilde{\lambda}_{2*}(\alpha) \le \lambda_2 \le \widetilde{\lambda}_2^*(\alpha) \right\} = \left[q_*^2(\alpha), q^{2*}(\alpha)\right].$$

The function

$$f_2(u'', u; \lambda_2) = e^{-\lambda_2 u'' - \frac{\mu u''^2}{2}} - e^{-\lambda_2 u - \frac{\mu u^2}{2}}$$

is increasing in $\lambda_2$ until a maximum which is attained for

$$\lambda_{2\max} = \frac{\ln(\frac{u}{u''}) - \mu \frac{u^2 - u''^2}{2}}{u - u''},$$

and after this point it is decreasing at infinite. It results that the minimum of the function $f_2$ is attained in $\lambda_{2*}(\alpha)$ if $\lambda_2 \le \lambda_{2\max}$ or in $\lambda_2^*(\alpha)$ for $\lambda_2 > \lambda_{2\max}$.

So, the minimum $\alpha$-*cut* of the fuzzy probability of the event $(u'' \le \widetilde{g}_2 \le u)$ is

$$q_*^2(\alpha) = \min\left\{e^{-\lambda_{2*}(\alpha)u'' - \frac{\mu u''^2}{2}} - e^{-\lambda_{2*}(\alpha)u - \frac{\mu u^2}{2}},\ e^{-\lambda_2^*(\alpha)u'' - \frac{\mu u''^2}{2}} - e^{-\lambda_2^*(\alpha)u - \frac{\mu u^2}{2}}\right\}. \quad \square$$

### 4.2. THE MODEL WITH $\mu$ - A FUZZY NUMBER AND $\lambda$ - REAL PARAMETER GIVEN

Let $\widetilde{\mu}$ be a fuzzy number. Also, let the variable $X$ a fuzzy random variable linear distributed with $\lambda$ real parameter given, and denote it with $\widetilde{X}$.

Lemma 3. *If $\widetilde{g}_1$ is a fuzzy random variable linear exponential distributed with $\lambda$ real parameter given, and $\widetilde{\mu}_1$ is a positive fuzzy number, then we have the following relation for the minimum $\alpha$-cut of the fuzzy probability of the event $(u \le \widetilde{g}_1 \le u')$:*

$$q_*^1(\alpha) = \min\left\{e^{-\lambda u - \frac{\mu_{1*}(\alpha)u^2}{2}} - e^{-\lambda u' - \frac{\mu_{1*}(\alpha)u'^2}{2}},\ e^{-\lambda u - \frac{\mu_1^*(\alpha)u^2}{2}} - e^{-\lambda u' - \frac{\mu_1^*(\alpha)u'^2}{2}}\right\}$$

*for $\alpha \in [0,1]$, where $\widetilde{\mu}_1[\alpha] = [\mu_{1*}(\alpha), \mu_1^*(\alpha)]$.*

*Proof.*

$$\widetilde{Pr}\left(u \le \widetilde{g}_1 \le u'\right)[\alpha] = \left\{\int_u^{u'} e^{-\lambda g_1 - \frac{\mu_1 g_1^2}{2}}(\lambda + \mu_1 g_1)\mathrm{d}g_1 \,\middle|\, \mu_1 \in \widetilde{\mu}_1[\alpha]\right\} =$$
$$= \left\{e^{-\lambda u - \frac{\mu_1 u^2}{2}} - e^{-\lambda u' - \frac{\mu_1 u'^2}{2}} \,\middle|\, \widetilde{\mu}_{1*}(\alpha) \le \mu_1 \le \widetilde{\mu}_1^*(\alpha)\right\} = \left[q_*^1(\alpha), q^{1*}(\alpha)\right].$$

The function

$$f_3(u, u'; \mu_1) = e^{-\lambda u - \frac{\mu_1 u^2}{2}} - e^{-\lambda u' - \frac{\mu_1 u'^2}{2}}$$

is increasing in $\mu_1$ until a maximum which is attained for

$$\mu_{1\max} = \frac{4\ln(\frac{u'}{u}) - 2\lambda(u' - u)}{u'^2 - u^2},$$

and after this point it is decreasing at infinite. It results that the minimum of the function $f_3$ is attained in $\mu_{1*}(\alpha)$ if $\mu_1 \le \mu_{1\max}$ or in $\mu_1^*(\alpha)$ for $\mu_1 > \mu_{1\max}$. So, it results that the minimum $\alpha$-*cut* of the fuzzy probability for the event $(u \le \widetilde{g}_1 \le u')$ is

$$q_*^1(\alpha) = \min\left\{e^{-\lambda u - \frac{\mu_{1*}(\alpha)u^2}{2}} - e^{-\lambda u' - \frac{\mu_{1*}(\alpha)u'^2}{2}},\ e^{-\lambda u - \frac{\mu_1^*(\alpha)u^2}{2}} - e^{-\lambda u' - \frac{\mu_1^*(\alpha)u'^2}{2}}\right\}. \quad \square$$

Lemma 4. *If $\widetilde{g}_2$ is a fuzzy random variable linear exponential distributed with $\lambda$ real parameter given, and $\widetilde{\mu}_2$ is a positive fuzzy number. Then we have* *the following relation for the minimum $\alpha$-cut of the fuzzy probability of the event* $(u'' \leq \widetilde{g}_2 \leq u)$ *is*

$$q_*^2(\alpha) = \min\left\{e^{-\lambda u''-\frac{\mu_{2*}(\alpha)u''^2}{2}} - e^{-\lambda u-\frac{\mu_{2*}(\alpha)u^2}{2}},\ e^{-\lambda u''-\frac{\mu_2^*(\alpha)u''^2}{2}} - e^{-\lambda u-\frac{\mu_2^*(\alpha)u^2}{2}}\right\}$$

*for* $\alpha \in [0,1]$, *where* $\widetilde{\mu}_2[\alpha] = [\mu_{2*}(\alpha), \mu_2^*(\alpha)]$.

*Proof.*

$$\widetilde{Pr}\left(u'' \leq \widetilde{g}_2 \leq u\right)[\alpha] = \left\{\int_{u''}^{u} e^{-\lambda g_2-\frac{\mu_2 g_2^2}{2}}(\lambda+\mu_2 g_2)\mathrm{d}g_2 \,\middle|\, \mu_2 \in \widetilde{\mu}_2[\alpha]\right\} =$$
$$= \left\{e^{-\lambda u''-\frac{\mu_2 u''^2}{2}} - e^{-\lambda u-\frac{\mu_2 u^2}{2}} \,\middle|\, \widetilde{\mu}_{2*}(\alpha) \leq \mu_2 \leq \widetilde{\mu}_2^*(\alpha)\right\} = \left[q_*^2(\alpha), q^{2*}(\alpha)\right].$$

The function

$$f_4(u'', u; \mu_2) = e^{-\lambda u''-\frac{\mu_2 u''^2}{2}} - e^{-\lambda u-\frac{\mu_2 u^2}{2}}$$

is increasing in $\mu_2$ until a maximum and and it is decreasing at infinite. So, it results that the minimum of the function $f_4$ is attained in $\mu_{2*}(\alpha)$ for $f_4$ increasing or in $\mu_2^*(\alpha)$ if $f_4$ is decreasing. Thus, the minimum $\alpha$-*cut* of the fuzzy probability for the event $(u'' \leq \widetilde{g}_2 \leq u)$ is

$$q_*^2(\alpha) = \min\left\{e^{-\lambda u''-\frac{\mu_{2*}(\alpha)u''^2}{2}} - e^{-\lambda u-\frac{\mu_{2*}(\alpha)u^2}{2}},\ e^{-\lambda u''-\frac{\mu_2^*(\alpha)u''^2}{2}} - e^{-\lambda u-\frac{\mu_2^*(\alpha)u^2}{2}}\right\}. \quad \square$$

## 5. AN ALGORITHM FOR SOLVING THE DETERMINISTIC EQUIVALENT OF THE FUZZY MINIMUM COST REBALANCING PROBLEM

### 5.1. The deterministic equivalent of the rebalancing problem

Depending on the choice of the parameters $\lambda_i$, $i = 1,2$ for the functions $f_1(u, u'; \lambda_1)$ and $f_2(u, u''; \lambda_2)$, we have more cases for fuzzy chance constraints.

In this paper, we will consider further on the case, where the functions, with unknown parameters $\lambda_i$, $i = 1,2$ are decreasing. The others cases will be treated likewise.

Theorem 2. *Assume that* $\widetilde{g}_i$ *is a fuzzy random variable linear exponential distributed with* $\mu$ *real parameter given, and* $\widetilde{\lambda}_i$, $i = 1,2$, *is a positive fuzzy number. Also, assume the following conditions* $\lambda_{1\max} < \lambda_1$ *and* $\lambda_{2\max} < \lambda_2$.

*Then the problem* $\big(P_3\big)$ *is equivalent with the following problem*

$$(P_4) \qquad \min \sum_{j=1}^{n} c_j(y_j)$$

*subject to*

$$e^{-\lambda_1^*(\alpha)u-\frac{\mu u^2}{2}} - e^{-\lambda_1^*(\alpha)u'-\frac{\mu u'^2}{2}} \geq q_1^*(\alpha),$$

$$e^{-\lambda_2^*(\alpha)u''-\frac{\mu u''^2}{2}} - e^{-\lambda_2^*(\alpha)u-\frac{\mu u^2}{2}} \geq q_2^*(\alpha),$$

$$M_2 \leq \sum_{j=1}^{n}(y_j + x_j^0) \leq M_1,$$

$$\gamma_j' \leq y_j + x_j^0 \leq \gamma_j, \quad j = 1, \dots, n,$$

$$w_0 \leq 2\sum_{t=1}^{T} z_t \leq W_0,$$

$$p_t \sum_{j=1}^{n}(r_{jt} - r_j)y_j - z_t \leq -p_t \sum_{j=1}^{n}(r_{jt} - r_j)x_j^0, \quad t = 1, \dots, T.$$

*Proof.* This theorem results from Lemma 1 and Lemma 2. □

The deterministic equivalent of the minimum cost rebalancing problem is a nonlinear programming problem. In what follows we will describe a convergent algorithm in order to find a solution of this problem.

### 5.2. **Subgradient method**

We solve the nonlinear programming problem using the subgradient method. The algorithm is simple, but it has a big running time. We consider the triangular fuzzy numbers $\widetilde{\lambda}_1 = (\lambda_1', \lambda_1, \lambda_1'')$, $\widetilde{\lambda}_2 = (\lambda_2', \lambda_2, \lambda_2'')$, $\widetilde{q}_1 = (q_1', q_1, q_1'')$ and $\widetilde{q}_2 = (q_2', q_2, q_2'')$.

From problem $(P_4)$ we obtain the following model.

$$(P_5) \qquad \min \sum_{j=1}^{n} c_j(y_j)$$

*subject to*

$$e^{\left[-\lambda_1''-\alpha(\lambda_1''-\lambda_1)\right]\left[\sum_{j=1}^{n} r_j(y_j-x_j^0)\right]-\frac{\mu\left(\sum_{j=1}^{n} r_j(y_j-x_j^0)\right)^2}{2}} -$$

$$- e^{\left[-\lambda_1''-\alpha(\lambda_1''-\lambda_1)\right]\left[\sum_{j=1}^n r_j'(y_j-x_j^0)\right]-\frac{\mu\left(\sum_{j=1}^n r_j'(y_j-x_j^0)\right)^2}{2}} -$$
$$- q_1'' + \alpha\big(q_1'' - q_1\big) + k_1 = 0,$$
$$e^{\left[-\lambda_2''-\alpha(\lambda_2''-\lambda_2)\right]\left[\sum_{j=1}^n r_j''(y_j-x_j^0)\right]-\frac{\mu\left(\sum_{j=1}^n r_j''(y_j-x_j^0)\right)^2}{2}} -$$
$$- e^{\left[-\lambda_2''-\alpha(\lambda_2''-\lambda_2)\right]\left[\sum_{j=1}^n r_j(y_j-x_j^0)\right]-\frac{\mu\left(\sum_{j=1}^n r_j(y_j-x_j^0)\right)^2}{2}} -$$
$$- q_1''2 + \alpha\big(q_2'' - q_2\big) + k_2 = 0,$$
$$\sum_{j=1}^n (y_j + x_j^0) - M_1 + k_3 = 0,$$
$$\sum_{j=1}^n (y_j + x_j^0) - M_2 - k_4 = 0,$$
$$y_j + x_j^0 - \gamma_j + k_{j+4} = 0, \quad j = 1, \dots, n,$$
$$y_j + x_j^0 - \gamma_j' - k_{j+n+4} = 0, \quad j = 1, \dots, n,$$
$$2\sum_{t=1}^T z_t - W_0 + k_{2n+5} = 0,$$
$$2\sum_{t=1}^T z_t - W_0 - k_{2n+6} = 0,$$
$$p_t \sum_{j=1}^n (r_{jt} - r_j) y_j - z_t + p_t \sum_{j=1}^n (r_{jt} - r_j) x_j^0 + k_{t+2n+6}, \quad t = 1, \dots, T.$$

The solution set of problem $(P_5)$ is denote by $S = \big\{\big(y', \alpha, k\big) \mid y' = \big(y_1, \dots, y_n, z_1, \dots, z_T\big)^t,\ k = \big(k_1, \dots, k_{2n+T+6}\big),\ 0 \le \alpha \le 1,\ k_i \ge 0\big\}$ and is a compact subset of $\mathbf{R}^{3n+2T+7}$. Let us consider $g_0 : \mathbf{R}^n \to \mathbf{R}$ the objective function of the problem $(P_5)$ and $g : \mathbf{R}^{3n+2T+7} \to \mathbf{R}^{2n+T+6}$ the constraints function of the problem $(P_5)$, where we assume that $g\big(y', \alpha, k\big) = \big(g_1, \dots, g_{2n+T+6}\big)^t$ is a continuous function.

The augmented Lagrangian $L$ associated with problem $(P_5)$ is
$$L : \mathbf{R}^{3n+2T+7} \times \mathbf{R}^{2n+T+6} \times \mathbf{R}_+ \to \mathbf{R}^{2n+T+6},$$
where
$$L\big(y', \alpha, k; u, c\big) = g_0\big(y', \alpha, k\big) + c\left\|g(y', \alpha, k)\right\| + \left\langle u, g\big(y', \alpha, k\big)\right\rangle,$$
with $\big(y', \alpha, k\big) \in S$, $u \in \mathbf{R}^{2n+T+6}$, $c \in \mathbf{R}_+$ and $\|\cdot\|$ is the Euclidean norm, and $\langle\ \rangle$ is the Euclidean inner product.

So,

$$L\big(y', \alpha, k; u, c\big) =$$

$$= \sum_{j=1}^{n} c_j(y_j) + c\Bigg[\Bigg(e^{\left[-\lambda_1''-\alpha(\lambda_1''-\lambda_1)\right]\left[\sum_{j=1}^{n} r_j(y_j-x_j^0)\right]-\frac{\mu\left(\sum_{j=1}^{n} r_j(y_j-x_j^0)\right)^2}{2}} -$$

$$-e^{\left[-\lambda_1''-\alpha(\lambda_1''-\lambda_1)\right]\left[\sum_{j=1}^{n} r_j'(y_j-x_j^0)\right]-\frac{\mu\left(\sum_{j=1}^{n} r_j'(y_j-x_j^0)\right)^2}{2}} - q_1'' + \alpha\big(q_1''-q_1\big) + k_1\Bigg)^2 +$$

$$+\Bigg(e^{\left[-\lambda_2''-\alpha(\lambda_2''-\lambda_2)\right]\left[\sum_{j=1}^{n} r_j''(y_j-x_j^0)\right]-\frac{\mu\left(\sum_{j=1}^{n} r_j''(y_j-x_j^0)\right)^2}{2}} -$$

$$-e^{\left[-\lambda_2''-\alpha(\lambda_2''-\lambda_2)\right]\left[\sum_{j=1}^{n} r_j(y_j-x_j^0)\right]-\frac{\mu\left(\sum_{j=1}^{n} r_j(y_j-x_j^0)\right)^2}{2}} - q_2'' + \alpha\big(q_2''-q_2\big) + k_2\Bigg)^2 +$$

$$+\Bigg(\sum_{j=1}^{n}(y_j + x_j^0) - M_1 + k_3\Bigg)^2 + \Bigg(\sum_{j=1}^{n}(y_j + x_j^0) - M_2 - k_4\Bigg)^2 +$$

$$+\sum_{j=1}^{n}\big(y_j + x_j^0 - \gamma_j + k_{j+4}\big)^2 + \sum_{j=1}^{n}\big(y_j + x_j^0 - \gamma_j' - k_{j+n+4}\big)^2 +$$

$$+\Bigg(2\sum_{t=1}^{T} z_t - W_0 + k_{2n+5}\Bigg)^2 + \Bigg(2\sum_{t=1}^{T} z_t - w_0 - k_{2n+6}\Bigg)^2 +$$

$$+\sum_{t=1}^{T}\Bigg(p_t\sum_{j=1}^{n}(r_{jt} - r_j)y_j - z_t + p_t\sum_{j=1}^{n}(r_{jt} - r_j)x_j^0 + k_{t+2n+6}\Bigg)^2\Bigg]^{\frac{1}{2}} +$$

$$u_1\Bigg(e^{\left[-\lambda_1''-\alpha(\lambda_1''-\lambda_1)\right]\left[\sum_{j=1}^{n} r_j(y_j-x_j^0)\right]-\frac{\mu\left(\sum_{j=1}^{n} r_j(y_j-x_j^0)\right)^2}{2}} -$$

$$-e^{\left[-\lambda_1''-\alpha(\lambda_1''-\lambda_1)\right]\left[\sum_{j=1}^{n} r_j'(y_j-x_j^0)\right]-\frac{\mu\left(\sum_{j=1}^{n} r_j'(y_j-x_j^0)\right)^2}{2}} - q_1'' + \alpha\big(q_1''-q_1\big) + k_1\Bigg) +$$

$$+u_2\Bigg(e^{\left[-\lambda_2''-\alpha(\lambda_2''-\lambda_2)\right]\left[\sum_{j=1}^{n} r_j''(y_j-x_j^0)\right]-\frac{\mu\left(\sum_{j=1}^{n} r_j''(y_j-x_j^0)\right)^2}{2}} -$$

$$-e^{\left[-\lambda_2''-\alpha(\lambda_2''-\lambda_2)\right]\left[\sum_{j=1}^{n} r_j(y_j-x_j^0)\right]-\frac{\mu\left(\sum_{j=1}^{n} r_j(y_j-x_j^0)\right)^2}{2}} - q_2'' + \alpha\big(q_2''-q_2\big) + k_2\Bigg) +$$

$$+u_3\Bigg(\sum_{j=1}^{n}(y_j + x_j^0) - M_1 + k_3\Bigg) + u_4\Bigg(\sum_{j=1}^{n}(y_j + x_j^0) - M_2 - k_4\Bigg) +$$

$$+\sum_{j=1}^{n} u_{j+4}\Bigg(y_j + x_j^0 - \gamma_j + k_{j+4}\Bigg) + \sum_{j=1}^{n} u_{j+n+4}\Bigg(y_j + x_j^0 - \gamma_j' - k_{j+n+4}\Bigg) +$$

$$+u_{2n+5}\left(2\sum_{t=1}^{T} z_t - W_0 + k_{2n+5}\right) + u_{2n+6}\left(2\sum_{t=1}^{T} z_t - w_0 - k_{2n+6}\right)+$$

$$+\sum_{t=1}^{T} u_{t+2n+6}\left(p_t\sum_{j=1}^{n}(r_{jt}-r_j)y_j - z_t + p_t\sum_{j=1}^{n}(r_{jt}-r_j)x_j^0 + k_{t+2n+6}\right).$$

The dual function $H$ is defined as

$$H(u,c) = \min_{(y',\alpha,k)\in S} L(y',\alpha,k;u;c), \quad u\in \mathbf{R}^{2n+T+6},\ c\in \mathbf{R}_+.$$

Then the dual problem of $(P_5)$ is given by

$$(P_5^*) \qquad \max_{(u,c)\in \mathbf{R^{2n+T+6}}\times \mathbf{R}_+} H(u,c).$$

*The modified subgradient algorithm* is devised for solving the dual problem. Let us outlines the steps of the algorithm:

**Step 0.** Choose a vector $(u^1, c^1)$ with $c^1 \geq 0$, let $p = 1$ and go to the Step 1.

**Step 1.** Given $(u^p, c^p)$, solve the following subproblem

$$\min_{(y',\alpha,k)\in S} L(y',\alpha,k;u^p;c^p)$$

and let $(y'^p, \alpha^p, k^p)$ any optimal solution. We have two situations:

- If $g\big(y'^p,\alpha^p,k^p\big) = 0$, then STOP; by Theorem 3, $(u^p, c^p)$ is a solution to the dual problem and $\big(y'^p,\alpha^p,k^p\big)$ is a solution to the primal problem;
- Otherwise, go to Step 2.

**Step 2.** Let

$$u^{p+1} = u^p - s^p g\big(y'^p,\alpha^p,k^p\big), \quad c^{p+1} = c^p + (s^p+\epsilon^p)\left\|g\big(y'^p,\alpha^p,k^p\big)\right\|,$$

where $s^p$ and $\epsilon^p$ are positive scalar stepsizes; replace $p$ by $p+1$ set $p = p+1$ and repeat Step 1.

For proving the stoping step of the algorithm, we need the following theorem from [11].

THEOREM 3 (Theorem 5 from [6]). *Let* $\inf(P_5) = \sup(P_5^*)$ *and suppose that for some*

$$(\bar{u},\bar{c}) \in \mathbf{R}^{2n+T+6}\times \mathbf{R}_+ \quad \textit{and} \quad (\bar{y'},\bar{\alpha},\bar{k}) \in \mathbf{R}^{3n+2T+7},$$

$$\min_{(y',\alpha,k)\in S} L(y',\alpha,k;\bar{u};\bar{c}) = g_0(\bar{y'},\bar{\alpha},\bar{k}) + \bar{c}\left\|g(\bar{y'},\bar{\alpha},\bar{k})\right\| + \left\langle \bar{u}, g(\bar{y'},\bar{\alpha},\bar{k})\right\rangle.$$

*Then* $(\bar{y'},\bar{\alpha},\bar{k})$ *is a solution to* $(P_5)$ *and* $(\bar{u},\bar{c})$ *is a solution to* $(P_5^*)$ *if and only if* $g(\bar{y'},\bar{\alpha},\bar{k}) = 0$.

The next proposition demonstrates that for the certain values of stepsizes $\epsilon^p$ and $s^p$, the distance between the points $(u^{p+1}, c^{p+1})$ generated by the algorithm and the solution $(\bar{u}, \bar{c})$ of the dual problem, decrease at each iteration.

Proposition 1 (stepsize). *Let $\big(u^p, c^p\big)$ be any iteration which is not a solution of the dual problem for any $p$, that is, from Theorem 3, $g\big(y'^p, \alpha^p, k^p\big) \neq 0$ for all $p$. Then for any dual solution $(\bar{u}, \bar{c})$, we have*

$$\big\|(\bar{u}, \bar{c}) - (u^{p+1}, c^{p+1})\big\| < \|(\bar{u}, \bar{c}) - (u^p, c^p)\|$$

*for all stepsizes $s^p$ such that*

$$0 < s^p < 2\frac{H(\bar{u}, \bar{c}) - H(u^{p+1}, c^{p+1})}{5\,\big\|g\big(y'^p, \alpha^p, k^p\big)\big\|^2}, \quad 0 < \epsilon^p < s^p.$$

*Proof.* We have

$$\begin{aligned}\big\|(\bar{u}, \bar{c}) - (u^{p+1}, c^{p+1})\big\|^2 = \big\|\bar{u} - u^{p+1}\big\|^2 + \big|\bar{c} - c^{p+1}\big|^2 = \\ = \big\|\bar{u} - u^p + s^p g\big(y'^p, \alpha^p, k^p\big)\big\|^2 + \big|\bar{c} - c^p - (s^p + \epsilon^p)\,\big\|g\big(y'^p, \alpha^p, k^p\big)\big\|\big|^2 = \\ = \|\bar{u} - u^p\|^2 + 2s^p(\bar{u} - u^p)^t g\big(y'^p, \alpha^p, k^p\big) + (s^p)^2\,\big\|g\big(y'^p, \alpha^p, k^p\big)\big\|^2 + (\bar{c} - c^p)^2 - \\ -2(\bar{c} - c^p)(s^p + \epsilon^p)\,\big\|g\big(y'^p, \alpha^p, k^p\big)\big\| + (s^p + \epsilon^p)^2\,\big\|g\big(y'^p, \alpha^p, k^p\big)\big\|^2.\end{aligned}$$

For

$$0 < \epsilon^p < s^p, \quad \bar{c} - c^p > 0, \quad \big\|g\big(y'^p, \alpha^p, k^p\big)\big\| > 0$$

we have the inequality

$$\begin{aligned}\big\|(\bar{u}, \bar{c}) - (u^{p+1}, c^{p+1})\big\|^2 = \\ = \|\bar{u} - u^p\|^2 + 2s^p(\bar{u} - u^p)^t g\big(y'^p, \alpha^p, k^p\big) + (s^p)^2\,\big\|g\big(y'^p, \alpha^p, k^p\big)\big\|^2 + \\ +(\bar{c} - c^p)^2 - 2(\bar{c} - c^p)s^p\,\big\|g\big(y'^p, \alpha^p, k^p\big)\big\| + (2s^p)^2\,\big\|g\big(y'^p, \alpha^p, k^p\big)\big\|^2.\end{aligned}$$

By using the subgradient inequality

$$H(\bar{u}, \bar{c}) - H(u^p, c^p) \leq (\bar{u} - u^p)^t\big(-g\big(y'^p, \alpha^p, k^p\big)\big) + (\bar{c} - c^p)\,\big\|g\big(y'^p, \alpha^p, k^p\big)\big\|$$

we obtain

$$\begin{aligned}\big\|(\bar{u}, \bar{c}) - (u^{p+1}, c^{p+1})\big\|^2 < \|\bar{u} - u^p\|^2 + \bar{c} - c^p)^2 - 2s^p\big(H(\bar{u}, \bar{c}) - H(u^p, c^p)\big) + \\ + 5(s^p)^2\,\big\|g\big(y'^p, \alpha^p, k^p\big)\big\|^2.\end{aligned}$$

The proof is complete. □

Proposition 2 (convergence of the algorithm). *Let $(u^p, c^p)$ be any iteration of the subgradient algorithm. Suppose that each new iteration $(u^{p+1}, c^{p+1})$ is calculated from Step 2 with $s^p = \frac{H(\bar{u},\bar{c}) - H(u^p, c^p)}{5\|g(y'^p, \alpha^p, k^p)\|^2}$, and $0 < \epsilon^p < s^p$, where $H(\bar{u}, \bar{c})$ is the optimal dual value. Then $H(u^p, c^p) \to H(\bar{u}, \bar{c})$.*

*Proof.* The proof of this proposition is similar with the proof of Theorem 9 from [6]. □

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