ON VEHICLE FOLLOWING CONTROL SYSTEMS WITH DELAYS

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In this paper, we consider the problem of vehicle following control with delay. To solve the problem of traffic congestion, one of the solutions to be considered consists in organizing the traffic into *platoons*, that is groups of vehicles including a leader and a number of followers "tightly" spaced, all moving in a longitudinal direction. Excepting the stability of individual cars, the problem of avoidance of slinky type effects will be explicitly discussed. Sufficient conditions on the set of control parameters to avoid such a phenomenon will be explicitly derived in a frequency-domain setting.

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1. INTRODUCTION

Traffic congestion (irregular flow of traffic) became an important problem in the last decade mainly to the exponential increasing of the transportation around medium- and large-size cities. One of the ideas to help solving this problem was the use of automatic control to replace human drivers and their low-predictable reaction with respect to traffic problems. As an example, human drivers have reaction time between 0.25–1.25 sec of around 30 m or more at 60 kms/hour (see, for instance, [19] for a complete description of human drivers reactions, and further comments on existing traffic flow models).

A way to solve this problem is to organize the traffic into *platoons*, consisting in groups of vehicles including a leader and a number of followers in a longitudinal direction. In this case, the controller of each vehicle of a platoon would use the sensor information to try to reach the speed and acceleration of the preceding vehicle. Another problem to be considered is the so-called *slinky-type effect* (see, e.g., [11], [20]). This is a phenomenon of amplification of the spacing errors between subsequent vehicles as vehicle index increases.

The aim of this paper is to propose an explicit control law guaranteeing simultaneously individual stability and the avoidance of the slinky-type effect phenomenon. We use a frequency-domain method to give necessary and sufficient conditions for the individual stability analysis by computing the explicit delay bounds guaranteeing asymptotic stability. Next, we shall explicitly compute bounds on the controller's gains ensuring the avoidance of the slinky effects.

The remaining paper is organized as follows: In Section 2, the problem formulation is presented. In Section 3, we state and prove our main results. In Section 4, two illustrative examples are presented. Finally, some concluding remarks end the paper.

2. PROBLEM FORMULATION

The closed-loop dynamics for each vehicle of a platoon is given by the following third order delay equation

$$(1)\qquad \frac{\mathrm{d}^3}{\mathrm{d}t^3}\,\delta_i(t) = -\alpha\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\delta_i(t) - k_s\delta_i(t-\tau_i) - (k_v+\lambda k_s)\frac{\mathrm{d}}{\mathrm{d}t}\,\delta_i(t-\tau_i) - \\ -\lambda k_v\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\delta_i(t-\tau_i) + k_s\delta_{i-1}(t-\tau_{i-1}) + k_v\frac{\mathrm{d}}{\mathrm{d}t}\,\delta_{i-1}(t-\tau_{i-1}),$$

where k_s and k_v are the controller gains and δ_i is the spacing error between the ith and $(i-1)$th vehicles. The spacing error is computed as

$$\delta_i(t) = x_{i-1}(t) - x_i(t) - (\lambda v_i + H_i),$$

where λ is a prescribed headway constant, H_i is the minimum separation distance allowable between the ith and $(i-1)$th vehicles and v_i is the velocity of the ith vehicle (see [8] and [9] for more details).

2.1. Individual stability

A basic control requirement for the overall system is the asymptotic stability of the ith vehicle outside the influence of the preceding one (i.e., the spacing errors verify $\delta_{i-1} = \dot{\delta}_{i-1} = 0$). In this case, the system is described by

$$(2)\qquad \frac{\mathrm{d}^3}{\mathrm{d}t^3}\,\delta_i(t) = -\alpha\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\delta_i(t) - k_s\delta_i(t-\tau_i) - \\ -(k_v+\lambda k_s)\frac{\mathrm{d}}{\mathrm{d}t}\,\delta_i(t-\tau_i) - \lambda k_v\frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta_i(t-\tau_i).$$

The characteristic equation is given by a third-order transcendental equation of the form

$$\begin{aligned}(3)\qquad \Gamma_i(s,\tau_i) &:= s^3+\alpha s^2+\big[\lambda k_v s^2+(k_v+\lambda k_s)s+k_s\big]\mathrm{e}^{-\tau_i s}\\ &= Q(s)+P(s)\mathrm{e}^{-s\tau}=0.\end{aligned}$$

Assumption 1. (a) $P(0)\neq 0$.
(b) *The polynomials* $P(s)$ *and* $Q(s)$ *do not have common zeros.*

If the first assumption is violated, then 0 is a zero of $\Gamma_i(s,\tau_i)$ for any $\tau_i\in\mathbb{R}_+$. Therefore, the system is never asymptotically stable. If Assumption 1(b) is not satisfied, $P(s)$ and $Q(s)$ have a common factor $c(s)\neq$ constant. Simplifying by $c(s)$ we get a system described by (3) which satisfies Assumption 1(b). It is noteworthy that $Q(s)$ has only one nonzero root ($s=-\alpha$) which is situated in the left half of the complex plane. The individual vehicle stability is guaranteed if and only if Γ has all its roots in the left half complex plane. This depends on the delay magnitude τ_i. Then the problem of stability can be formulated as a research of parameters α,λ,k_s and k_v such that this condition is ensured.

2.2. Avoiding slinky effect

The second part of the multi-objective problem previously defined consist in controlling the slinky effect. The goal is to find sufficient conditions to guarantee that we avoid such a phenomenon. Let us define

$$(4)\qquad G(s)=\delta_i(s)/\delta_{i-1}(s)=\frac{(k_s+sk_v)\mathrm{e}^{-\tau_{i-1}s}}{(k_s+(k_v+\lambda k_s)s+\lambda k_v s^2)\mathrm{e}^{-\tau_i s}+\alpha s^2+s^3}.$$

We have no *slinky-type effect* if

$$(5)\qquad |G(j\omega)|=\left|\frac{\delta_i(j\omega)}{\delta_{i-1}(j\omega)}\right|<1$$

for any $\omega>0$ (see [11], [20], [21]). Then the problem turns out in finding the set of parameters (k_s,k_v) and the delays τ_i such that the stability of the system (2) is guaranteed and the condition (5) is satisfied (to avoid slinky-effect).

3. MAIN RESULTS

3.1. Delay stability margin

Before proceeding further, we consider the case without delay. Analyzing the asymptotic stability of the system (1) free of delay turns out to check when the polynomial $\Gamma_i(s,0)$, with $\tau_i = 0$, is Hurwitz. Since α, λ, k_s and k_v are strictly positive, the third-order polynomial

$$s^3 + (\alpha + \lambda k_v)s^2 + (k_v + \lambda k_s)s + k_s = 0 \tag{6}$$

is Hurwitz if and only if

$$(\alpha + \lambda k_v)(k_v + \lambda k_s) > k_s, \tag{7}$$

which is equivalent to

$$\lambda k_v^2 + (\alpha + \lambda^2 k_s)k_v + (\alpha\lambda - 1)k_s > 0. \tag{8}$$

Since α, λ, k_s and k_v are strictly positive the solution of the inequality (8) is the set

$$\left\{(k_v, k_s)\in\mathbb{R}_+^2 \mid k_v > \max\left\{0, \frac{1-\alpha\lambda}{\lambda^2}\right\}\right\} \cup \left(0, \frac{1-\alpha\lambda}{\lambda^2}\right) \times \left(0, \frac{\lambda k_v^2 + \alpha k_v}{1 - \alpha\lambda - \lambda^2 k_v}\right).$$

Denote now by Ω the set of crossing frequencies, that is the set of reals $\omega > 0$, such that $\pm j\omega$ is a solution of the characteristic equation (3). We have the following:

PROPOSITION 1. *Consider the characteristic equation* (3) *associated to the system* (2). *Then:*

(a) *the crossing frequency set* Ω *is not empty, and*

(b) *the system is asymptotically stable for all delays* $\tau_i \in (0, \tau^\star)$ *where* $\tau^\star$ *is defined by*

$$\tau^\star = \frac{1}{\omega} \arccos\left(\frac{\alpha(k_s - \lambda k_v\omega^2)\omega^2 + (k_v + \lambda k_s)\omega^4}{(k_s - \lambda k_v\omega^2)^2 + (k_v + \lambda k_s)^2\omega^2}\right), \tag{9}$$

where ω *is the unique element of* Ω.

The condition (a) above simply says that the corresponding system cannot be delay-independent asymptotically stable, and the condition (b) above gives an explicit expression of the delay margin $\tau^\star$. In order to have a self-contained paper, a proof of the Proposition above is included in the Appendix. For a different proof, see, for instance, [18].

3.2. Stability analysis in controller parameter space (k_v, k_s)

In the sequel, we study the behavior of the system for a fixed delay value τ. More precisely, for a given τ we search the crossing frequencies ω and the corresponding *crossing points* in the parameter space (k_v, k_s) defined by the control law such that $Q(j\omega, k_v, k_s, \tau) + P(j\omega, k_v, k_s, \tau)\mathrm{e}^{-j\omega\tau} = 0$.

According to the continuity of zeros with respect to the delay parameters, the number of roots in the right-half plane (RHP) can change only when some zeros appear and cross the imaginary axis. Thus, it is natural to consider the *frequency crossing set* Ω consisting of all real positive ω such that there exist at least a pair (k_v, k_s) for which

$$H(j\omega, k_v, k_s, \tau) := Q(j\omega) + P(j\omega)\mathrm{e}^{-j\omega\tau} = 0. \tag{10}$$

Remark 1. Using the conjugate of a complex number we get

$$H(j\omega, k_v, k_s, \tau) = 0 \Leftrightarrow H(-j\omega, k_v, k_s, \tau) = 0.$$

Therefore, it is natural to consider only positive frequencies, that is $\Omega \subset (0, \infty)$.

Considering that the set Ω and the parameters α, λ are known we can easily derive all the crossing points in the parameter space (k_v, k_s).

PROPOSITION 2. *For a given $\tau > 0$ and $\omega \in \Omega$ the corresponding crossing point (k_v, k_s) is given by*

$$k_v = \frac{\omega^2(1 - \alpha\lambda)\cos\omega\tau + \omega(\alpha + \lambda\omega^2)\sin\omega\tau}{1 + \lambda^2\omega^2}, \tag{11}$$

$$k_s = \frac{\omega^2(\lambda\omega^2 + \alpha)\cos\omega\tau + \omega^3(\alpha\lambda - 1)\sin\omega\tau}{1 + \lambda^2\omega^2}. \tag{12}$$

Proof. Using the decomposition of the equation (10) into real and imaginary part, straightforward computation lead us to

$$k_v + \lambda k_s = \omega(\omega\cos\omega\tau + \alpha\sin\omega\tau), \tag{13}$$

$$k_s - \lambda k_v\omega^2 = \omega^2(\alpha\cos\omega\tau - \omega\sin\omega\tau) \tag{14}$$

and further we can derive the result stated above. $\square$

As an example, let us consider the case where $\alpha = 5$, $\lambda = 1$ and $\tau = 0.5$. Then for each $\omega \in \Omega$ the corresponding crossing points (k_v, k_s) are represented in the following figure.

Remark 2. For all $\omega \in \Omega$ we have $P(j\omega) \neq 0$. Indeed, it is easy to see that if $\omega \in \Omega$, then there exists at least one pair (k_v, k_s) such that $H(j\omega, k, T, \tau) = 0$. Therefore, assuming that $P(j\omega) = 0$ we get also $Q(j\omega) = 0$ which contradicts Assumption 1(b).

Since we are interested in finding the crossing points (k_v, k_s) such that k_v and k_s are *finite* the frequency crossing set Ω is characterized by the following:

PROPOSITION 3. *The frequency crossing set Ω consists of a finite number of intervals of finite length.*

Proof. It is obvious from the equations (11) and (12) that the controller parameters k_v and k_s approach infinity when $\omega \to \infty$. Thus, in order to have finite values for k_v and k_s we have to impose an upper limit for the variation of ω. On the other hand, considering $\Omega \subset (0, M]$, it is clear that the inequalities $k_v > 0$ and $k_s > 0$ are simultaneously satisfied for ω into a finite number of intervals included in $(0, M]$. $\square$

Let us suppose that $\Omega = \bigcup_{\ell=1}^{N} \Omega_\ell$. Then (11) and (12) define a continuous curve. Using the notations introduced in the previous paragraph and the technique developed in [6] and [15], we can easily derive the crossing direction corresponding to this curve.

More exactly, let us denote $\mathcal{T}_\ell$ the curve defined above and consider the following decompositions into real and imaginary parts

$$\begin{aligned} R_0 + jI_0 &= \frac{j}{s}\frac{\partial H(s,k_v,k_s,\tau)}{\partial s}\bigg|_{s=j\omega}, \\ R_1 + jI_1 &= -\frac{1}{s}\frac{\partial H(s,k_v,k_s,\tau)}{\partial k_v}\bigg|_{s=j\omega}, \\ R_2 + jI_2 &= -\frac{1}{s}\frac{\partial H(s,k_v,k_s,\tau)}{\partial k_s}\bigg|_{s=j\omega}. \end{aligned}$$

Then, since $H(s, k_v, k_s, \tau)$ is an analytic function of s, k_v and k_s, the implicit function theorem indicates that the tangent of $\mathcal{T}_\ell$ can be expressed as

$$\begin{pmatrix} \dfrac{\mathrm{d}k_v}{\mathrm{d}\omega} \\ \dfrac{\mathrm{d}k_s}{\mathrm{d}\omega} \end{pmatrix} = \frac{1}{R_1 I_2 - R_2 I_1} \begin{pmatrix} R_1 I_0 - R_0 I_1 \\ R_0 I_2 - R_2 I_0 \end{pmatrix}, \tag{15}$$

provided that

$$R_1 I_2 - R_2 I_1 \neq 0. \tag{16}$$

It follows that $\mathcal{T}_\ell$ is smooth everywhere except possibly at the points where either (16) is not satisfied, or when

$$\frac{\mathrm{d}k_v}{\mathrm{d}\omega} = \frac{\mathrm{d}k_s}{\mathrm{d}\omega} = 0. \tag{17}$$

From the above discussions, we can conclude with the following:

Proposition 4. *The curve $\mathcal{T}_\ell$ is smooth everywhere except possibly at the point corresponding to $s = j\omega$ such that $s = j\omega$ is a multiple solution of* (10).

Proof. If (17) is satisfied then straightforward computations show us that $R_0 = I_0 = 0$. In other words, $s = j\omega$ is a multiple solution of (10).

On the other hand,

$$R_1 I_2 - R_2 I_1 = -\omega(1 + \lambda^2\omega^2) < 0, \quad \forall\omega > 0. \quad \square$$

The next paragraph focuses on the characterization of the crossing direction corresponding to each of the curves defined by (11) and (12) (see, for instance, [13] or [14] for similar results for different problems):

We will call the direction of the curve that corresponds to increasing ω the *positive direction*. We will also call the region on the left hand side as we head in the positive direction of the curve *the region on the left*.

PROPOSITION 5. *Assume* $\omega \in \Omega_\ell$, k_v, k_s *satisfy* (11) *and* (12) *respectively, and* ω *is a simple solution of* (10) *and* $H(j\omega', k_v, k_s, \tau) \neq 0$, $\forall\omega' > 0$, $\omega' \neq \omega$ *(i.e.,* (k_v, k_s) *is not an intersection point of two curves or different sections of a single curve).*

Then a pair of solutions of (10) *cross the imaginary axis to the right, through* $s = \pm j\omega$ *if* $R_1 I_2 - R_2 I_1 > 0$. *The crossing is to the left if the inequality is reversed.*

Remark 3. In the proof of Proposition 4 we have shown that $R_1 I_2 - R_2 I_1$ is always negative. Thus, a system described by (10) may have more than one stability region in controller parameter space (k_v, k_s) if one of the following two items are satisfied:

- it has one or more crossing curves with some turning points (the direction of $\mathcal{T}_\ell$ in controller parameter space changes);
- it has at least two different crossing curves with opposite direction in (k_v, k_s)-space.

3.3. **Avoiding slinky effects**

Now, we treat the second part of the multi-objective problem under consideration. This correspond to the characterization of the conditions guaranteeing that we avoid slinky-effects. The condition (5) can be rewritten as:

$$A(\omega, \tau_i)(\omega) = \omega^2 B(\omega, \tau_i) \geq 0, \tag{18}$$

with

$$\begin{gathered} B(\omega, \tau_i)(\omega) = \omega^4 - 2\lambda k_v \sin(\omega\tau_i)\omega^3 + \\ +(\lambda^2 k_v^2 + \alpha^2 + 2(\alpha\lambda k_v - k_v - \lambda k_s)\cos(\omega\tau_i))\omega^2 + \\ +2(k_s - \alpha(k_v + \lambda k_s))\sin(\omega\tau_i)\omega + \lambda^2 k_s^2 - 2\alpha k_s \cos(\omega\tau_i), \end{gathered}$$

which should be satisfied for all $\omega \in \mathbb{R}$.

The objective is to define conditions on the parameters of the controller, in order to satisfy this constraint.

Consider first the case $\tau_i = 0$. Then, we have

$$B(\omega,0) = \omega^4 + \left[(\lambda k_v + \alpha)^2 - 2(k_v + \lambda k_s)\right]\omega^2 + \lambda^2 k_s^2 - 2\alpha k_s. \tag{19}$$

A necessary condition for the positivity of $B(\omega,0)$ is

$$\lambda^2 k_s^2 - 2\alpha k_s > 0, \tag{20}$$

which implies that

$$k_s \in \left(\frac{2\alpha}{\lambda^2}, +\infty\right). \tag{21}$$

Under this condition, the positivity of $B(\omega,0)$ is guaranteed if

$$\left[(\lambda k_v + \alpha)^2 - 2(k_v + \lambda k_s)\right]^2 \le 4(\lambda^2 k_s^2 - 2\alpha k_s) \tag{22}$$

which leads to

$$-2k_s\lambda\sqrt{1 - \frac{2\alpha}{\lambda^2 k_s}} \le (\lambda k_v + \alpha)^2 - 2(k_v + \lambda k_s) \le 2k_s\lambda\sqrt{1 - \frac{2\alpha}{\lambda^2 k_s}}. \tag{23}$$

In order to complete this analysis, we want to characterize the set of parameters k_v guaranteeing the previous inequality under the constraint (21). Considering the second inequality in (23), which is equivalent to

$$\lambda^2 k_v^2 + 2(\lambda\alpha - 1)k_v + \alpha^2 - 2\lambda k_s\left(1 + \sqrt{1 - \frac{2\alpha}{\lambda^2 k_s}}\right) \le 0$$

we can remark that if

$$k_s > \max\left\{\frac{2\alpha}{\lambda^2}, \frac{2\alpha\lambda - 1}{2\lambda^3}\right\} \tag{24}$$

then there exists at least one positive value k_v, such that the second inequality in (23) is satisfied. Moreover, k_v should satisfy

$$\max\left\{0, \frac{1 - \alpha\lambda - \sqrt{\Delta_1}}{\lambda^2}\right\} \le k_v \le \frac{1 - \alpha\lambda + \sqrt{\Delta_1}}{\lambda^2}, \tag{25}$$

where

$$\Delta_1 = 1 - 2\alpha\lambda + 2\lambda^3 k_s\left(1 + \sqrt{1 - \frac{2\alpha}{\lambda^2 k_s}}\right).$$

The left inequality in (23) can be rewritten as

$$\lambda^2 k_v^2 + 2(\lambda\alpha - 1)k_v + \alpha^2 - 2\lambda k_s\left(1 - \sqrt{1 - \frac{2\alpha}{\lambda^2 k_s}}\right) \ge 0.$$

This leads to the following condition on k_v

$$k_v \in \left(-\infty, \frac{1-\alpha\lambda-\sqrt{\Delta_2}}{\lambda^2}\right] \cup \left[\frac{1-\alpha\lambda+\sqrt{\Delta_2}}{\lambda^2}, +\infty\right), \tag{26}$$

where

$$\Delta_2 = 1 - 2\alpha\lambda + 2\lambda^3 k_s \left(1 - \sqrt{1 - \frac{2\alpha}{\lambda^2 k_s}}\right)$$

is assumed to be positive. If $\Delta_2 < 0$, then the first inequality in (23) will be satisfied for all positive k_v. Finally, using the conditions (25) and (26) function of the sign of Δ_2, it follows that k_v must be chosen in the intersection of the intervals defined by (25) and (26).

Now we analyze the sign of $B(\omega, \tau_i)$ when $\tau_i \in (0, \tau^\star)$ with $\tau^\star$ defined in (9). We consider again $B(\omega, \tau_i)$ given by (18). For the terms involving $\cos(\omega\tau_i)$, we have

$$-2\alpha k_s \cos(\omega\tau_i) \geq -2\alpha k_s$$

and

$$2(\alpha\lambda k_v - k_v - \lambda k_s)\cos(\omega\tau_i) \geq -2|\alpha\lambda k_v - k_v - \lambda k_s|.$$

Concerning the terms involving $\sin(\omega\tau_i)$, since $\sin(\omega\tau_i) \leq \omega\tau_i$ for $\omega > 0$ then

$$-2\lambda k_v \sin(\omega\tau_i)\omega^3 \geq -2\lambda k_v \tau_i \omega^4 \geq -2\lambda k_v \tau^\star \omega^4$$

and

$$\begin{aligned} 2(k_s - \alpha(k_v + \lambda k_s))\sin(\omega\tau_i)\omega &\geq -2|k_s - \alpha(k_v + \lambda k_s)|\tau_i\omega^2 \\ &\geq -2|k_s - \alpha(k_v + \lambda k_s)|\tau^\star\omega^2. \end{aligned}$$

Therefore,

$$\begin{aligned} B(\omega, \tau_i) \geq{}& (1 - 2\lambda k_v \tau^\star)\omega^4 + \big[\lambda^2 k_v^2 + \alpha^2 - 2|\alpha\lambda k_v - k_v - \lambda k_s| - \\ &-2\tau^\star|k_s - \alpha(k_v + \lambda k_s)|\big]\omega^2 + \lambda^2 k_s^2 - 2\alpha k_s \geq (1 - 2\lambda k_v\tau^\star)\omega^4 + \\ &+\big[(\lambda k_v - \alpha)^2 - 2k_v - 2\lambda k_s - 2\tau^\star k_s - 2\tau^\star\alpha(k_v + \lambda k_s)\big]\omega^2 + \lambda^2 k_s^2 - 2\alpha k_s \geq 0. \end{aligned}$$

Let us set

$$\begin{aligned} C(\omega, \tau^\star) ={}& (1 - 2\lambda k_v \tau^\star)\omega^4 + \big[(\lambda k_v - \alpha)^2 - 2k_v - \\ &-2\lambda k_s - 2\tau^\star k_s - 2\tau^\star\alpha(k_v + \lambda k_s)\big]\omega^2 + \lambda^2 k_s^2 - 2\alpha k_s. \end{aligned}$$

We suppose that

$$1 - 2\lambda k_v \tau^\star > 0. \tag{27}$$

Then the positivity of $C(\omega, \tau^\star)$ is ensured if (21) is satisfied and if we have

$$\begin{aligned} &[(\lambda k_v - \alpha)^2 - 2k_v - 2\lambda k_s - 2\tau^\star k_s - 2\tau^\star\alpha(k_v + \lambda k_s)]^2 \leq \\ &\leq 4(1 - 2\lambda k_v \tau^\star)(\lambda^2 k_s^2 - 2\alpha k_s). \end{aligned} \tag{28}$$

This leads to the condition

$$-2k_s\lambda\sqrt{\left(1-\frac{2\alpha}{\lambda^2 k_s}\right)(1-2\lambda k_v\tau^\star)} \le (\lambda k_v-\alpha)^2-2k_v-2\lambda k_s- \tag{29}$$
$$-2\tau^\star(k_s+\alpha(k_v+\lambda k_s)) \le 2k_s\lambda\sqrt{\left(1-\frac{2\alpha}{\lambda^2 k_s}\right)(1-2\lambda k_v\tau^\star)}.$$

Now, we search to define the set of parameters k_v which satisfy these inequalities. Since $1-2\lambda k_v\tau^\star \le 1$ and $1-\frac{2\alpha}{\lambda^2 k_s}\le 1$ one has

$$\begin{aligned}&\lambda^2k_v^2-2(1+\alpha\lambda+\alpha\tau^\star)k_v+\alpha^2-2\tau^\star(k_s+\alpha\lambda k_s)-\\&-2\lambda k_s\left(1+\sqrt{\left(1-\frac{2\alpha}{\lambda^2k_s}\right)(1-2\lambda k_v\tau^\star)}\right)\le\\&\le\lambda^2k_v^2-2(1+\alpha\lambda+\alpha\tau^\star)k_v+\alpha^2-2\tau^\star(k_s+\alpha\lambda k_s)-\\&-2\lambda k_s\left(1+(1-2\lambda k_v\tau^\star)\left(1-\frac{2\alpha}{\lambda^2k_s}\right)\right).\end{aligned} \tag{30}$$

Thus, if we can find k_v such that

$$\begin{aligned}&\lambda^2k_v^2-2(1+\alpha\lambda+5\alpha\tau^\star-2\tau^\star\lambda^2k_s)k_v+\\&+\alpha^2-2\tau^\star(1+\alpha\lambda)k_s-4\lambda k_s+\frac{4\alpha}{\lambda}\le 0,\end{aligned} \tag{31}$$

then the second inequality in (29), will be satisfied. A necessary condition to guarantee the previous condition is to have

$$\begin{aligned}\Delta_{1,\tau^\star}&=\left(1+\alpha\lambda+5\alpha\tau^\star-2\tau^\star\lambda^2k_s\right)^2-\\&-\lambda^2\left(\alpha^2-2\tau^\star(1+\alpha\lambda)k_s-4\lambda k_s+\frac{4\alpha}{\lambda}\right)\ge 0\end{aligned} \tag{32}$$

and under this condition, we choose k_v as follows

$$\max\left\{0,\frac{a_1-\sqrt{\Delta_{1,\tau^\star}}}{\lambda^2}\right\}\le k_v\le\frac{a_1+\sqrt{\Delta_{1,\tau^\star}}}{\lambda^2}, \tag{33}$$

where $a_1=1+\alpha\lambda+5\alpha\tau^\star-2\tau^\star\lambda^2k_s$. We can remark that (32) can be rewritten as

$$4{\tau^\star}^2\lambda^4k_s^2+2\lambda^2\left(2\lambda-\tau^\star(1+10\alpha\tau^\star+\alpha\lambda)\right)k_s+(1+5\alpha\tau^\star)^2+2\alpha\lambda[5\alpha\tau^\star-1]\ge 0$$

which leads to

$$k_s\in\left(-\infty,\xi_1\right]\cup[\xi_2,+\infty\right), \tag{34}$$

where

$$\xi_1 = \frac{\lambda^2\big(2\lambda - \tau^\star(1+10\alpha\tau^\star+\alpha\lambda)\big) - \sqrt{\overline{\Delta_{1,\tau^\star}}}}{4{\tau^\star}^2\lambda^4},$$
$$\xi_2 = \frac{\lambda^2\big(2\lambda - \tau^\star(1+10\alpha\tau^\star+\alpha\lambda)\big) + \sqrt{\overline{\Delta_{1,\tau^\star}}}}{4{\tau^\star}^2\lambda^4}$$

and

$$\overline{\Delta_{1,\tau^\star}} = \lambda^4\big(2\lambda - \tau^\star(1+10\alpha\tau^\star+\alpha\lambda)\big)^2 - 4{\tau^\star}^2\lambda^4[(1+5\alpha\tau^\star)^2 + 2\alpha\lambda(5\alpha\tau^\star-1)]$$

which is supposed to be positive. If $\overline{\Delta_{1,\tau^\star}} < 0$, then the condition (32) is verified for all $k_s \geq 0$.

The first inequality in (29) can be rewritten as

$$0 \leq \lambda^2 k_v^2 - 2(1+\alpha\lambda+\alpha\tau^\star)k_v + \alpha^2 - 2\tau^\star(k_s+\alpha\lambda k_s) - \\ -2\lambda k_s\left(1 - \sqrt{\left(1-\frac{2\alpha}{\lambda^2 k_s}\right)(1-2\lambda k_v\tau^\star)}\right). \tag{35}$$

Proceeding as above, we have

$$\lambda^2 k_v^2 - 2(1+\alpha\lambda+\alpha\tau^\star)k_v + \alpha^2 - 2\tau^\star(k_s+\alpha\lambda k_s) - \\ -2\lambda k_s\left(1-(1-2\lambda k_v\tau^\star)\left(1-\frac{2\alpha}{\lambda^2 k_s}\right)\right) \leq \\ \leq \lambda^2 k_v^2 - 2(1+\alpha\lambda+\alpha\tau^\star)k_v + \alpha^2 - 2\tau^\star(k_s+\alpha\lambda k_s) - \\ -2\lambda k_s\left(1 - \sqrt{\left(1-\frac{2\alpha}{\lambda^2 k_s}\right)(1-2\lambda k_v\tau^\star)}\right). \tag{36}$$

If there exists k_v such that

$$0 \leq \lambda^2 k_v^2 - 2\left(1+\alpha\lambda+\alpha\tau^\star+2\tau^\star\lambda^2 k_s\left(1-\frac{2\alpha}{\lambda^2 k_s}\right)\right)k_v + \\ +\alpha^2 - 2\tau^\star(k_s+\alpha\lambda k_s) - 2\lambda k_s\left(1-\left(1-\frac{2\alpha}{\lambda^2 k_s}\right)\right) \tag{37}$$

then the first inequality in (29), will be verified. This inequality can be simplified as

$$0 \leq \lambda^2 k_v^2 - 2(1+\alpha\lambda-3\alpha\tau^\star+2\tau^\star\lambda^2 k_s)k_v + \alpha^2 - 2\tau^\star(1+\alpha\lambda)k_s - \frac{4\alpha}{\lambda} \tag{38}$$

and is satisfied for all k_v such that

$$k_v \in \left(-\infty, \frac{1+\alpha\lambda-3\alpha\tau^\star+2\tau^\star\lambda^2 k_s-\sqrt{\Delta_{2,\tau^\star}}}{\lambda^2}\right] \cup \cup \left[\frac{1+\alpha\lambda-3\alpha\tau^\star+2\tau^\star\lambda^2 k_s+\sqrt{\Delta_{2,\tau^\star}}}{\lambda^2}, +\infty\right), \tag{39}$$

where

$$\Delta_{2,\tau^\star} = \left(1+\alpha\lambda-3\alpha\tau^\star+2\tau^\star\lambda^2 k_s\right)^2 - \lambda^2\left(\alpha^2-2\tau^\star(1+\alpha\lambda)k_s-\frac{4\alpha}{\lambda}\right) \tag{40}$$

is supposed to be positive. If $\Delta_{2,\tau^\star} < 0$ the inequality (37) and by consequence (35), would be satisfied for all $k_v \geq 0$. The positivity of $\Delta_{2,\tau^\star}$ can be rewritten as

$$4{\tau^\star}^2\lambda^4 k_s^2 + 6\lambda^2\tau^\star[1+\alpha-2\alpha\tau^\star]k_s + (1-3\alpha\tau^\star)^2 + 6\alpha\lambda(1-\alpha\tau^\star) \geq 0$$

which leads to the condition on k_s given by

$$k_s \in \left(-\infty, \frac{3\lambda^2\tau^\star(2\alpha\tau^\star-1-\alpha)-\sqrt{\overline{\Delta_{2,\tau^\star}}}}{4\lambda^4{\tau^\star}^2}\right] \cup \cup \left[\frac{3\lambda^2\tau^\star(2\alpha\tau^\star-1-\alpha)+\sqrt{\overline{\Delta_{2,\tau^\star}}}}{4\lambda^4{\tau^\star}^2}, +\infty\right) \tag{41}$$

if

$$\overline{\Delta_{2,\tau^\star}} = 9\lambda^4{\tau^\star}^2[1+\alpha-2\alpha\tau^\star]^2 - 4\lambda^4{\tau^\star}^2[(1-3\alpha\tau^\star)^2+6\alpha\lambda(1-\alpha\tau^\star)] \tag{42}$$

is positive. It is clear that if $\overline{\Delta_{2,\tau^\star}}$ is negative, then the positivity of $\Delta_{2,\tau^\star}$ would be satisfied for all $k_s \geq 0$.

The negativity of $\Delta_{2,\tau^\star}$ implies that the first inequality in (29) is satisfied for all k_v positive. Moreover, $\Delta_{2,\tau^\star} \leq 0$ is satisfied for

$$\max\left\{0, \frac{3\lambda^2\tau^\star(2\alpha\tau^\star-1-\alpha)-\sqrt{\overline{\Delta_{2,\tau^\star}}}}{4\lambda^4{\tau^\star}^2}\right\} \leq k_s \leq \leq \frac{3\lambda^2\tau^\star(2\alpha\tau^\star-1-\alpha)+\sqrt{\overline{\Delta_{2,\tau^\star}}}}{4\lambda^4{\tau^\star}^2}, \tag{43}$$

where $\overline{\Delta_{2,\tau^\star}}$ is assumed to be positive.

Summarizing, the parameters k_v and k_s guaranteeing that (29) is satisfied, verify both:

- k_v in the interval defined by (33) under the necessary condition that $\Delta_{1,\tau^\star}$ is positive;
- $k_v > 0$ if $\Delta_{2,\tau^\star} \geq 0$ or k_v in the interval defined by (39) if $\Delta_{2,\tau^\star} \leq 0$.

It is noteworthy that $\Delta_{1,\tau^\star}$ and $\Delta_{2,\tau^\star}$ are function of k_s. Their sign are conditioned by the sign of $\overline{\Delta_{1,\tau^\star}}$ and $\overline{\Delta_{2,\tau^\star}}$.

4. **SIMULATION RESULTS**

We consider a platoon of 4 following vehicles. We suppose that initially these vehicles travel at the steady-state velocity of $v_0 = 20$ m/s and the safety distance is characterized by $\lambda = 1$ and $H_i = 2$ m with $\alpha = 5$. We choose the controller parameters $k_s = 19$ and $k_v = 0.12$. Then by Proposition 1, we obtain the *optimal delay margin* equal to $\tau^* = 0.215$. The system (2) is then asymptotically stable for all delays $\tau < 0.215$.

We arrive to the same conclusion by using the Matlab package DDE-BIFTOOL (see [3]) to represent the rightmost roots of the characteristic equation. Indeed, if we choose the limit value of the delay $\tau = 0.215$ then we can observe that rightmost roots of the characteristic equation are on the imaginary axis. When we choose a larger delay, the system becomes unstable.

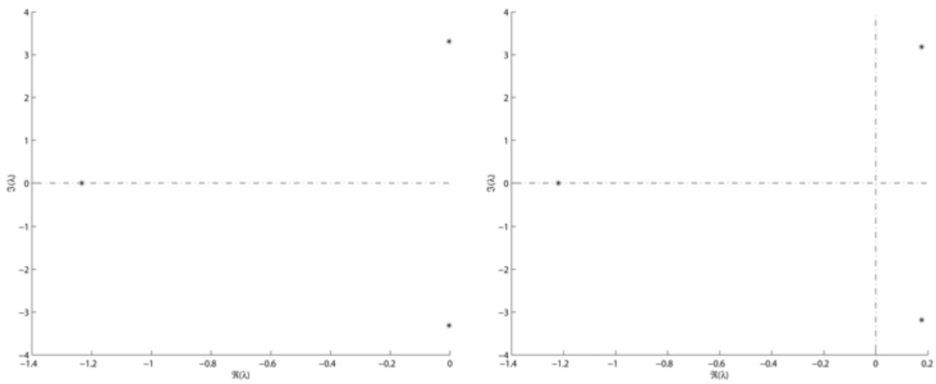

Fig. 1. Left: rightmost roots of the characteristic equation for $\tau = 0.215$.
Right: rightmost roots of the characteristic equation for $\tau = 0.25$.

The upper delay bound guaranteeing no slinky effects is $\tau = 0.0504$. Therefore, if we set the delay value $\tau = 0.2$, the slinky effect is present while for $\tau = 0.05$ one has no slinky effect (see Figure 2). Precisely, the spacings, velocities and accelerations stop oscillating when $\tau = 0.05$ meaning that we arrive to a consensus in terms of distance, velocity and acceleration. In other words, the slinky effect is not encountered. On the other hand, when $\tau = 0.2$, the spacings, velocities and accelerations do not stop oscillating which indicate the presence of the slinky effect.

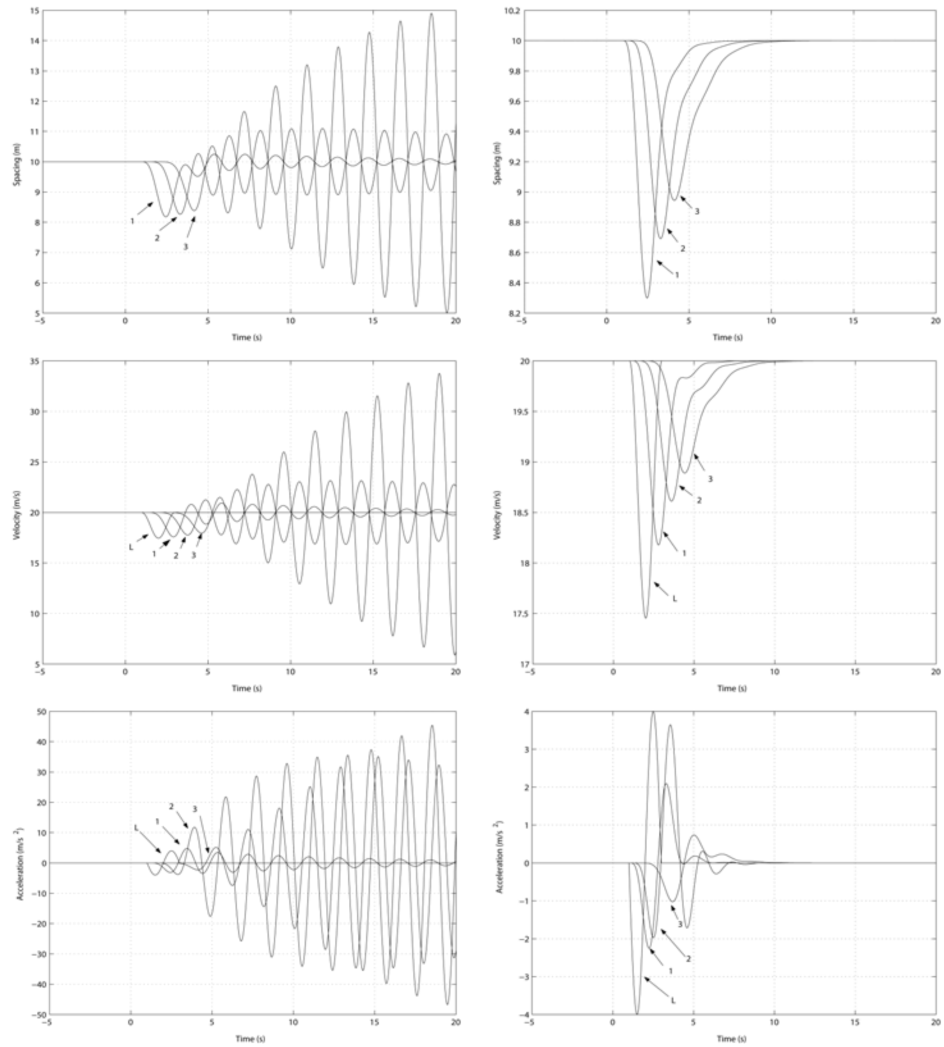

Fig. 2. Left: control responses of 4 following vehicles with time delay 0.2 s,
Right: control responses of 4 following vehicles with time delay 0.05 s.

Thus, in order to guarantee the individual stability of vehicles of the platoon and to avoid the slinky effect phenomenon, it suffices to choose the delay $\tau \leq \min(0.215, 0.0504) = 0.0504$.

5. CONCLUSIONS

In this paper, we have considered the problem of vehicle following control system. For a given controller structure, we have developed conditions guaranteeing the individual stability of each vehicle of the platoon, and the derived conditions depend on the size of the delay. Moreover, we considered the problem of slinky-effect phenomenon, and we proposed sufficient conditions to avoid it. We have given an explicit characterization of some sets of controller parameters which solve the problem.

A. PROOF OF PROPOSITION 1

(a) Straightforward. Assume by contradiction that the delay-independent stability holds. As discussed in [16], a necessary condition for delay-independent stability is the Hurwitz stability of Q, and this is not the case.

(b) Since the system free of delay is asymptotically stable, the conclusion of (a) leads to the existence of a delay margin $\tau^\star$, such that the system is asymptotically stable for all delays $\tau \in [0, \tau^\star)$. Furthermore at $\tau = \tau^\star$, the characteristic equation (3) has at least one root $s = jw$ on the imaginary axis, with $w \in \Omega$ (crossing frequency). Since

$$\frac{P(jw)}{Q(jw)} = -\mathrm{e}^{-jw\tau} = -\cos(w\tau) + j\sin(w\tau) \tag{44}$$

this implies that

$$\cos(w\tau) = -\Re\left(\frac{P(jw)}{Q(jw)}\right).$$

We compute the right hand side of this equation with

$$\frac{P(jw)}{Q(jw)} = -\frac{\alpha(k_s - \lambda k_v w^2)w^2 + (k_v + \lambda k_s)w^4}{(k_s - \lambda k_v w^2)^2 + (k_v + \lambda k_s)^2 w^2} - \\ -\frac{j(k_s - \lambda k_v w^2)w^3 - j\alpha(k_v + \lambda k_s)w^3}{(k_s - \lambda k_v w^2)^2 + (k_v + \lambda k_s)^2 w^2}. \tag{45}$$

Therefore,

$$\tau^\star = \frac{1}{w}\arccos\left(\frac{\alpha(k_s - \lambda k_v w^2)w^2 + (k_v + \lambda k_s)w^4}{(k_s - \lambda k_v w^2)^2 + (k_v + \lambda k_s)^2 w^2}\right), \tag{46}$$

where w is a *crossing frequency.*

In the sequel, we explicitly determinate the expression of the crossing frequencies by solving the equation

$$w^6 + (\alpha^2 - \lambda^2 k_v^2)w^4 - (k_v^2 + \lambda^2 k_s^2)w^2 - k_s^2 = 0. \tag{47}$$

For this equation in w^2, we have one real solution (and two complex roots) or three real roots. We have to analyze their sign to consider only the positive candidates.

If we denote by r_i, $i = 1, \dots, 3$, the roots of the equation, we know that they are solutions of

$$x^3 - Sx^2 + \Pi_2 x - \Pi_3 = 0,$$

where $S = \sum_{i=1}^{3} r_i$, $\Pi_2 = \prod_{i \neq j \in \{1,\dots,3\}} r_i r_j$, $\Pi_3 = \prod_{i \in \{1,\dots,3\}} r_i$.

Since $\Pi_3 = k_s^2 > 0$, if we have only one real root (the others are complex and conjugate), this root is positive and if we have three real roots, we have one positive root and two real roots with the same sign. In the latter case, we only take into account only the case where the three real roots are positive. Moreover, with $\Pi_2 = -(k_v^2 + \lambda^2 k_s^2) < 0$, we can remark that we cannot have three positive real roots. Finally, we can have only one positive real root (square of the crossing frequency). Now we apply the method of Cardan to define the form of this crossing frequency. We can establish that if

$$\Big(\alpha^4 + \lambda^2\big(\lambda^2 k_v^4 + 3k_s^2 - 2\alpha^2 k_v^2\big) + 3k_v^2\Big)^3 <$$
$$< \frac{1}{4}\Big((\alpha^2 - \lambda^2 k_v^2)[2(\alpha^2 - \lambda^2 k_v^2) + 9(\lambda^2 k_s^2 + k_v^2)] - 27k_s^2\Big)^2,$$

then the crossing frequency is of the form

$$w_f = \sqrt{\Big(-\frac{w_1}{54}\Big)^{\frac{1}{3}} + \Big(-\frac{w_2}{54}\Big)^{\frac{1}{3}} - \frac{\alpha^2 - \lambda^2 k_v^2}{3}}, \tag{48}$$

where

$$w_1 = \gamma_1 + \sqrt{\zeta_1} \quad \text{and} \quad w_2 = \gamma_1 - \sqrt{\zeta_1}, \tag{49}$$

with

$$\gamma_1 = \big((\alpha^2 - \lambda^2 k_v^2)\big[2(\alpha^2 - \lambda^2 k_v^2)^2 + 9(\lambda^2 k_s^2 + k_v^2)\big] - 27k_s^2\big),$$

and

$$\zeta_1 = \gamma_1^2 - 4\big((\alpha^2 - \lambda^2 k_v^2)^2 + 3(\lambda^2 k_s^2 + k_v^2)\big)^3.$$

If

$$\Big(\alpha^4 + \lambda^2\big(\lambda^2 k_v^4 + 3k_s^2 - 2\alpha^2 k_v^2\big) + 3k_v^2\Big)^3 >$$
$$> \frac{1}{4}\Big((\alpha^2 - \lambda^2 k_v^2)\big[2(\alpha^2 - \lambda^2 k_v^2) + 9(\lambda^2 k_s^2 + k_v^2)\big] - 27k_s^2\Big)^2,$$

then it is of the form

$$w_f = \sqrt{\Big(-\frac{\tilde{w_1}}{54}\Big)^{\frac{1}{3}} + \Big(-\frac{\tilde{w_2}}{54}\Big)^{\frac{1}{3}} - \frac{\alpha^2 - \lambda^2 k_v^2}{3}}, \tag{50}$$

where

$$\tilde{w_1} = \gamma_1 + j\sqrt{-\zeta_1} \quad \text{and} \quad \tilde{w_2} = \gamma_1 - j\sqrt{-\zeta_1}. \tag{51}$$

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