GRADIENT FLOWS WITH JUMPS ASSOCIATED WITH NONLINEAR HAMILTON-JACOBI EQUATIONS WITH JUMPS

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We analyze gradient flows with jumps generated by a finite set of complete vector fields in involution using some Radon measures $u \in \mathcal{U}_a$ as admissible perturbations. Both the evolution of a bounded gradient flow $\{x^u(t,\lambda) \in B(x^*, 3\gamma) \subseteq \mathbb{R}^n : t \in [0,T],\ \lambda \in B(x^*, 2\gamma)\}$ and the unique solution $\lambda = \psi^u(t,x) \in B(x^*, 2\gamma) \subseteq \mathbb{R}^n$ of integral equation $x^u(t,\lambda) = x \in B(x^*,\gamma)$, $t \in [0,T]$, are described using the corresponding gradient representation associated with flow and Hamilton-Jacobi equations.

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1. INTRODUCTION

For a given finite set of complete vector fields $\{g_1, \ldots, g_m\} \subseteq \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R}^n)$ consider the corresponding local flows $\{G_1(t_1)[x], \ldots, G_m(t_m)[x] : |t_i| \leq a_i,$ $x \in B(x^*, 3\gamma) \leq \mathbb{R}^n,\ 1 \leq i \leq m\}$ generated by $\{g_1, \ldots, g_m\}$ correspondingly and satisfying

$$(1) \qquad |G_i(t_i)[x] - x| \leq \frac{\gamma}{2m}, \quad x \in B(x^*, 3\gamma),\ |t_i| \leq a_i,\ 1 \leq i \leq m$$

for some fixed constants $a_i > 0$ and $\gamma > 0$.

Denote by $\mathcal{U}_a$ the set of admissible perturbations consisting of all piecewise right-continuous mappings (of $t \geq 0$) $u(t,x) : [0,\infty) \times \mathbb{R}^n \to \bigsqcup = \prod_{i=1}^{m} [-a_i, a_i]$ fulfilling

$$(2) \qquad u(0,\lambda) = 0, \quad u(t,\cdot) \in \mathcal{C}_b^1(\mathbb{R}^n; \mathbb{R}^n) \quad \text{and}$$
$$|\partial_\lambda u_i(t,\lambda)| \leq K_1, \quad t \geq 0,\ \lambda \in \mathbb{R}^n,\ 1 \leq i \leq m,$$

for some fixed constant $K_1 > 0$.

For each admissible perturbation $u \in \mathcal{U}_a$, we associate a piecewise right-continuous trajectory (for $t \geq 0$)

$$x^u(t,\lambda) = G\big(u(t,\lambda)\big)[\lambda], \quad t \geq 0,\ \lambda \in B(x^*, 2\gamma), \tag{3}$$

where the smooth mapping $G(p)[x] : \bigsqcup \times B(x^*, 2\gamma) \to B(x^*, 3\gamma)$ is defined by

$$G(p)[x] = G_1(t_1) \circ \cdots \circ G_m(t_m)[x], \quad p = (t_1, \ldots, t_m) \in \bigsqcup,\ x \in B(x^*, 2\gamma) \tag{4}$$

verifying $G(p)[x] \in B(x^*, 3\gamma)$ (see (1)).

We are going to introduce some nonlinear ODE with jumps fulfilled by the bounded flow $\{x^u(t,\lambda) : t \in [0,T],\ \lambda \in B(x^*, 2\gamma)\}$ defined in (3), when $u \in \mathcal{U}_a$ has a bounded variation property. In addition, the unique solution $\{\lambda = \psi(t,x) \in B(x^*, 2\gamma) : t \in [0,T],\ x \in B(x^*, \gamma)\}$ of the integral equation

$$x^u(t,\lambda) = x \in B(x^*, \gamma), \quad t \in [0,T] \tag{5}$$

fulfils a quasilinear Hamilton-Jacobi (H-J) equation on each continuity interval $t \in [t_k, t_{k+1}) \subseteq [0,T]$. These result are contained in the last section of this paper (see Theorems 3.1, 3.3 and 3.4). In the case that we assume $\{g_1, \ldots, g_m\} \subset \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R}^n)$ are commuting using Lie bracket then the result are more or less contained in [1].

Here, in this paper, the vector fields $\{g_1, \ldots, g_m\} \subset \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R}^n)$ are supposed to be in involution over reals which lead us to make use of algebraic representation for gradient systems in a finite dimensional Lie algebra (see [1]) without involving a global nonsingularity or local times. The analysis performed here reveals the meaningful connection between dynamical systems and partial differential equations.

2. FORMULATION OF PROBLEMS AND SOME AUXILIARY RESULTS

Consider a finite set of complete vector fields $g_i \in \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R}^n)$, $1 \leq i \leq m$, and let $\{G_i(t_i)[x] : |t_i| \leq a_i,\ x \in B(x^*, 3\gamma) \subseteq \mathbb{R}^n\}$ be the local flow generated by g_i satisfying

$$|G_i(t_i)[x] - x| \leq \frac{\gamma}{2m}, \quad x \in B(x^*, 3\gamma),\ |t_i| \leq a_i,\ 1 \leq i \leq m \tag{6}$$

for some fixed constants $a_i > 0$ and $\gamma > 0$.

Denote by $\mathcal{U}_a$ the set of admissible perturbations consisting of all piecewise right-continuous mappings (of $t \geq 0$) $u(t,\lambda) : [0,\infty) \times \mathbb{R}^n \to \bigsqcup = \prod_{i=1}^{m} [-a_i, a_i]$ fulfilling

$$\begin{gathered} u(0,\lambda) = 0, \quad u(t,\cdot) \in \mathcal{C}_b^1(\mathbb{R}^n; \mathbb{R}^n) \quad \text{and} \\ |\partial_\lambda u_i(t,\lambda)| \leq K_1, \quad t \geq 0,\ \lambda \in \mathbb{R}^n,\ 1 \leq i \leq m, \end{gathered} \tag{7}$$

for some fixed constant $K_1 > 0$. For each admissible perturbation $u \in \mathcal{U}_a$, we associate a piecewise right-continuous trajectory (for $t \geq 0$)

$$x^u(t,\lambda) = G\big(u(t,\lambda)\big)[\lambda], \quad t \geq 0,\ \lambda \in B(x^*, 2\gamma). \tag{8}$$

Here the gradient smooth mapping $G(p)[x] : \bigcup \times B(x^*, 2\gamma) \to B(x^*, 3\gamma)$ is defined as follows

$$G(p)[x] = G_1(t_1) \circ \cdots \circ G_m(t_m)[x], \quad p = (t_1, \ldots, t_m) \in \bigcup,\ x \in B(x^*, 2\gamma), \tag{9}$$

and satisfies (see (6)) $G(p)[\lambda] \in B(x^*, 3\gamma)$ for any $p \in \bigcup$ and $\lambda \in B(x^*, 2\gamma)$.

The flow with jumps represented as in (8) stands for a gradient flow with jumps and it relies on the smooth mapping defined in (9) which is the unique solution of an associated integrable gradient system

$$\begin{cases} \partial_{t_1} y = g_1(y),\ \partial_{t_2} y = Y_2(t; y), \ldots, \partial_{t_m} y = Y_m(t_1, \ldots, t_{m-1}; y), \\ y(0) = \lambda \in B(x^*, 2\gamma), \quad p = (t_1, \ldots, t_m) \in \bigsqcup,\ y \in \mathbb{R}^n. \end{cases} \tag{10}$$

We are looking for sufficient condition on $\{g_1, \ldots, g_m\}$ (see [2]) such that the vector fields with parameters given in (10) can be represented as follows

$$\{g_1, Y_2(t_1), \ldots, Y_m(t_1, \ldots, t_{m-1})\}(y) = \{g_1, \ldots, g_m\}(y) A(p),\ y \in B(x^*, 3\gamma), \tag{11}$$

$p = (t_1, \ldots, t_m) \in \bigsqcup$, where the $(m \times m)$ matrix $A(p)$ satisfies

$$A(0) = I_m, \quad A(p) = [b_1 b_2(t_1) \ldots b_m(t_1, \ldots, t_{m-1})], \tag{12}$$

$$b_j \in \mathcal{C}^\infty\big(\bigsqcup, \mathbb{R}^n\big), \quad b_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

This algebraic representation help us to define each integral

$$\int_0^t b_j^i\big(u_1(s,\lambda), \ldots, u_{j-1}(s,\lambda)\big) d_s u_j(s,\lambda) = \alpha_{ij}^u(t,\lambda), \tag{13}$$

$1 \leq j \leq m$, $1 \leq i \leq m$, $t \in [0,T]$, as a bounded variation function with respect to $t \in [0,T]$, provided we assume that

(14) each $u_i(t,\lambda),\ t \in [0,T],\ 1 \leq i \leq m,$ has a bounded variation property.

In addition, the algebraic representation (11) (see [2]) can be obtained assuming $g_i \in \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R}^n)$, $1 \leq i \leq m$, and

$$\begin{cases} \{g_1(x), \ldots, g_m(x) : x \in B(x^*, 3\gamma) \subseteq \mathbb{R}^n\} \text{ are in involution} \\ \text{over reals, i.e., each Lie bracket can be written as} \\ [g_i, g_j](x) = \sum_{k=1}^m \gamma_k^{ij} g_k(x) \text{ for } x \in B(x^*, 3\gamma), \text{ using } \gamma_k^{ij} \in \mathbb{R}. \end{cases} \tag{15}$$

Let $[t_k, t_{k+1}) \subset [0,T]$, $0 \leq k \leq N-1$, be the continuity intervals of $u \in \mathcal{U}_a$.

Problem P_1. Under the hypothesis (14) and (15), describe the evolution of the gradient flow with jumps in (8) as follows

$$(16)\quad \begin{cases} d_t x^u(t,\lambda)=\sum\limits_{k=1}^{m} g_i\big(x^u(t,\lambda)\big)d_t\beta_i^u(t,\lambda),\ t\in[t_k,t_{k+1}),\ 0\le k\le N-1,\\ x^u(0,\lambda)=\lambda\in B(x^*,2\gamma),\ \text{where}\, \beta_i^u(t,\lambda)=\sum\limits_{k=1}^{m}\alpha_{ij}^u(t,\lambda),\ 1\le i\le m. \end{cases}$$

Here the matrix $\{\alpha_{ij}^u(t,\lambda) : i,j\in\{1,\dots,m\},\ t\in[0,T]\}$ of bounded variation and piecewise right-continuous function of $t\in[0,T]$ are defined in (13).

Problem P_2. Under the hypothesis (14), (15) and for $K_1>0$ sufficiently small (see (7)), prove that the integral equations (with respect to $\lambda\in B(x^*,2\gamma)$ see (8))

$$(17)\quad x^u(t,\lambda)=x\in B(x^*,\gamma),\ \text{are reversibly with respect to}\ \lambda\in B(x^*,2\gamma).$$

The unique bounded variation and piecewise right-continuous solution $\{\lambda=\psi^u(t,x)\in B(x^*,2\gamma);\ t\in[0,T]\}$ is first order continuously differentiable of $x\in \operatorname{int} B(x^*,\gamma)$.

Remark 2.1. One may wonder about the Hamilton-Jacobi equation with jumps satisfied by the unique solution $\{\lambda=\psi^u(t,x)\in B(x^*,2\gamma) : t\in[0,T],\ x\in B(x^*,\gamma)\}$ found in (P_2). This will be presented at the end of the following section. The next Lemma lead us to the solution of the problem (P_1).

LEMMA 2.2. *Assume that the hypothesis* (14) *and* (15) *are fulfilled and consider the gradient flow with jumps* $\{x^u(t,\lambda) : t\in[0,T],\lambda\in B(x^*,2\gamma)\}$ *defined in* (8)*, where* $u\in\mathcal{U}_a$ *and* $T>0$ *are fixed arbitrarily. Then there exists an* $(m\times m)$ *matrix composed by bounded variation and piecewise right-continuous functions* $\{\alpha_{ij}^u(t,\lambda) : \alpha_{ij}^u(0,\lambda)=0,\ 1\le i,j\le m,\ t\in[0,T],\ \lambda\in B(x^*,2\gamma)\}$ *(see* (13)*) such that*

$$(18)\quad \begin{cases} d_t x^u(t,\lambda)=\sum\limits_{i=1}^{m} g_i\big(x^u(t,\lambda)\big)d_t\beta_i^u(t,\lambda) \qquad t\in[t_k\cdot t_{k+1}),\ 0\le k\le N-1,\\ x^u(0,\lambda)=\lambda,\ \text{where}\ \beta_i^u\overset{def}{=}\sum\limits_{j=1}^{m}\alpha_{ij}^u(t,\lambda)\quad 1\le i\le m, \end{cases}$$

and $[t_k,t_{k+1})\subseteq[0,T]$, $0\le k\le N-1$, *are the continuity intervals of* $u\in\mathcal{U}_a$.

Proof. By definition, $x^u(t,\lambda)\in B(x^*,3\gamma)$, $t\ge 0$, $\lambda\in B(x^*,2\gamma)$ (see (8)) where $x^u(t,\lambda)=G(u(t,\lambda))[\lambda]$ defined in (9) fulfils the integrable gradient system given in (10) (see [2]). In addition, using the hypothesis (15) (see [2]) we may and do represent the vector fields of (10) as in (11). As far as $x^u(t,\lambda)=y_\lambda(u(t,\lambda))$, $t\in[0,T]$, $\lambda\in B(x^*,2\gamma)$, where $\{y_\lambda(p) : p\in\bigsqcup\}$ is the unique solution of (10), we get the conclusion (18) provided the algebraic representation (11) and (12) is used. The proof is complete. □

Remark 2.3. For solving integral equation $x^u(t,\lambda) = x \in B(x^*,\gamma)$ (for some fixed $u \in \mathcal{U}_a$) using integral representation (8), we notice that these are equivalent with the following integral equations

$$\lambda = H\big(u(t,\lambda)\big)[x], \quad t \in [0,T],\ x \in B(x^*,\gamma) \tag{19}$$

with respect to $\lambda \in B(x^*, 2\gamma)$. Here $H(p)[x] = [G(p)]^{-1})(x)$ satisfies

$$H(p)[x] \overset{\text{def}}{=} G_m(-t_m)\circ\cdots\circ G_1(-t_1)[x] \in B(x^*,2\gamma), \text{ for any } p=(t_1,\dots,t_m)\in \bigsqcup \tag{20}$$

and $x \in B(x^*,\gamma)$.

In addition, using the hypothesis (15) and writing the corresponding integrable gradient system for $y(p;\lambda) = G(p)[\lambda]$ (see (10) and (11)) we get each $\partial_{t_i}(H(p)[x])$ as follows

$$\partial_{t_1}H(p)[x] = -\partial_x\big(H(p)[x]\big)g_1(x),\ \partial_{t_2}H(p)[x] = -\partial_x\big(H(p)[x]\big)Y_2(t_1;0;x), \tag{21}$$

$$\dots,\partial_{t_m}H(p)[x] = -\partial_x\big(H(p)[x]\big)Y_m(t_1,\dots,t_{m-1};x).$$

Here a direct computation is applied to the identity $H(p)[G(p)(\lambda)] = \lambda$ and write $0 = \partial_{t_i}H(p)[x] + \partial_x\big(H(p)[x]\big)Y_i(t_1,\dots,t_{i-1;0;x})$ for each $i \in \{1,\dots,m\}$, where $Y_1(x) = g_1(x)$ and (see (10) and (11))

$$\{g_1(x), Y_2(t_1;x),\dots,Y_m(t_1,\dots,t_{m-1};x)\} = \{g_1(x),\dots,g_m(x)\}A(p),\ p \in \bigsqcup. \tag{22}$$

Denote $z(p,x) = H(p)[x]$.

Lemma 2.4. *Assume that the hypothesis* (15) *is satisfied and define* $H(p)[x] = [G(p)]^{-1}(x) = G_m(-t_m)\circ\cdots\circ G_1(-t_1)[x]$, $x \in B(x^*,\gamma)$, $p = (t_1,\dots,t_m) \in \bigsqcup$, *where* $y = G(p)[\lambda]$, $p \in \bigsqcup$, $\lambda \in B(x^*,2\gamma)$, *verifies* (9) *and is the unique solution of the integrable gradient system* (10) *and* (11)*. Then there exists an* $(m\times m)$ *analytic matrix* $A(p)$ *verifying* (22) *such that the following system of* (*H-J*) *equation is fulfilled*

$$\begin{cases} \partial_p z(p;x) + \partial_x(z(p;x))\{g_1(x),\dots,g_m(x)\}A(p) = 0,\ p \in \bigsqcup,\ x \in B(x^*,\gamma) \\ z(0;x) = x \end{cases} \tag{23}$$

Proof. A direct computation applied to the identity $H(p)[G(p)(\lambda)] = \lambda$ lead us to the following system of (H-J) equations (see $z(p;x) = H(p)[x]$)

$$\partial_{t_i}z(p;x) + \partial_x(z(p;x))Y_i(t_1,\dots,t_{i-1};x) = 0,\ 1 \le i \le m,\ \forall\, p \in \bigsqcup,\ x \in B(x^*,\gamma). \tag{24}$$

Here the vector fields with parameters $\{Y_1,\dots,Y_m\}$ are defined in (10) and fulfils the algebraic representation given in (11). Using (11), we rewrite (24) as follows

$$\begin{cases} \partial_p z(p;x) + \partial_x(z(p;x))\{g_1(x),\dots,g_m(x)\}A(p) = 0,\ p \in \bigsqcup,\ x \in B(x^*,\gamma) \\ z(0;x) = x \end{cases} \tag{25}$$

and the proof is complete. $\square$

Lemma 2.5. *Under the conditions assumed in Lemma* 2.4, *define*

$$V^u(t,x;\lambda) = z\big(u(t,\lambda);x\big), \quad t \geq 0,\ \lambda \in \mathbb{R}^n,\ x \in B(x^*,\lambda), \tag{26}$$

where $u \in \mathcal{U}_a$ *is fixed and* $u(p;x)$, $p \in \bigsqcup$, $x \in B(x^*,\lambda)$, *satisfies* (H-J) *equations* (23). *Then the* $(n \times n)$ *matrix* $M^u(t,x;\lambda) \stackrel{def}{=} \partial_\lambda V^u(t,x;\lambda)$, *verifies the following inequality*

$$|M^u(t,x;\lambda)| \leq C_1C_2K_1, \quad t \geq 0,\ \lambda \in \mathbb{R}^n,\ x \in B(x^*,\gamma), \tag{27}$$

where $K_1 > 0$ *is fixed in* (7) (*see definition of* $\mathcal{U}_a$) *and*

$$C_1 \stackrel{def}{=} \max\{|\partial_x(z(p;x))g_i(x)| : p \in \bigsqcup,\ x \in B(x^*,\gamma),\ 1 \leq i \leq m\}, \tag{28}$$

$$C_2 \stackrel{def}{=} \max\{|A(p)| : p \in \bigsqcup\}\ (A(p) \text{ is given in } (22) \text{ and used in } (23)). \tag{29}$$

Proof. By hypothesis, the mapping $z(p;x) = H(p)[x]$ defined in Lemma 2.4 fulfils (H-J) equation (23) and for an arbitrary $u \in \mathcal{U}_a$, we get

$$M^u(t,x;\lambda) = \partial_p z(u(t,\lambda);x)\partial_\lambda u(t,\lambda), \quad t \geq 0,\ \lambda \in \mathbb{R}^n,\ x \in B(x^*,\gamma). \tag{30}$$

Here $u(t,\lambda) \in \bigsqcup \subseteq \mathbb{R}^m$ and $|\partial_\lambda u_i(t,\lambda)| \leq K_1$, $1 \leq i \leq m$, for any $t \geq 0$, $\lambda \in \mathbb{R}^n$ (see definition of $\mathcal{U}_a$ in (7)). On the other hand, using (23) of Lemma 2.4, the following inequality is valid

$$|\partial_p z(u(t;\lambda);x)| \leq C_1C_2, \quad t \geq 0,\ \lambda \in \mathbb{R}^n,\ x \in B(x^*,\gamma), \tag{31}$$

where the constants C_1, C_2 are given in (28), (29). A direct computation applied to (30) leads us to

$$|M^u(t,x;\lambda)| \leq C_1C_2K_1, \quad t \geq 0,\ \lambda \in \mathbb{R}^n,\ x \in B(x^*,\gamma) \tag{32}$$

and the proof is complete. $\square$

Lemma 2.6. *Assume that* $u \in \mathcal{U}_a$ *and* $\{g_1,\dots,g_m\} \subseteq \mathcal{C}^\infty(\mathbb{R}^n;\mathbb{R}^n)$ *satisfies* (14) *and* (15). *Consider* $z(p;x) = H(p)[x]$ *which verifies* (H-J) *equations* (23) *of Lemma* 2.4 *and define*

$$V^u(t,x;\lambda) = z(u(t,\lambda);x), \quad t \in [0,T],\ \lambda \in \mathbb{R}^n,\ x \in B(x^*,\gamma) \subseteq \mathbb{R}^n. \tag{33}$$

Let $\{\alpha_{ij}^u(t,\lambda) : t \in [0,T],\ \lambda \in \mathbb{R}^n,\ 1 \leq i,j \leq m\}$ *be the* $(m \times m)$ *matrix given in* (13) *and define new bounded variation piecewise right-continuous function* $\beta_i^u(t,\lambda) = \sum_{j=1}^m \alpha_{ij}^u(t,\lambda)$, $t \in [0,T]$, $1 \leq i \leq m$. *Then* $\{V^u(t,x;\lambda) : t \in [0,T]\}$ *is* *a bounded variation piecewise right-continuous mapping satisfying the following (H-J) equations with jumps*

$$(34)\quad \begin{cases} d_t V^u(t,x;\lambda) + \partial_x V^u(t,x;\lambda)\Big[\sum\limits_{j=1}^{m} g_i(x) d_t \beta_i^u(t,\lambda)\Big] = 0, \\ V^u(0,x;\lambda) = x,\ t\in[t_k,t_{k+1}),\ x\in \operatorname{Int} B(x^*,\gamma),\ \lambda\in\mathbb{R}^n,\ 0\le k\le N-1, \end{cases}$$

where $[t_k,t_{k+1})\subseteq[0,T]$, $0\le k\le N-1$, *are the continuity intervals of* $u\in\mathcal{U}_a$.

Proof. By hypothesis, the conclusion (23) of Lemma 2.4 is valid. By a direct computation, we get $V^u(0,x;\lambda)=x$ and

$$(35)\qquad d_t V^u(t,x;\lambda) = \sum_{j=1}^{m} \partial_{t_i} z(u(t,\lambda);x) d_t u_i(t,\lambda), \quad t\in[t_k,t_{k+1}).$$

Using (23), rewrite (35) as follows
(36)

$$d_t V^u(t,x;\lambda) = -\partial_x V^u(t,x;\lambda)\{g_1(x),\dots,g_m(x)\} A(u(t,\lambda)) \begin{pmatrix} d_t u_1(t,\lambda) \\ \vdots \\ d_t u_m(t,\lambda) \end{pmatrix},$$

where $A(p) = [b_1, b_2(t_1),\dots,b_m(t_1,\dots,t_{m-1})]$, $b_j\in\mathcal{C}^\infty(\bigsqcup,\mathbb{R}^m)$, $t\in[t_k,t_{k+1})$. Using (13), write

$$(37)\quad \alpha_{ij}^u(t,\lambda) = \int_0^t b_j^i\big(u_1(s-,\lambda),\dots,u_{j-1}(s-,\lambda)\big) d_s u_j(s,\lambda), \quad 1\le i,j\le m,$$

for $t\in[0,T]$, $\lambda\in\mathbb{R}^n$. Rewrite (36) (using (37)) and we get conclusion (34). The proof is complete. □

Remark 2.7. The (H-J) equations with jumps satisfied by the unique solution of Problem (P_2) are strongly connected with Lemmas 2.4 and 2.6. On the other hand, the existence of a solution for Problem (P_2) relies on Lemma 2.5 and it will be analyzed in the next Lemma assuming that $K_1>0$ satisfies

$$(38)\qquad C_1 C_2 K_1 = \rho \in [0, \tfrac{1}{2}].$$

LEMMA 2.8. *Assume that* $u\in\mathcal{U}_a$ *and* $\{g_1,\dots,g_m\}\subseteq\mathcal{C}^\infty(\mathbb{R}^n;\mathbb{R}^n)$ *fulfil* (14), (15) *and* (38). *Then there exists a unique bounded variation piecewise right continuous (of* $t\in[0,T]$*) mapping* $\{\lambda=\psi^u(t,x)\in B(x^*,2\gamma) : t\in[0,T],\ x\in B(x^*,\gamma)\}$ *which is first order continuously differentiable of* $x\in \operatorname{int} B(x^*,\gamma)$, *satisfying integral equations*

$$\begin{cases} x^u(t-,\psi^u(t-,x)) = x\in B(x^*,\gamma), & t\in[0,T], \\ \psi^u(t-,x) = V^u(t-,x;\psi(t-,x)),\ \psi^u(t,x) = V^u(t,x;\psi^u(t-,x)), & t\in[0,T]. \end{cases}$$

Proof. By hypothesis, the conclusion of Lemma 2.5 is valid for $V^u(t,x;\lambda) = z(u(t,\lambda);x)$, $t \geq 0$, $\lambda \in \mathbb{R}^n$, $x \in B(x^*,\gamma)$. Notice that $x^u(t,\lambda) = x$ can be rewritten as

$$\lambda = V^u(t,x;\lambda), \quad t \in [0,T],\ x \in B(x^*,\gamma), \tag{39}$$

where the $(n\times n)$ matrix $\partial_\lambda V^u(t,x;\lambda) = M^u(t,x;\lambda)$ fulfils the conclusion (27), i.e.,

$$|M^u(t,x;\lambda)| \leq C_1C_2K_1, \quad t \geq 0,\ \lambda \in \mathbb{R}^n,\ x \in B(x^*,\gamma). \tag{40}$$

Assuming that $K_1 > 0$ is sufficiently small such that

$$\rho = C_1C_2K_1 \in [0, \tfrac{1}{2}] \quad \text{(see (38))}, \tag{41}$$

then the contraction mapping theorem can be applied for solving integral equations (39). Construct the convergent sequence $\{\lambda_k(t,x) : t \in [0,T],\ x \in B(x^*,\gamma)\}$ such that

$$\begin{cases} \lambda_k(t,x) \in B(x^*,2\gamma), \quad \lambda_0(t,x) = x, \\ \lambda_{k+1}(t,x) = V^u(t,x;\lambda_k(t-,x)), \quad k \geq 0. \end{cases} \tag{42}$$

Notice that the following estimates are valid

$$\begin{cases} |\lambda_{k+1}(t,x) - \lambda_k(t,x)| \leq \rho^k|\lambda_1(t-,x) - \lambda_0(t,x)|, \quad k \geq 0, \\ |\lambda_1(t,x) - \lambda_0(t,x)| \leq \max\{|G(p)[x] - x| : x \in B(x^*,\gamma)\} \leq \frac{\gamma}{2}, \end{cases} \tag{43}$$

(see (9)) which lead us to

$$|\lambda_k(t,x) - x| \leq \frac{1}{1-\rho}\left(\frac{\gamma}{2}\right) \leq \gamma, \quad t \in [0,T],\ x \in B(x^*,\gamma),\ k \geq 0. \tag{44}$$

Combining (43) and (44), we get $\{\lambda_k(t,x)\}_{k\geq 0}$ fulfils (42) and passing $k \to \infty$ in (42), we obtain

$$\psi^u(t,x) = \lim_{k\to\infty} \lambda_k(t,x) \in B(x^*,2\gamma), \quad t \in [0,T],\ x \in B(x^*,\gamma) \tag{45}$$

satisfying integral equations

$$\begin{cases} \psi^u(t,x) = V^u(t,x;\psi(t-,x)), \\ \psi^u(t-,x) = V^u(t-,x;\psi(t-,x)), \quad t \in [0,T],\ x \in B(x^*,\gamma)\}. \end{cases} \tag{46}$$

The proof is complete. □

3. MAIN THEOREMS

With the same notations as in Section 1, we reconsider the problems P_1 and P_2 in more general setting. Consider the local flow $\{G_0(t)[x] : |t| \le T, x \in B(x^*, 3\gamma) \subseteq \mathbb{R}^n\}$ generated by a complete vector field $g_0 \in \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R}^n)$. Assume that

$$I_1 = \begin{cases} \{g_1, \dots, g_m\} \subseteq \mathcal{C}^\infty(\mathbb{R}^n;\mathbb{R}^n) \text{ satisfies (15) and } [g_0, g_i] = 0,\ 1 \le i \le m, \\ \text{for any } x \in B(x^*, 3\gamma). \end{cases}$$

$$I_2 = \begin{cases} u \in \mathcal{U}_a \text{ fulfils (14) where } T > 0 \text{ and } \bigsqcup = \prod_1^m [-a_i, a_i] \subseteq \mathbb{R}^n, \\ \text{are fixed such that } |G_0(t[x]) - x| \le \frac{\gamma}{2(m+1)},\ |G_i(t_i)[x] - x| \le \frac{\gamma}{2(m+1)}. \end{cases}$$

Let $[t_k, t_{k+1}) \subseteq [0, T]$, $0 \le k \le N-1$, be the continuity intervals of $u \in \mathcal{U}_a$.

Problem (R_1). Under the hypothesis (I_1) and (I_2), describe the evolution of the gradient flow with jumps

$$\{y^u(t,\lambda) \stackrel{\text{def}}{=} G_0(t) \circ G(u(t,\lambda))[\lambda] : t \in [0,T],\ \lambda \in B(x^*, 2\gamma) \subseteq \mathbb{R}^n\} \tag{47}$$

(see $G(p) = G_1(t_1) \circ \cdots \circ G_m(t_m)$) as a solution of the following system with jumps

(48)
$$\begin{cases} d_t y^u(t,\lambda) = g_0(y^u(t,\lambda))dt + \sum\limits_{i=1}^m g_i(y^u(t,\lambda))dt\beta_i^u(t,\lambda),\ y^u(t,\lambda) \in B(x^*, 3\gamma), \\ y^u(0,\lambda) = \lambda \in B(x^*, 2\gamma),\ t \in [t_k, t_{k+1}],\ \beta_i^u(t,\lambda) \stackrel{\text{def}}{=} \sum\limits_{j=1}^m \alpha_{ij}^u(t,\lambda),\ 1 \le k \le N-1, \end{cases}$$

where the matrix $\{a_{ij}^u(t,\lambda) : 1 \le i,j \le m\}$ is defined in (13).

Theorem 3.1. *Assume that* $\{g_0, g_1, \dots, g_m\} \subseteq \mathcal{C}^\infty(\mathbb{R}^n;\mathbb{R}^n)$ *and* $u \in \mathcal{U}_a$ *fulfil the hypothesis* (I_1) *and* (I_2). *Then the gradient flow with jumps* $\{y^u(t,\lambda)\}$ *defined in* (47) *verifies* $y^u(t,\lambda) \in B(x^*, 3\gamma)$ *and is a solution of the system* (48).

Proof. By hypothesis, the gradient flow with jumps $\{y^u(t,\lambda)\}$ can be rewritten $y^u(t,\lambda) = G(u(t,\lambda)) \circ G_0(t)[\lambda]$ and using (I_2), we get $y^u(t,\lambda) \in B(x^*, 3\gamma)$, $t \in [0,T]$, $\lambda \in B(x^*, 2\gamma)$. On the other hand, a direct computation applied to $\{y^u(t,\lambda)\}$ lead us to the following equations

(49)
$$d_t y^u(t,\lambda) = g_0\big(y^u(t,\lambda)\big)dt + \sum_{i=1}^m \partial_{t_i} G(u(t,\lambda))[G_0(t)(\lambda)]d_t u_i(t,\lambda),\ t \in [t_k, t_{k+1}),$$

where $y(p;\mu) \stackrel{\text{def}}{=} G(p)[\mu]$ satisfies an integrable gradient system (see (10)),

$$\partial_{t_1} y = g_1(y),\ \partial_{t_2} y = Y_2(t;y), \dots, \partial_{t_m} = Y_m(t_1, \dots, t_{m-1}; y), \tag{50}$$

fulfilling the algebraic representation given in (11) and (12). As a consequence, we may and do rewrite the second term in the right-hand side of (49) as a follows

(51)
$$\sum_{i=1}^{m} \partial_{t_i} G(u(t,\lambda))[G_0(t)(\lambda)] d_t u_i(t,\lambda) = \sum_{i=1}^{m} g_i(y^u(t,\lambda)) d_t \beta_i^u(t,\lambda),\ t \in [t_k, t_{k+1}),$$

$\lambda \in B(x^*, 2\gamma)$, $0 \le k \le N-1$ where $\beta_i^u(t,\lambda) \overset{\text{def}}{=} \sum_{j=1}^{m} \alpha_{ij}^u(t,\lambda)$ and the matrix $\{\alpha_{ij}^u(t,\lambda) : 1 \le i,j \le m\}$ is defined in (13). The proof is complete. □

Remark 3.2. The evolution of the gradient flow defined in (47) satisfies the same system with jumps given in (48) if the commutative condition $[g_0, g_i](x) = 0$, $1 \le i \le m$, $x \in B(x^*, 3\gamma) \subseteq \mathbb{R}^n$, assumed in (I_1) is replaced by

$$[g_0, g_i](x) = \sum_{k=1}^{m} \gamma_k^i g_k(x), \quad x \in B(x^*, 3\gamma),\ 1 \le i \le m, \tag{52}$$

where $\gamma_k^i \in \mathbb{R}$ are same constants.

The only change which appears is reflected in the algebraic representation corresponding to the gradient integrable system associated with smooth mapping

$$y(t,p)[\lambda] = G_0(t) \circ G(p)[\lambda],\ |t| \le T,\ p = (t_1, \ldots, t_m) \in \bigsqcup,\ \lambda \in B(x^*, 2\gamma). \tag{53}$$

We get

$$\begin{gathered} \partial_t y = g_0(y),\ \partial_{t_1} y = Y_1(t; y),\ \partial_{t_2} y = Y_2(t, t_1; y), \ldots, \\ \partial_{t_m} y = Y_m(t, t_1, \ldots, t_{m-1}; y). \end{gathered} \tag{54}$$

(55)
$$\{g_0, Y_1(t), Y_2(t,t_1), \ldots, Y_m(t, t_1, \ldots, t_{m-1})\}(y) = \{g_0, g_1, \ldots, g_m\}(y) V(t, \mu).$$

This time, the $(m+1) \times (m+1)$ analytic matrix $V(t,p)$ has the following structure

$$\begin{cases} V(t,p) = [V_0, V_1(t), V_2(t,t_1), \ldots, V_m(t, t_1, \ldots, t_{m-1})] & V_j \in \mathbb{R}^{m+1}, \\ V_0^1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ and } V_i(t, t_1, \ldots, t_{i-1}) = \begin{pmatrix} 0 \\ b_i(t, t_1, \ldots, t_{i-1}) \end{pmatrix} & b_i \in \mathbb{R}^m, \end{cases} \tag{56}$$

$1 \leq i \leq m$. A standard computation applied to $y^u(t,\lambda) = y(t,u(t,\lambda))[\lambda] = G_0(t) \circ G(u(t,\lambda))[\lambda]$, $t \in [0,T]$, leads to

$$(57)\quad \begin{aligned} d_t y^u(t,\lambda) &= \partial_t y(t,u(t,\lambda))[\lambda]dt + \sum_{i=1}^{m} \big(\partial_{t_i} y(t,u(t,\lambda))[\lambda]\big) d_t u_i(t,\lambda) \\ &= g_0(y^u(t,\lambda))dt + \sum_{i=1}^{m} g_i(y^u(t,\lambda)) d_t \beta_i^u(t,\lambda), \quad t \in [t_k, t_{k+1}), \end{aligned}$$

$0 \leq k \leq N-1$, where $\beta_i^u(t,\lambda) = \sum_{j=1}^{m} \alpha_{ij}^u(t,\lambda)$ and

$$\alpha_{ij}^u(t,\lambda) \stackrel{\text{def}}{=} \int_0^t b_j^i(s, u_1(s-,\lambda), \ldots, u_{i-1}(s-,\lambda)) d_s u_j(s,\lambda), \quad 1 \leq i,j \leq m.$$

Here $\{b_1(t), b_2(t,t_1), \ldots, b_m(t,t_1,\ldots,t_{m-1})\} \subseteq \mathbb{R}^m$ are given in (56).

Problem (R_2). Under the hypothesis (I_1) and (I_2) and for $K_1 > 0$ sufficiently small (see (38) of Lemma 2.8) prove that the integral equation with respect to $\lambda \in B(x^*, 2\gamma)$ (see (47))

$$(58)\qquad y^u(t,\lambda) \stackrel{\text{def}}{=} G_0(t) \circ G(u(t,\lambda))[\lambda] = x \in B(x^*,\gamma), \quad t \in [0,T]$$

has a unique bounded variation and piecewise right continuous solution $\{\lambda = \psi^u(t,x) \in B(x^*, 2\gamma) : t \in [0,T]\}$ of (58) which is first order continuously differentiable of $x \in \operatorname{int} B(x^*,\gamma)$.

In addition, the following equations are valid

$$(59)\quad \begin{cases} V^u(t,x;\lambda) \stackrel{\text{def}}{=} H(u(t,\lambda))[G_0(-t;x)],\ H(p) = [G(p)]^{-1},\ t \in [0,T],\ p \in \bigsqcup, \\ \psi^u(t-,x) = V^u(t-,x;\psi(t-,x)),\ \psi^u(t,x) = V^u(t,x,\psi(t-,x)), \\ y^u(t-,\psi^u(t-,x)) = x \in B(x^*,\gamma),\ \text{for any } t \in [0,T]. \end{cases}$$

Define the following two constants

$$(60)\qquad \begin{cases} C_1 \stackrel{\text{def}}{=} \max\big\{|\partial_y(z(p;y))g_i(y)| : p \in \bigsqcup,\ y \in B(x^*,2\gamma)\big\}, \\ C_2 \stackrel{\text{def}}{=} \max\big\{|A(p)| : p \in \bigsqcup\big\}, \end{cases}$$

where $z(p;y) = H(p)[y]$ and the analytic $(m \times m)$ matrix $A(p)$ is given in (2). Assume that $K_1 > 0$ used in the definition of admissible set $\mathcal{U}_a$ (see (7)) satisfies

$$(61)\qquad C_1 C_2 K_1 = \rho \in [0, \tfrac{1}{2}].$$

THEOREM 3.3. *Assume that* $\{g_0, g_1, \ldots, g_m\} \subseteq \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^n)$ *and* $u \in \mathcal{U}_a$ *fulfil the hypothesis* (I_1), (I_2) *and* (61). *Then there exists a unique bounded variation and piecewise right-continuous solution* $\{\lambda = \psi^u(t,x) \in B(x^*) : t \in [0,T]\}$ *of* (58) *which is first order continuously differentiable of* $x \in$

$\operatorname{int} B(x^*,\gamma)$ *such that the integral equations* (59) *are satisfied. In addition* $V(t,x) \overset{def}{=} V^u(t,x;\lambda)$ *satisfy the following system of* (*H-J*) *equations with jumps*

$$d_tV(t,x)+\partial_xV(t,x)\left[g_0(x)dt+\sum_{i=1}^{m}g_i(x)d_t\beta_i^u(t,\lambda)\right]=0,\ x\in \operatorname{int}B(x^*,\gamma), \tag{62}$$

$t\in[t_k,t_{k+1}]$, $0\le k\le N-1$, *where* $\beta_i^u(t,\lambda)$, $1\le i\le m$, *are defined in* (48) (*see Problem* (R_1)).

Proof. Define $\widehat{V}^u(t,y;\lambda)=H(u(t,\lambda))[y]$, where $H(p)=[G(p)]^{-1}$, $t\in[0,T]$, $p\in\bigsqcup$ and $y\in B(x^*,2\gamma)$. The integral equation (58) can be replaced by the following

$$\lambda=\widehat{V}^u(t,y_0(t,x);\lambda)\overset{\text{not}}{=}V^u(t,x;\lambda), \tag{63}$$

where $y_0(t,x)=G_0(-t)(x)\in B(x^*,\gamma_1)$, for any $t\in[0,T]$, $x\in B(x^*,\gamma)$ and $\gamma_1=\gamma(1+\frac{1}{2(m+1)})$ (see (I_2)). It allows us to get the unique solution $\lambda=\psi^u(t,x)$ as a composition

$$\psi^u(t,x)=\widehat{\psi}^u(t,y_0(t,x)),\quad t\in[0,T],\ x\in B(x^*,\gamma), \tag{64}$$

where $\lambda=\widehat{\psi}^u(t,y)$, $t\in[0,T]$, $y\in B(x^*,\gamma_1)$, is the unique solution of the following integral equations

$$\lambda=\widehat{V}^u(t,y;\lambda),\quad \lambda\in B(x^*,2\gamma),\ y\in B(x^*,\gamma_1),\ t\in[0,T]. \tag{65}$$

By hypothesis, the mapping $\{\widehat{V}^u(t,y;\lambda),\ t\in[0,T],\ y\in B(x^*,\gamma_1)\}$ fulfils the hypothesis (14), (15) and (38) of Lemmas 2.6 and 2.8 for any $\lambda\in\mathbb{R}^n$. We get the corresponding (H-J) equations (see (34) of Lemma 2.6).

$$\begin{cases} d_t\widehat{V}^u(t,y;\lambda)+\partial_y\widehat{V}(t,y;\lambda)\left[\sum_{i=1}^{m}g_i(y)d_t\beta_i^u(t,\lambda)\right]=0,\\ \widehat{V}^u(0,y;\lambda)=y,\ t\in[t_k,t_{k+1}],\ y\in\operatorname{int}B(x^*,\gamma_1),\ \lambda\in\mathbb{R}^n,\ 0\le k\le N-1.\end{cases} \tag{66}$$

In addition, there exists a unique bounded variation piecewise right-continuous (of $t\in[0,T]$) mapping $\{\widehat{\lambda}=\widehat{\psi}^u(t,y)\in B(x^*,2\gamma):\ t\in[0,T],\ y\in B(x^*,\gamma_1)\}$ (see Lemma 2.8)

$$\begin{cases} \widehat{\psi}^u(t-,y)=\widehat{V}^u(t-,y;\widehat{\psi}^u(t-,y)) & t\in[0,T],\ y\in B(x^*,\gamma_1),\\ \widehat{\psi}^u(t,y)=\widehat{V}^u(t,y;\widehat{\psi}^u(t-,y)) & t\in[0,T],\ y\in B(x^*,\gamma_1),\end{cases} \tag{67}$$

notice that $\widetilde{\lambda}=\widetilde{\psi}^u(t,y)$ is first order continuously differentiable of $y\in\operatorname{int}B(x^*,\gamma_1)$ and, using (66) and (67), we get the corresponding equations satisfied by $\lambda=\psi^u(t,x)\overset{\text{def}}{=}\widehat{\psi}^u(t,G_0(-t)(x))$, $t\in[0,T]$, as follows

$$\psi^u(t-,y)=V^u(t-,x;\widehat{\psi}^u(t-,y)),\psi^u(t,x)=V^u(t,x;\widehat{\psi}^u(t-,y)),\ t\in[0,T], \tag{68}$$

$\forall\, x \in B(x^*, \gamma_1)$ where $\widehat{V}(t, x; \lambda) = \widehat{V}^u(t, G_0(-t)(x); \lambda) = H(u(t, \lambda))[G_0(-t)(x)]$. The equations (68) stands for integral equation (59) and the first conclusion is proved. For the (H-J) equations (62), we notice that

(69)
$$d_t V^u(t, x; \lambda) = d_t V^u(t, G_0(-t)(x); \lambda) - \partial_y \widehat{V}^u(t, G_0(-t)(x); \lambda) g_0(G_0(-t)(x)) dt,$$

for any $t \in [t_k, t_{k+1})$, $0 \le k \le N-1$. In addition, using (I_1), we get

$$(70) \quad \begin{cases} \partial_y \widehat{V}^u(t, G_0(-t)(x); \lambda) g_i(G_0(-t)(x)) = \partial_x V^u(t, x; \lambda)[\partial_x(G_0(-t)(x))]^{-1} \cdot \\ \cdot g_i(G_0(-t)) = \partial_x V^u(t, x; \lambda) g_i(x), \ t \in [0, T], \ 0 \le i \le m. \end{cases}$$

Rewrite (69) (using (70) and (66)) we get (H-J) equations (62). The proof is complete. □

Theorem 3.4. *Under the hypothesis of Theorem* 3.3 *and assume that* $u \in \mathcal{U}_a$ *is continuously differentiable on each continuity interval* $[t_k, t_{k+1}) \subseteq [0, T]$, $0 \le k \le N-1$. *Then the unique solution* $\{\lambda = \psi^u(t, x) \in B(x^*, 2\gamma) : [t_k, t_{k+1}), x \in \operatorname{int} B(x^*, \gamma)\}$ *of integral equations* (58) (*see* (59) *also*) *satisfies the following* (*H-J*) *equations*

$$(71) \qquad \partial_t \psi^u(t, x) + \partial_x \psi^u(t, x)\left[g_0(x) + \sum_{i=1}^m g_i(x) \partial_t \beta_i^u(t, \psi^u(t, x))\right] = 0,$$

$t \in [t_k, t_{k+1})$, $x \in \operatorname{int} B(x^*, \gamma)$, $0 \le k \le N-1$. *Here* $\beta_i^u(t, \lambda)$, $1 \le i \le m$, *are defined in Problem* (R_1) (*see* (48)).

Proof. By hypothesis, the conclusion of Theorem 3.3 are valid and, in particular, the (H-J) equations (62) can be rewritten

(72)
$$\partial_t V(t, x) + \partial_x V(t, x)\left[g_0(x) + \sum_{i=1}^m g_i(x) \partial_t \beta_i^u(t, \psi^u(t, x))\right] = 0, \ t \in [t_k, t_{k+1}),$$

$x \in \operatorname{int} B(x^*, \gamma)\}$, $0 \le k \le N-1$, where $V(t, x) \overset{\text{not}}{=} V^u(t, x; \lambda)$, $\lambda \in B(x^*, 2\gamma)$. On the other hand, using integral equations (59), we get

(73)
$$\psi^u(t, x) = V^u(t, x; \psi^u(t, x)), \quad t \in [t_k, t_{k+1}), \ x \in B(x^*, \gamma)\}, \ 0 \le k \le N-1.$$

By a direct derivation, from (73) we obtain

$$(74) \qquad \begin{cases} \partial_t \psi^u(t, x) = [I_n - \partial_\lambda V^u(t, x; \psi^u(t, x))]^{-1} \partial_t V^u(t, x; \psi^u(t, x)), \\ \partial_x \psi^u(t, x) = [I_n - \partial_\lambda V^u(t, x; \psi^u(t, x))]^{-1} \partial_x V^u(t, x; \psi^u(t, x)), \end{cases}$$

for any $t \in [t_k, t_{k+1})$, $x \in \operatorname{int} B(x^*, \gamma)$, $0 \le k \le N-1$. Here the nonsingular matrix used in the right-hand side of (74) relies on (27) in Lemma 2.5 and

(61) of Theorem 3.3. The equations (72) are valid for any $\lambda \in B(x^*, 2\gamma)$ and, in particular for $\lambda = \psi^u(t,x) \in B(x^*, 2\gamma)$, we get

(75)
$$\partial_t V(t,x;\psi^u(t,x)) + \partial_x V^u(t,x,\psi^u(t,x))\left[g_0(x) + \sum_{i=1}^m g_i(x)\partial_t \beta_i^u(t,\psi^u(t,x))\right] = 0,$$

for any $t \in [t_k, t_{k+1})$, $x \in \operatorname{int} B(x^*, \gamma)$, $0 \le k \le N-1$.

Using (75) and multiplying the second equation in (74) by $[g_0(x) + \sum_{i=1}^m g_i(x)\partial_t \beta_i^u(t,\psi^u(t,x))]$, we obtain the conclusion (71). The proof is complete. $\square$

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