CLOSEDNESS OF THE SOLUTION MAP FOR PARAMETRIC VECTOR EQUILIBRIUM PROBLEMS WITH TRIFUNCTIONS

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In this paper we introduce new definitions of vector topological pseudomonotonicity to study the parametric vector equilibrium problems with trifunctions. The main result gives sufficient conditions for closedness of the solution map defined on the set of parameters. The Hadamard well-posedness of parametric vector equilibrium problems is also analyzed.

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1. INTRODUCTION

Bogdan and Kolumbán [3] gave sufficient conditions for closedness of the solution map defined on the set of parameters. They considered the parametric equilibrium problems governed by topological pseudomonotone maps depending on a parameter. In this paper we generalize this result for parametric vector equilibrium problems with trifunctions.

Let X and Y be Hausdorff topological spaces and P, the set of parameters, another Hausdorff topological space, $T : X \to 2^Y$ be a multi-valued mapping.

Generalized vector equilibrium problems ($GVEP$ for short) are obtained from generalized equilibrium problems by considering trifunctions on $K \times D \times K$ into a real topological vector space $\mathcal{Z}$ with an ordering cone. By an ordering cone $C \subset \mathcal{Z}$ we mean that C is a closed convex cone in $\mathcal{Z}$ with $\operatorname{Int} C \neq \emptyset$ and $C \neq \mathcal{Z}$, where $\operatorname{Int} C$ denotes the interior of C.

Let $f_p : X \times Y \times X \to \mathcal{Z}$ be a trifunction. For a given $p \in P$, we consider the following problem $(GVEP)_p$:

Find a pair $(x_p, y_p) \in K_p \times T(x_p)$ such that

$$f_p(x_p, y_p, u) \in (-\operatorname{Int} C)^c \text{ for all } u \in K_p,$$

where $(-\operatorname{Int} C)^c$ is the complement of $-\operatorname{Int} C$ in $\mathcal{Z}$ and K_p is a nonempty subset of X. Such an x_p will be called a strong solution of the problem $(GVEP)_p$ in the sense that y_p does not depend on $u \in K_p$.

Let us denote by $S(p)$ the set of the strong solutions for a fixed p. Suppose that $S(p) \neq \emptyset$, for all $p \in P$. Some existence results for $GVEP$ are given in [7, 9, 10].

The paper is organized as follows. In Section 2, we recall the notions of the vector topological pseudomonotonicity and the Mosco convergence of the sets. Section 3 is devoted to the study of the closedness of solution map for parametric vector equilibrium problems with trifunctions. In the final section, we investigate the generalized Hadamard well-posedness of parametric vector equilibrium problems with trifunctions.

2. PRELIMINARIES

In this section, we will introduce two new definitions of the vector topologically pseudomonotone trifunctions with values in $\mathcal{Z}$. First, the definition of the suprema and the infima of subsets of $\mathcal{Z}$ are given. Following [1], for a subset A of $\mathcal{Z}$ the suprema of A with respect to C is defined by

$$\operatorname{Sup} A = \left\{z \in \bar{A} : A \cap (z + \operatorname{Int} C) = \emptyset\right\}$$

and the infima of A with respect to C is defined by

$$\operatorname{Inf} A = \left\{z \in \bar{A} : A \cap (z - \operatorname{Int} C) = \emptyset\right\}.$$

For more details see [6].

Let $(z_i)_{i\in I}$ be a net in $\mathcal{Z}$. Let $A_i = \{z_j : j \geq i\}$ for every i in the index set I. The limit inferior of $(z_i)_{i\in I}$ is given by

$$\operatorname{Lim\,inf} z_i = \operatorname{Sup}\left(\bigcup_{i\in I} \operatorname{Inf} A_i\right).$$

Similarly, the limit superior of $(z_i)_{i\in I}$ can be defined as

$$\operatorname{Lim\,sup} z_i = \operatorname{Inf}\left(\bigcup_{i\in I} \operatorname{Sup} A_i\right).$$

We will use the following result.

THEOREM 2.1 ([8, Theorem 2.1]). *Let $(z_i)_{i\in I}$ be a net in $\mathcal{Z}$ convergent to z and let $A_i = \{z_j : j \geq i\}$.*

i) *If there is an index i_0 such that, for every $i \geq i_0$, there exists $j \geq i$ with $\operatorname{Inf} A_j \neq \emptyset$, then $z \in \operatorname{Lim\,inf} z_i$.*

ii) *If there is an index i_0 such that, for every $i \geq i_0$, there exists $j \geq i$ with $\operatorname{Sup} A_j \neq \emptyset$, then $z \in \operatorname{Lim\,sup} z_i$.*

We introduce two new definitions of vector topologically pseudomonotonicity which play a central role in our main results.

Definition 2.2. Let (X,σ_1) and (Y,σ_2) be two Hausdorff topological spaces, let $f : X \times Y \times X \to \mathcal{Z}$ be a trifunction. Then f is said to be of class (SPM_1) if for every $u \in X$, $w \in \operatorname{Int} C$ and for each net $(x_i, y_i)_{i\in I}$ in $X \times Y$ satisfying $(x_i, y_i) \xrightarrow{\sigma_1,\sigma_2} (x,y) \in X \times Y$ (i.e., $(x_i) \xrightarrow{\sigma_1} x \in X$ and $(y_i) \xrightarrow{\sigma_2} y \in Y$) and

$$\operatorname{Lim\,inf} f\left(x_i, y_i, x\right) \cap \left(-\operatorname{Int} C\right) = \emptyset,$$

there is $j_0 \in I$ such that

$$\overline{\left\{f\left(x_i, y_i, u\right) : i \geq j\right\}} \subset f\left(x, y, u\right) + w - \operatorname{Int} C$$

for all $j \geq j_0$.

Definition 2.3. Let (X,σ_1) and (Y,σ_2) be two Hausdorff topological spaces, let $f : X \times Y \times X \to \mathcal{Z}$ be a trifunction. Then f is said to be of class (SPM_2) if for every $u \in X$, $w \in \operatorname{Int} C$ and for each net $(x_i, y_i)_{i\in I}$ in $X \times Y$ satisfying $(x_i, y_i) \xrightarrow{\sigma_1,\sigma_2} (x,y) \in X \times Y$ and

$$\operatorname{Lim\,inf} f\left(x_i, y_i, x\right) = \emptyset \quad \text{or} \quad \operatorname{Lim\,inf} f\left(x_i, y_i, x\right) \cap \left(-\operatorname{Int} C\right)^c \neq \emptyset,$$

there is $j_0 \in I$ such that

$$\overline{\left\{f\left(x_i, y_i, u\right) : i \geq j\right\}} \subset f\left(x, y, u\right) + w - \operatorname{Int} C$$

for all $j \geq j_0$.

The Definition 2.2 is a slight generalization of the notion of vector topological pseudomonotonicity given by Chiang, Chadli and Yao in [7].

The above definitions represents extensions to a vector framework of the classical pseudomonotonicity notion introduced by Brézis [4].

Remark 2.4. Every function of class (SPM_2) is a function of class (SPM_1). The inverse relation does not take place in generally.

Example 2.5. Let the $T : X \to 2^Y$ set-valued be defined by $T(x) = \{1\}$ for every $x \in X$, and real vector function $f : X \times Y \times X \to \mathbb{R}^2$, where $X = [0,1]$ and $Y = [0,1]$ given with

$$f(x,y,u) = \begin{cases} (yx - u, y - x) & \text{if } x > 0, \\ (u, y) & \text{if } x = 0, \end{cases}$$

where the ordering cone C of $\mathbb{R}^2$ is the third quadrant, i.e.,

$$C = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 0,\ x_2 \leq 0\right\}.$$

The function f is of class (SPM_1), but is not belonging to the class (SPM_2). Indeed, if we make the substitutions, the example become Example 7 in [14].

Let us consider σ_1 and τ two topologies on X. Suppose that τ is stronger than σ_1 on X.

For the parametric domains in $(GVEP)_p$ we shall use a slight generalization of Mosco's convergence [11].

Definition 2.6 ([3, Definition 2.2]). Let K_p be subsets of X for all $p \in P$. The sets K_p converge to K_{p_0} in the Mosco sense ($K_p \xrightarrow{M} K_{p_0}$) as $p \to p_0$ if:

i) for every subnet $(x_{p_i})_{i\in I}$ with $x_{p_i} \in K_{p_i}$, $p_i \to p_0$ and $x_{p_i} \xrightarrow{\sigma_1} x$ imply $x \in K_{p_0}$;

ii) for every $x \in K_{p_0}$, there exist $x_p \in K_p$ such that $x_p \xrightarrow{\tau} x$ as $p \to p_0$.

3. CLOSEDNESS OF THE SOLUTION MAP

This section is devoted to prove the closedness of the solution map for parametric generalized vector equilibrium problems with trifunctions.

THEOREM 3.1. *Let X and (Y, σ_2) Hausdorff topological spaces, the space X is endowed with two topologies σ_1 and τ, where $\sigma_1 \subseteq \tau$. Let K_p be nonempty sets of X and let $p_0 \in P$ be fixed. Suppose that $S(pt) \neq \emptyset$ for each $p \in P$ and the following conditions hold:*

i) $K_p \xrightarrow{M} K_{p_0}$ as p tends to p_0;

ii) for each net of elements $(p_i, (x_{p_i}, y_{p_i})) \in \operatorname{Graph} S$, if $p_i \to p_0$, $(x_{p_i}, y_{p_i}) \xrightarrow{\sigma_1,\sigma_2} (x, y)$, $u_{p_i} \in K_{p_i}$, $u \in K_{p_0}$, and $u_{p_i} \xrightarrow{\tau} u$ then

$$\operatorname{Lim\,inf}\left(f_{p_i}\left(x_{p_i}, y_{p_i}, u_{p_i}\right) - f_{p_0}\left(x_{p_i}, y_{p_i}, u\right)\right) \cap (-\operatorname{Int} C) \neq \emptyset,$$

where $y_{p_i} \in T\left(x_{p_i}\right)$;

iii) $T : X \to 2^Y$ is closed at x;

iv) $f_{p_0} : X \times Y \times X \to \mathcal{Z}$ is of class (SPM_1).

Then the solution map $p \to S\left(p\right)$ is closed at p_0, i.e., for each net of elements $(p_i, (x_{p_i}, y_{p_i})) \in \operatorname{Graph} S$, $p_i \to p_0$ and $(x_{p_i}, y_{p_i}) \xrightarrow{\sigma_1,\sigma_2} (x, y)$ imply $(p_0, (x, y)) \in \operatorname{Graph} S$.

Proof. Let $(p_i, (x_{p_i}, y_{p_i}))_{i\in I}$ be a net of elements $(p_i, (x_{p_i}, y_{p_i})) \in \operatorname{Graph} S$, i.e.,

$$f_{p_i}\left(x_{p_i}, y_{p_i}, u\right) \in \left(-\operatorname{Int} C\right)^c, \quad \forall u \in K_{p_i}, \tag{3.1}$$

with $p_i \to p_0$ and $(x_{p_i}, y_{p_i}) \xrightarrow{\sigma_1,\sigma_2} (x, y)$. By the Mosco convergence of the sets K_{p_i}, we get $x \in K_{p_0}$. Moreover, there exists a net $\left(u_{p_i}\right)_{i\in I}$, $u_{p_i} \in K_{p_i}$ such that $u_{p_i} \xrightarrow{\tau} x$. From the assumption ii) we obtain that

$$\operatorname{Lim\,inf}\left(f_{p_i}\left(x_{p_i}, y_{p_i}, u_{p_i}\right) - f_{p_0}\left(x_{p_i}, y_{p_i}, x\right)\right) \cap \left(-\operatorname{Int} C\right) \neq \emptyset.$$

Since $-\operatorname{Int} C$ is an open cone, it follows that there exists a subnet $\left(x_{p_i}, y_{p_i}\right)_{i \in I}$, denoted by the same indexes, such that

$$f_{p_i}\left(x_{p_i}, y_{p_i}, u_{p_i}\right) - f_{p_0}\left(x_{p_i}, y_{p_i}, x\right) \in -\operatorname{Int} C, \quad \forall\, i \in I. \tag{3.2}$$

By replacing u with u_{p_i} in (3.1) we get

$$f_{p_i}\left(x_{p_i}, y_{p_i}, u_{p_i}\right) \in \left(-\operatorname{Int} C\right)^c. \tag{3.3}$$

From (3.2) and (3.3) we obtain that

$$f_{p_0}\left(x_{p_i}, y_{p_i}, x\right) \in \left(-\operatorname{Int} C\right)^c, \text{ for all } i \in I.$$

Since $\left(-\operatorname{Int} C\right)^c$ is closed, it follows

$$\operatorname{Lim\,inf} f_{p_0}\left(x_{p_i}, y_{p_i}, x\right) \subset \left(-\operatorname{Int} C\right)^c.$$

Now we can apply iv) and we obtain that for every $u \in K_{p_0}$, $w \in \operatorname{Int} C$, there exists $j_1 \in I$ such that

$$\overline{\left\{f_{p_0}\left(x_{p_i}, y_{p_i}, u\right) : i \geq j\right\}} \subset f_{p_0}\left(x, y, u\right) + w - \operatorname{Int} C, \quad \forall j \geq j_1, \tag{3.4}$$

where $y \in T\left(x\right)$ which is true since $y_i \in T\left(x_i\right)$ and T is closed at x.

We have to prove that

$$f_{p_0}\left(x, y, u\right) \in \left(-\operatorname{Int} C\right)^c, \quad \forall\, u \in K_{p_0}.$$

Assume the contrary, that there exists $\overline{u} \in K_{p_0}$ such that

$$f_{p_0}\left(x, y, \overline{u}\right) \in -\operatorname{Int} C.$$

Let be $f_{p_0}\left(x, y, \overline{u}\right) = -w$ where $w \in \operatorname{Int} C$. From (3.4) we obtain that there exists $j_1 \in I$ such that

$$\overline{\left\{f_{p_0}\left(x_{p_i}, y_{p_i}, \overline{u}\right) : i \geq j\right\}} \subset -w + w - \operatorname{Int} C = -\operatorname{Int} C, \quad \forall j \geq j_1. \tag{3.5}$$

Since $\overline{u} \in K_{p_0}$ from the Mosco convergence of the sets K_{p_i} there exists $\left(\overline{u}_{p_i}\right)_{i \in I} \subset K_{p_i}$ such that $\overline{u}_{p_i} \xrightarrow{\tau} \overline{u}$. By using again the assumption ii), it follows that there exists a subnet $\left(x_{p_i}, y_{p_i}\right)_{i \in I}$, denoted by the same indexes, for which

$$f_{p_i}\left(x_{p_i}, y_{p_i}, \overline{u}_{p_i}\right) - f_{p_0}\left(x_{p_i}, y_{p_i}, \overline{u}\right) \in -\operatorname{Int} C, \text{ for all } i \in I. \tag{3.6}$$

From (3.5) and (3.6) it follows that

$$f_{p_i}\left(x_{p_i}, y_{p_i}, \overline{u}_{p_i}\right) \in -\operatorname{Int} C, \quad i \in I,$$

contradicting (3.1). Hence $\left(p_0, (x, y)\right) \in \operatorname{Graph} S$. $\square$

Remark 3.2. The Theorem 3.1 generalizes the Theorem 3.1 in [12] but it does not imply the Theorem 1 in [3] since the assumption ii) cannot be replaced by

ii′) For each net of elements $(p_i, (x_{p_i}, y_{p_i})) \in \operatorname{Graph} S$, if $p_i \to p_0$, $(x_{p_i}, y_{p_i}) \xrightarrow{\sigma_1,\sigma_2} (x, y)$, $u_{p_i} \in K_{p_i}$, $u \in K_{p_0}$, and $u_{p_i} \xrightarrow{\tau} u$ then

$$\operatorname{Lim\,inf}\left(f_{p_i}\left(x_{p_i}, y_{p_i}, u_{p_i}\right) - f_{p_0}\left(x_{p_i}, y_{p_i}, u\right)\right) \cap (-C) \neq \emptyset.$$

If we replace the assumption ii) with ii′) we have to give a stronger condition to assumption iv).

Theorem 3.3. *Let X and (Y, σ_2) Hausdorff topological spaces, the space X is endowed with two topologies σ_1 and τ, where $\sigma_1 \subseteq \tau$. Let K_p be nonempty sets of X and let $p_0 \in P$ be fixed. Suppose that $S(p) \neq \emptyset$ for each $p \in P$ and the following conditions hold:*

i) $K_p \xrightarrow{M} K_{p_0}$ as p tends to p_0;

ii′) for each net of elements $(p_i, (x_{p_i}, y_{p_i})) \in \operatorname{Graph} S$, if $p_i \to p_0$, $(x_{p_i}, y_{p_i}) \xrightarrow{\sigma_1,\sigma_2} (x, y)$, $u_{p_i} \in K_{p_i}$, $u \in K_{p_0}$, and $u_{p_i} \xrightarrow{\tau} u$ then

$$\operatorname{Lim\,inf}\left(f_{p_i}\left(x_{p_i}, y_{p_i}, u_{p_i}\right) - f_{p_0}\left(x_{p_i}, y_{p_i}, u\right)\right) \cap (-C) \neq \emptyset,$$

where $y_{p_i} \in T(x_{p_i})$;

iii) $T : X \to 2^Y$ is closed at x;

iv) $f_{p_0} : X \times Y \times X \to \mathcal{Z}$ is of class (SPM_2).

Then the solution map $p \to S(p)$ is closed at p_0.

Proof. The proof is given in the following three steps.

Step 1. Let $(p_i, (x_{p_i}, y_{p_i}))_{i\in I}$ be a net of elements $(p_i, (x_{p_i}, y_{p_i})) \in \operatorname{Graph} S$, i.e.,

$$f_{p_i}\left(x_{p_i}, y_{p_i}, u\right) \in (-\operatorname{Int} C)^c, \quad \forall u \in K_{p_i} \tag{3.7}$$

with $p_i \to p_0$ and $(x_{p_i}, y_{p_i}) \xrightarrow{\sigma_1,\sigma_2} (x, y)$. By the Mosco convergence of the sets K_{p_i} we get $x \in K_{p_0}$. Moreover, there exists a net $(u_{p_i})_{i\in I}$, $u_{p_i} \in K_{p_i}$ such that $u_{p_i} \xrightarrow{\tau} x$. From the assumption ii′) we obtain that

$$\operatorname{Lim\,inf}\left(f_{p_i}\left(x_{p_i}, y_{p_i}, u_{p_i}\right) - f_{p_0}\left(x_{p_i}, y_{p_i}, u\right)\right) \cap (-C) \neq \emptyset. \tag{3.8}$$

Step 2. We will prove that (3.8) and (3.7) imply

$$\operatorname{Lim\,inf} f_{p_0}\left(x_{p_i}, y_{p_i}, x\right) = \emptyset \quad \text{or} \quad \operatorname{Lim\,inf} f_{p_0}\left(x_{p_i}, y_{p_i}, x\right) \cap (-\operatorname{Int} C)^c \neq \emptyset.$$

For this we can distinguish two cases:

Case 1. $\operatorname{Lim\,inf}\left(f_{p_i}\left(x_{p_i}, y_{p_i}, u_{p_i}\right) - f_{p_0}\left(x_{p_i}, y_{p_i}, x\right)\right) \cap (-\operatorname{Int} C) \neq \emptyset$.

Since $-\operatorname{Int} C$ is an open cone, it follows that there exists a subnet, denoted by the same indexes, such that

$$f_{p_i}\left(x_{p_i}, y_{p_i}, u_{p_i}\right) - f_{p_0}\left(x_{p_i}, y_{p_i}, x\right) \in -\operatorname{Int} C, \text{ for all } i \in I. \tag{3.9}$$

By replacing u with u_{p_i} in (3.7) we get

$$f_{p_i}(x_{p_i}, y_{p_i}, u_{p_i}) \in (-\operatorname{Int} C)^c. \tag{3.10}$$

From (3.10) and (3.9) we obtain that

$$f_{p_0}(x_{p_i}, y_{p_i}, x) \in (-\operatorname{Int} C)^c, \text{ for all } i \in I.$$

Since $(-\operatorname{Int} C)^c$ is closed, it follows

$$\operatorname{Lim\,inf} f_{p_0}(x_{p_i}, y_{p_i}, x) \in (-\operatorname{Int} C)^c$$

consequently

$$\operatorname{Lim\,inf} f_{p_0}(x_{p_i}, y_{p_i}, x) = \emptyset \quad \text{or} \quad \operatorname{Lim\,inf} f_{p_0}(x_{p_i}, y_{p_i}, x) \cap (-\operatorname{Int} C)^c \neq \emptyset.$$

Case 2. $\operatorname{Lim\,inf}(f_{p_i}(x_{p_i}, y_{p_i}, u_{p_i}) - f_{p_0}(x_{p_i}, y_{p_i}, x)) \cap (-\operatorname{Int} C) = \emptyset$.

We can suppose that

$$f_{p_i}(x_{p_i}, y_{p_i}, u_{p_i}) - f_{p_0}(x_{p_i}, y_{p_i}, x) \in (-\operatorname{Int} C)^c, \quad \forall i \in I \tag{3.11}$$

and

$$f_{p_0}(x_{p_i}, y_{p_i}, x) \in -\operatorname{Int} C, \quad \forall i \in I \tag{3.12}$$

otherwise we get back the first case.

Since $\operatorname{Lim\,inf}(f_{p_i}(x_{p_i}, y_{p_i}, u_{p_i}) - f_{p_0}(x_{p_i}, y_{p_i}, x)) \cap (-\operatorname{Int} C) = \emptyset$, from (3.8) and (3.11) it follows that, there exists a subnet (x_{p_i}, y_{p_i}), denoted by the same indexes, for which

$$(f_{p_i}(x_{p_i}, y_{p_i}, u_{p_i}) - f_{p_0}(x_{p_i}, y_{p_i}, x))_{i \in I} \text{ converges to the boundary of cone } -C. \tag{3.13}$$

Indeed, otherwise it must exist $i_0 \in I$ such that

$$\overline{\{f_{p_i}(x_{p_i}, y_{p_i}, u_{p_i}) - f_{p_0}(x_{p_i}, y_{p_i}, x) : i \geq i_0\}} \subset (-C)^c$$

then from the definition of the limit inferior, we obtain that

$$\operatorname{Lim\,inf}(f_{p_i}(x_{p_i}, y_{p_i}, u_{p_i}) - f_{p_0}(x_{p_i}, y_{p_i}, x)) \subset (-C)^c,$$

which is in contradiction with assumption ii$'$).

From (3.12) and (3.13) we obtain that there exists a subnet (x_{p_i}, y_{p_i}), denoted by the same indexes, such that

$$(f_{p_0}(x_{p_i}, y_{p_i}, x))_{i \in I} \text{ converges to an element in the boundary of the cone } -C. \tag{3.14}$$

To prove this statement, let us suppose the contrary, that

$$\overline{\{f_{p_0}(x_{p_i}, y_{p_i}, x) : i \in I\}} \subset -\operatorname{Int} C.$$

Then from (3.13) we obtain that

$$f_{p_i}(x_{p_i}, y_{p_i}, u_{p_i}) \text{ converges to an element in } -\operatorname{Int} C.$$

Since $-\operatorname{Int} C$ is an open cone, it follows that there exists $i_1 \in I$ such that

$$f_{p_i}(x_{p_i}, y_{p_i}, u_{p_i}) \in -\operatorname{Int} C, \text{ for all } i \geq i_1,$$

contradicting (3.7).

By applying the Theorem 2.1 for the subnet in (3.14) we obtain that

$$\operatorname{Lim\,inf} f_{p_0}(x_{p_i}, y_{p_i}, x) \cap (-\partial C) \neq \emptyset,$$

or there exists $i_2 \in I$ such that

$$\operatorname{Inf}\{f_{p_0}(x_{p_i}, y_{p_i}, x) : i \geq i_2\} = \emptyset.$$

This implies that

$$\operatorname{Lim\,inf} f_{p_0}(x_{p_i}, y_{p_i}, x) \cap (-\operatorname{Int} C)^c \neq \emptyset \quad \text{or} \quad \operatorname{Lim\,inf} f_{p_0}(x_{p_i}, y_{p_i}, x) = \emptyset.$$

So, in both cases, we can apply iv) and we obtain that for every $u \in K_{p_0}$ and $w \in \operatorname{Int} C$, there exists $j_0 \in I$ such that

$$\overline{\{f_{p_0}(x_i, y_i, u) : i \geq j\}} \subset f_{p_0}(x, y, u) + w - \operatorname{Int} C, \quad \forall j \geq j_0, \tag{3.15}$$

where $y \in T(x)$ which is true since $y_i \in T(x_i)$ and T is closed at x.

Step 3. We have to prove that

$$f_{p_0}(x, y, u) \in (-\operatorname{Int} C)^c, \quad \forall u \in K_{p_0}.$$

Assume the contrary, that there exists $\overline{u} \in K_{p_0}$ such that

$$f_{p_0}(x, y, \overline{u}) \in -\operatorname{Int} C.$$

Let be $f_{p_0}(x, y, \overline{u}) = -w$ where $w \in \operatorname{Int} C$. From (3.15) we obtain that there exists $j_0 \in I$ such that

$$\overline{\{f_{p_0}(x_i, y_i, \overline{u}) : i \geq j\}} \subset -w + w - \operatorname{Int} C = -\operatorname{Int} C, \quad \forall j \geq j_0. \tag{3.16}$$

Since $\overline{u} \in K_{p_0}$ from the Mosco convergence of the sets K_{p_i}, we have that there exists $(\overline{u}_{p_i})_{i\in I} \subset K_{p_i}$ such that $\overline{u}_{p_i} \xrightarrow{\tau} \overline{u}$. By using again the assumption ii′), it follows that one of the next cases, corresponding to (3.9) and (3.13) respectively, hold: there exists a subnet (x_{p_i}, y_{p_i}), denoted by the same indexes, such that

$$f_{p_i}(x_{p_i}, y_{p_i}, \overline{u}_{p_i}) - f_{p_0}(x_{p_i}, y_{p_i}, \overline{u}) \in -\operatorname{Int} C, \quad \forall i \in I \tag{3.17}$$

or there exists a subnet (x_{p_i}, y_{p_i}), denoted by the same indexes, for which

$$\begin{gathered}(f_{p_i}(x_{p_i}, y_{p_i}, \overline{u}_{p_i}) - f_{p_0}(x_{p_i}, y_{p_i}, \overline{u}))_{i\in I} \text{ converges} \\ \text{to the boundary of cone } -C.\end{gathered} \tag{3.18}$$

From (3.16), (3.17) and (3.18) it follows that there exists $j_1 \in I$ such that

$$f_{p_i}(x_{p_i}, y_{p_i}, \overline{u}_{p_i}) \in -\operatorname{Int} C, \quad i \geq j_1 \geq j_0,$$

but on other side $(p_i, (x_{p_i}, y_{p_i})) \in \operatorname{Graph} S$, and

$$f_{p_i}(x_{p_i}, y_{p_i}, \overline{u}_{p_i}) \in (-\operatorname{Int} C)^c$$

which is a contradiction. Hence $(p_0, (x, y)) \in \operatorname{Graph} S$. $\square$

Remark 3.4. Theorem 3.3 implies Theorem 1 in [3] and Theorem 10 in [14].

Example 3.5. Let $\sigma_1 = \sigma_2 = \tau$ be the natural topology on $X = Y = [0,1]$. Let $P = \mathbb{N} \cup \{\infty\}$, $p_0 = \infty$, (∞ means $+\infty$ from real analysis), $K_n = (0,1)$, $n \in \mathbb{N}$ and $K_\infty = [0,1]$. On P we consider the topology induced by the metric d given by $d(m,n) = |1/m - 1/n|$, $d(n,\infty) = d(\infty,n) = 1/n$, for $m,n \in \mathbb{N}$, and $d(\infty,\infty) = 0$. Let us consider the third quadrant as the ordering cone C in $\mathbb{R}^2$. The multi-valued mapping $T : X \to 2^Y$ be defined by $T(x) = [0,1]$ for every $x \in X$.

Let the real vector functions $f_n : [0,1] \times [0,1] \times [0,1] \to \mathbb{R}^2$ be given by $f_n(x,y,u) = (x - u - 1/n, 1 + x + y)$, $n \in \mathbb{N}$ and the function $f_\infty : [0,1] \times [0,1] \times [0,1] \to \mathbb{R}^2$ be defined by $f_\infty(x,y,u) = (x - 2u, 2x + y + u)$.

The function f_∞ is of class (SPM_2), since it is continuous. The mapping T is closed at each x from X.

Only the assumption ii$'$) has to be verified. Let $x_n, u_n \in (0,1)$, $x_n \to x$ and $u_n \to u$. One has

$$\operatorname{Lim\,inf}\left(f_n(x_n, y_n, u_n) - f_\infty(x_n, y_n, u)\right) =$$
$$= \operatorname{Lim\,inf}\left\{(-1/n - u_n + 2u, 1 - x_n - u),\ n \geq 1\right\},$$

by Theorem 2.1 it follows that

$$(u, 1 - x - u) \in \operatorname{Lim\,inf}\left(f_n(x_n, y_n, u_n) - f_\infty(x_n, y_n, u)\right).$$

The $S(n) = \{(x,y) \in (0,1) \times [0,1] : x \in (0,1/n]\}$ for each $n \in \mathbb{N}$. Since $1 + x + y > 0$ we obtain that

$$x - u - 1/n \geq 0 \text{ for every } u \in (0,1)$$

from where it follows $x \in (0,1/n]$. Hence every sequence (x_n) satisfying $(n, (x_n, y_n)) \in \operatorname{Graph} S$ has to converge to $x = 0$. From $(u, 1-u) \in -\operatorname{Int} C$ it follows that the assumption ii$'$) takes place. By Theorem 3.3 we obtain that the solution mapping S is closed at ∞.

4. HADAMARD WELL-POSEDNESS

Let us recall some classical definitions from set-valued analysis. Let X, Y be topological spaces. The map $T : X \to 2^Y$ is said to be *upper semi-continuous* at $u_0 \in \operatorname{dom} T := \{u \in X \mid T(u) \neq \emptyset\}$ if for each neighborhood V of $T(u_0)$, there exists a neighborhood U of u_0 such that $T(U) \subset V$.

Closedness and upper semi-continuity of a multifunction are closely related.

PROPOSITION 4.1 ([2, Proposition 1.4.8, 1.4.9]). *Let $T : X \to 2^Y$ be a set-valued map.*

i) *If T has closed values and is upper semi-continuous then T is closed.*

ii) *If Y is compact and T is closed at $x \in X$ then T is upper semi-continuous at $x \in X$.*

Now we recall the notion of generalized Hadamard well-posedness.

Definition 4.2. Let (X, σ_1) and (Y, σ_2) be two Hausdorff topological spaces. The problem $(GVEP)_p$ is said to be Hadamard well-posed (briefly, H-wp) at $p_0 \in P$ if $S(p_0) = \{(x_{p_0}, y_{p_0})\}$ and for any $(x_p, y_p) \in S(p)$ one has $(x_p, y_p) \stackrel{\sigma_1,\sigma_2}{\longrightarrow} (x_{p_0}, y_{p_0})$, as $p \to p_0$. The problem $(GVEP)_p$ is said to be generalized Hadamard well-posed (briefly, gH-wp) at $p_0 \in P$ if $S(p_0) \neq \emptyset$ and for any $(x_p, y_p) \in S(p)$, if $p \to p_0$, (x_p, y_p) must have a subsequence (σ_1, σ_2)-converging to an element of $S(p_0)$.

With the help of the next result we are able to establish the relationship between upper semi-continuity and Hadamard well-posedness.

PROPOSITION 4.3 ([15, Theorem 2.2]). *Let $T : X \to 2^Y$ be a set-valued map. If T is upper semi-continuous at $x \in X$ and $T(x)$ is compact, then T is gH-wp at x. If more, $T(x) = \{y^*\}$, then T is H-wp at x.*

In the following we prove that the solution map of $(GVEP)_p$ has closed value at p_0.

PROPOSITION 4.4. *Let K_{p_0} be closed with respect to the σ_1 topology and $T : X \to 2^Y$ be a closed set-valued map. If $f_{p_0} : X \times Y \times X \to \mathcal{Z}$ is of class (SPM_1), then $S(p_0)$ is closed with respect to the (σ_1, σ_2) topology pair.*

Proof. Let $S(p_0) \neq \emptyset$ and $(x_i, y_i) \in S(p_0)$, with $(x_i, y_i) \stackrel{\sigma_1,\sigma_2}{\longrightarrow} (x, y)$. Since K_{p_0} is closed with respect to the σ_1 topology, we have $x \in K_{p_0}$. From $(x_i, y_i) \in S(p_0)$ it follows that

$$f_{p_0}(x_i, y_i, x) \in (-\operatorname{Int} C)^c, \quad \forall i \in I.$$

Since $(-\operatorname{Int} C)^c$ is closed, we get

$$\operatorname{Lim\,inf} f_{p_0}(x_i, y_i, x) \in (-\operatorname{Int} C)^c.$$

By using that f_{p_0} is of class (SPM_1) we obtain that for every $w \in \operatorname{Int} C$ there is j_0 in the index set I such that

$$\overline{\{f_{p_0}(x_i, y_i, u) : i \geq j\}} \subset f(x, y, u) + w - \operatorname{Int} C, \quad \forall j \geq j_0,\ \forall u \in K_{p_0}. \tag{4.1}$$

We have to prove that $(x,y) \in S(p_0)$, i.e.,

$$f_{p_0}(x,y,u) \in (-\operatorname{Int} C)^c, \quad \forall\, u \in K_{p_0}.$$

Assume the contrary, that there exists $\overline{u} \in K_{p_0}$ such that

$$f_{p_0}(x,y,\overline{u}) \in -\operatorname{Int} C.$$

Let $f_{p_0}(x,y,\overline{u}) = -w$ where $w \in \operatorname{Int} C$. From (4.1) we obtain that

$$\overline{\{f_{p_0}(x_i,y_i,\overline{u}) : i \geq j\}} \subset -w + w - \operatorname{Int} C = -\operatorname{Int} C, \quad \forall j \geq j_0$$

which is a contradiction to $(x_i,y_i) \in S(p_0)$. Thus $(x,y) \in S(p_0)$. □

Now we can formulate the following results.

Corollary 4.5. *Let (X,σ_1) be a compact Hausdorff topological space and P be a Hausdorff topological space. Let K_p be nonempty sets of X, and K_{p_0} be a closed subset of X. If the hypotheses of Theorem* 3.1 *are satisfied, then $(GVEP)_p$ is generalized Hadamard well-posed at p_0. Furthermore, if $S(p_0) = \{(x,y)\}$ (a singleton), then $(GVEP)_p$ is Hadamard well-posed at p_0.*

Proof. From Theorem 3.1 we obtain that the solution map S is closed at p_0. By using Proposition 4.1 ii) it follows that S is upper semi-continuous at p_0. The set $S(p_0)$ is closed by Proposition 4.4, hence it is compact. The conclusion follows from Proposition 4.3. □

From Remark 2.4 and Corollary 4.5 we obtain:

Corollary 4.6. *Let (X,σ_1) be a compact Hausdorff topological space and P be a Hausdorff topological space. Let K_p be nonempty sets of X and K_{p_0} be a closed subset of X. If the hypotheses of Theorem* 3.3 *are satisfied, then $(GVEP)_p$ is generalized Hadamard well-posed at p_0. Furthermore, if $S(p_0) = \{(x,y)\}$ (a singleton), then $(GVEP)_p$ is Hadamard well-posed at p_0.*

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