UNIQUENESS AND QUALITATIVE PROPERTIES OF THE SOLUTIONS TO THE FUNCTIONAL EQUATION $f \circ f + af + b1_{\mathbb{R}} = 0$

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We deal with the functional equation in the title for real and nonzero a, b. Namely, in this paper we give some qualitative properties for the continuous solutions of the aforementioned equation, in all cases.

AMS 2010 Subject Classification: 39B12, 30D05.

Key words: iterative functional equation, continuous solution, homeomorphism, compact sets.

1. INTRODUCTION

In this part results from [2] and [3] (without proof) which will be used throughout this paper are introduced.

Iterative polinomial equations are solved in [1] and [9] with rather similar methods.

Let a, b be real numbers $a \neq 0$, $b \neq 0$. We shall be concerned with the functional equation (called *fundamental equation.*)

$$f \circ f + af + b1_{\mathbb{R}} = 0.$$

Namely, we want to find a continuous function $f : \mathbb{R} \to \mathbb{R}$ having the property that, for any $x \in \mathbb{R}$

$$f\left(f(x)\right) + af(x) + bx = 0.$$

Such a function (in case it exists) will be called *a solution of the fundamental equation* (or, simply *a solution*). In the sequel, the fundamental equation will be written in the form

$$f \circ f + af + bx = 0.$$

It is seen that a solution must be a homeomorphism. Moreover, the function $g = f^{-1}$ satisfies the equation

$$g \circ g + \frac{a}{b}\, g + \frac{1}{b}\, 1_{\mathbb{R}} = 0.$$

Incidentally, the fundamental equation will be written alternatively

$$f \circ f \pm af \pm bx = 0.$$

with positive a and b.

The *characteristic equation* of the problem is the quadratic equation

$$x^2 + ax + b = 0.$$

with (complex) roots r_1, r_2 and discriminant $\Delta = a^2 - 4b$. Actually, in this paper we shall study the case when r_1, r_2 are real, because, for non real r_1, r_2 the fundamental equation has no solutions.

THEOREM 1.a (Calibration Theorem). *Let us assume that f is a solution and $|r_1| \leq |r_2|$. Then, for any real x, y one has*

$$|r_1| \cdot |x - y| \leq |f(x) - f(y)| \leq |r_2| \, |x - y|.$$

LEMMA 1.1. *Let us assume $1 < r_1 < r_2$. For any solution f we have the following properties*:

a) $f(0) = 0$.

b) *For any $x_0 \in \mathbb{R}$ one has*

$$r_1 x_0 \leq f(x_0) \leq r_2 x_0 \quad \text{if } x_0 \geq 0,$$
$$r_2 x_0 \leq f(x_0) \leq r_1 x_0 \quad \text{if } x_0 < 0.$$

LEMMA 1.2. *Let us assume that $r_2 < r_1 < 0$. For any solution f, we have the properties*:

a) $f(0) = 0$.

b) *For any $x_0 \in \mathbb{R}$,*

$$r_2 x_0 \leq f(x_0) \leq r_1 x_0 \quad \text{if } x_0 \geq 0,$$
$$r_1 x_0 \leq f(x_0) \leq r_2 x_0 \quad \text{if } x_0 < 0.$$

LEMMA 1.3. *Let us assume that $r_2 < r_1 < -1$. Let $0 \neq x_0 \in \mathbb{R}$ and $x_1 \in [r_2 x_0, r_1 x_0]$ (in case $x_0 > 0$) or $x_1 \in [r_1 x_0, r_2 x_0]$ (in case $x_0 < 0$). Using the coefficients of the fundamental equation we define the sequences $(x_n)_{n \geq 0}$ and $(x_{-n})_{n \geq 0}$ as follows.*

a) $x_{n+2} = -ax_{n+1} - bx_n$ *with starting terms x_0 and x_1. Such a sequence is the sequence given via $x_{n+1} = f(x_n)$, with starting term x_0 (see Lemma 1.2 for $x_1 = f(x_0)$).*

b) *The sequence $(x_{-n})_n$ is defined in two steps. First, we define the sequence $(y_n)_{n \geq 0}$ by*

$$y_{n+2} = -\frac{a}{b} y_{n+1} - \frac{1}{b} y_n$$

with starting terms $y_0 = x_1$ and $y_1 = x_0$.

Next, we write $x_{-n} = y_{n+1}$ *for all natural* n*. Hence*

$$x_{-n-2} = -\frac{a}{b}\,x_{-n-1} - \frac{1}{b}\,x_{-n}.$$

with starting terms $x_0 = y_1$ *and* $x_{-1} = y_2$*. Such a sequence is the sequence given via* $x_{-n-1} = f^{-1}(x_{-n})$ *with starting term* x_0 *(see Lemma 1.2 for* $x_1 = f(x_0) \Leftrightarrow x_0 = f^{-1}(x_1)$*).*

In case $x_0 > 0$ *we have* $x_{2n} \uparrow \infty$ *(strictly),* $x_{2n+1} \downarrow -\infty$ *(strictly),* $x_{-2n} \downarrow 0$ *(strictly) and* $x_{-2n+1} \uparrow 0$ *(strictly). This implies*

$$\bigcup_{n\geq 0}\left([x_{2n}, x_{2n+2}] \cup [x_{-2n}, x_{-2n+2}]\right) = (0, \infty),$$

$$\bigcup_{n>0}\left([x_{2n+1}, x_{2n-1}] \cup [x_{-2n+1}, x_{-2n-1}]\right) = (-\infty, 0).$$

The case $x_0 < 0$ *is symmetric (e.g.,* $x_{2n} \downarrow -\infty$ *strictly ...).*

LEMMA 1.4. *Let us assume that* $1 < r_1 < r_2$*. Let* $0 \neq x_0 \in \mathbb{R}$ *and* $x_1 \in [r_1x_0, r_2x_0]$ *(in case* $x_0 > 0$*) or* $x_1 \in [r_2x_0, r_1x_0]$ *(in case* $x_0 < 0$*).*

Define the sequences $(x_n)_{n\geq 0}$ *and* $(x_{-n})_{n\geq 0}$ *as in Lemma 1.3. In particular we can take* $x_{n+1} = f(x_n)$ *with starting term* x_0 *and* $x_{-n-1} = f^{-1}(x_{-n})$ *with starting term* x_0 *(see Lemma 1.1).*

In case $x_0 > 0$ *we have* $x_n \uparrow \infty$ *(strictly),* $x_{-n} \downarrow 0$ *(strictly). In case* $x_0 < 0$ *we have* $x_n \downarrow -\infty$ *(strictly) and* $x_{-n} \uparrow 0$ *(strictly). This implies*

$$\bigcup_{n\in\mathbb{Z}} [x_n, x_{n+1}] = (0, \infty) \quad \textit{if } x_0 > 0,$$

$$\bigcup_{n\in\mathbb{Z}} [x_n, x_{n+1}] = (-\infty, 0) \quad \textit{if } x_0 < 0.$$

LEMMA 1.5. *Let us assume that* $0 < r_1 < 1 < r_2$*. Let* $x_0 \in \mathbb{R}$ *and* $x_1 > r_1x_0$*,* $x_1 > r_2x_0$*. Define the sequence* $(x_n)_{n\geq 0}$ *and* $(x_{-n})_{n\geq 0}$ *exactly like in Lemma 1.3. In particular, we can take* $x_{n+1} = f(x_n)$*, with starting term* x_0 *and* $x_{-n-1} = f^{-1}(x_{-n})$*, with starting term* x_0*.*

Then $x_n \uparrow \infty$ *(strictly),* $x_{-n} \downarrow -\infty$ *(strictly) and*

$$\bigcup_{n\in\mathbb{Z}} [x_n, x_{n+1}] = \mathbb{R}.$$

THEOREM 1.6 (Case $1 < r_1 < r_2$). *We shall write the fundamental equation in the form*

$$f \circ f - af + bx = 0.$$

All the solutions $f : \mathbb{R} \to \mathbb{R}$ *are of the form*

$$f(x) = \begin{cases} F_1(x) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ F_2(x) & \text{if } x < 0, \end{cases}$$

where F_1 *and* F_2 *are constructed as follows*:

1. *Construct the sequences* $(x_n)_{n\geq 0}$ *and* $(x_{-n})_{n\geq 0}$ *according to Lemma* 1.4 *starting with an arbitrary* $x_0 > 0$ *and* $x_1 \in [r_1x_0, r_2x_0]$. *Consider a bijection* $f_0 : [x_0, x_1] \to [x_1, x_2]$ *having the property that for any* $x > y$ *in* $[x_0, x_1]$ *one has*

$$r_1(x - y) \leq f_0(x) - f_0(y) \leq r_2(x - y). \tag{1.1}$$

Then, for any natural n, *one can construct the bijections*

$$f_n : [x_n, x_{n+1}] \to [x_{n+1}, x_{n+2}]$$

and

$$f_{-n} : [x_{-n}, x_{-n+1}] \to [x_{-n+1}, x_{-n+2}]$$

defined via

$$f_{n+1}(x) = ax - bf_n^{-1}(x) \quad \text{and} \quad f_{-n-1}^{-1}(x) = \frac{a}{b}x - \frac{1}{b}f_{-n}^{-1}(x). \tag{1.2}$$

Finally, for any

$$x \in (0, \infty) = \bigcup_{n\in\mathbb{Z}} [x_n, x_{n+1}]$$

we have, for some natural n, *either* $x \in [x_n, x_{n+1}]$ *and* $F_1(x) = f_n(x)$ *or* $x \in [x_{-n}, x_{-n+1}]$ *and* $F_1(x) = f_{-n}(x)$.

The values at the common endpoints coincide.

2. *Construct the sequences* $(x_n)_{n\geq 0}$ *and* $(x_{-n})_{n\geq 0}$ *according to Lemma* 1.4 *starting with an arbitrary* $x_0 < 0$ *and* $x_1 \in [r_2x_0, r_1x_0]$. *Consider a bijection* $f_0 : [x_1, x_0] \to [x_2, x_1]$ *having the property* (1.1) *for any* $x > y$ *în* $[x_1, x_0]$.

Then, for any natural n, *one can construct the bijections*

$$f_n : [x_{n+1}, x_n] \to [x_{n+2}, x_{n+1}]$$

and

$$f_{-n} : [x_{-n+1}, x_{-n}] \to [x_{-n+2}, x_{-n+1}]$$

defined via (1.2).

Finally, for any

$$x \in (-\infty, 0) = \bigcup_{n\in\mathbb{Z}} [x_{n+1}, x_n]$$

we have, for some natural n, *either* $x \in [x_{n+1}, x_n]$ *and* $F_2(x) = f_n(x)$ *or* $x \in [x_{-n+1}, x_{-n}]$ *and* $F_2(x) = f_{-n}(x)$. *The values at the common endpoints coincide.*

Theorem 1.7 (Case $r_2 < r_1 < -1$). *All the solutions are obtained as follows. Start with an arbitrary $x_0 > 0$, and we choose $x_1 \in [r_2 x_0, r_1 x_0]$. Apply Lemma 1.3 and construct the sequences $(x_n)_n$ and $(x_{-n})_n$. Let $f_0 : [x_0, x_2] \to [x_3, x_1]$ be a strictly decreasing bijection having the property*

$$-r_1(x-y) \le f_0(y) - f_0(x) \le -r_2(x-y) \tag{1.3}$$

for all $x > y$ in $[x_0, x_2]$.

Construct the following strictly decreasing bijections (for any natural n):

$$f_{2n} : [x_{2n}, x_{2n+2}] \to [x_{2n+3}, x_{2n+1}]\,,$$

$$f_{2n}(x) = -ax - bf^{-1}_{2n-1}(x) \tag{1.4}$$

$$f_{2n+1} : [x_{2n+3}, x_{2n+1}] \to [x_{2n+2}, x_{2n+4}]\,,$$

$$f_{2n+1}(x) = -ax - bf^{-1}_{2n}(x) \tag{1.5}$$

$$f_{-2n} : [x_{-2n}, x_{-2n+2}] \to [x_{-2n+3}, x_{-2n+1}]\,,$$

$$f^{-1}_{-2n}(x) = -\frac{a}{b}x - \frac{1}{b}f_{-2n+1}(x) \tag{1.6}$$

$$f_{-2n-1} : [x_{-2n+1}, x_{-2n-1}] \to [x_{-2n}, x_{-2n+2}]\,,$$

$$f^{-1}_{-2n-1}(x) = -\frac{a}{b}x - \frac{1}{b}f_{-2n}(x). \tag{1.7}$$

Since the reunion of all above mentioned intervals is equal to $\mathbb{R} \setminus \{0\}$, we can construct $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ f_n(x) & \text{if } x \neq 0, \end{cases}$$

where $0 \neq x$ belongs to one of the above mentioned intervals which is the domain of definition for f_n, $n \in \mathbb{Z}$. The values at the common endpoints coincide.

Then f is a solution and all the solutions can be obtained in this way.

Theorem 1.8. *Assume $0 < r_1 < 1 < r_2$ and let f be a solution with the property $f(0) \neq 0$. Then either $f(x) > x$ for any $x \in \mathbb{R}$ or $f(x) < x$ for any $x \in \mathbb{R}$.*

I. *Assume that $f(x) > x$ for all $x \in \mathbb{R}$. Then f can be obtained as follows: Construct the sequences $(x_n)_n$ and $(x_{-n})_n$ according to Lemma 1.5, where we take $x_0 = 0$ and $x_1 > 0$ arbitrary (the conditions of Lemma 1.5 are fulfilled). Consider a strictly increasing bijection $f_0 : [0, x_1] \to [x_1, x_2]$ such that*

$$r_1(x-y) \le f_0(x) - f_0(y) \le r_2(x-y)$$

for all $x > y$ in $[0, x_1]$.

Then, for any natural n *one can construct the bijections* $f_n : [x_n, x_{n+1}] \to [x_{n+1}, x_{n+2}]$ *and* $f_{-n} : [x_{-n}, x_{-n+1}] \to [x_{-n+1}, x_{-n+2}]$ *defined by*

$$f_{n+1}(x) = ax - bf_n^{-1}(x) \quad and \quad f_{-n-1}^{-1}(x) = \frac{a}{b}x - \frac{1}{b}f_{-n}(x).$$

Finally, for any

$$x \in \mathbb{R} = \bigcup_{n\in\mathbb{Z}} [x_n, x_{n+1}]$$

we have, for some natural n*, either* $x \in [x_n, x_{n+1}]$ *and* $f(x) = f_n(x)$*, or* $x \in [x_{-n}, x_{-n+1}]$ *and* $f(x) = f_{-n}(x)$*.*

II. *Assume that* $f(x) < x$ *for any* $x \in \mathbb{R}$*. Then* $f^{-1}(x) > x$ *for any* $x \in \mathbb{R}$ *and* f^{-1} *can be constructed according to part* I.

Remark. The characteristic equation for the problem concerning the inverse $g = f^{-1}$ is

$$bx^2 + ax + 1 = 0$$

and has the roots r_1^{-1}, r_2^{-1} where r_1, r_2 are the roots of the characteristic equation for f. Consequently, Theorems 1.6 and 1.7 cover the cases $0 < r_1 < r_2 < 1$, $-1 < r_1 < r_2 < 0$, too.

2. SUFFICIENT CONDITIONS FOR THEUNIQUENESS OF THE SOLUTIONS

We consider the functional equation

$$f \circ f(x) + af(x) + bx = 0, \tag{E1}$$

where the signs of a and b are taken according to the convention from the beginning of part 1.

We shall establish which conditions guarantee the uniqueness of continuous solutions of this functional equation. More precisely, if two solutions coincide on a non degenerate interval under some conditions, then they coincide everywhere. We shall see which conditions must fulfill this interval in each case.

Theorem 2.1. *Let us consider the functional equation* (E1) *in case* $\Delta > 0$*.*

a) *If* $1 < r_1 < r_2$ *and two solutions coincide on* $I = [a', b']$*,* $I \subset (0, \infty)$ *and*

$$\frac{b'}{a'} \geq r_2,$$

then they coincide on $(0, \infty)$*. A similar result holds if* $I \subset (-\infty, 0)$*. If* $I = [0, a']$ *the solutions coincide on* $(0, \infty)$*. A similar result holds if* $I = [a', 0]$*.*

b) *If* $r_2 < r_1 < -1$ *and two solutions coincide on* $I = [a', b']$, $I \subset (0, \infty)$ *and*

$$\frac{b'}{a'} \geq r_2^2$$

(*also if* $I \subset (-\infty, 0)$ *and* $I = [a', b']$ *and* $\frac{a'}{b'} \geq r_2^2$), *then they coincide on* $\mathbb{R}$. *If* $0 \in I$ *the two solutions coincide on* $\mathbb{R}$.

c) *If* $r_1 < 1 < r_2$ *two solutions* f *and* g *which coincide on* $[0, a']$ *and have the property that* $f(0) \neq 0$, $g(0) \neq 0$, *coincide on* $\mathbb{R}$. *We have a similar result for* $[a', 0]$.

Proof. Because the solutions are continuous, we can use the corresponding existence theorems from the previous section.

a) We consider the equation $f \circ f - af(x) + bx = 0$ with solutions given by Theorem 1.6. Let f and g two solutions that coincide on $[a', b']$, $[a', b'] \subset (0, \infty)$. We shall prove that there exist $x_0, x_1 \in [a', b']$ such that the sequence $(x_n)_{n \in \mathbb{Z}}$ is that one of Theorem 1.6. We choose $x_0 = a'$ and prove that

$$[x_0 r_1, x_0 r_2] \subset \left[a', b'\right].$$

Indeed

$$x_0 r_1 > a' \Leftrightarrow x_0 r_1 > x_0 \Leftrightarrow r_1 > 1$$

and

$$x_0 r_2 < b' \Leftrightarrow b' > a' r_2.$$

These conditions are fulfilled from the hypothesis. Then, because

$$(\alpha) \qquad r_1(x - y) \leq f(x) - f(y) \leq r_2(x - y)$$

(the same for g), we can take

$$f_0 = f|_{[x_0, x_1]} = g|_{[x_0, x_1]} = g_0.$$

We define $(x_n)_{n \in \mathbb{Z}}$ as in Theorem 1.6, $f_0 : [x_0, x_1] \to [x_1, x_2]$ is increasing, bijective and satisfies (α). The same for g_0, hence f_0 and g_0 fulfill the condition from Theorem 1.6. We shall prove inductively that $f_n(x) = g_n(x)$ for $x \in [x_n, x_{n+1}]$, $n \geq 0$, where f_n and g_n are those of Theorem 1.6. We shall prove that the functions obtained in this way are increasing, bijective, continuous, and satisfy (α) for all $n \geq 0$. Let us suppose $f_{n-1} = g_{n-1}$. But

$$f_n(x) = ax - bf_{n-1}^{-1}(x)$$

and

$$g_n(x) = ax - bf_{n-1}^{-1}(x).$$

Because $f_{n-1}^{-1} = g_{n-1}^{-1}$ we have $f_n(x) = g_n(x)$, for all $x \in [x_n, x_{n+1}]$. Then $f = g$ on $[x_n, x_{n+1}]$, i.e., $f = g$ on $[x_0, \infty)$. Let us prove now that $f_{-n} = g_{-n}$, for all $n \geq 0$. Assume inductively $f_{-n+1} = g_{-n+1}$ (f_{-n} and g_{-n} are those of Theorem 1.6). But

$$f_{-n}^{-1}(x) = \frac{1}{b}\left(ax - f_{-n+1}(x)\right) \quad \text{for } x \in [x_{-n+1}, x_{-n+2}]$$

and

$$g_{-n}^{-1}(x) = \frac{1}{b}\left(ax - g_{-n+1}(x)\right) \quad \text{for } x \in [x_{-n+1}, x_{-n+2}]\,.$$

Because $f_{-n+1} = g_{-n+1}$ we have $f_{-n}^{-1} = g_{-n}^{-1}$, hence

$$f_{-n}(x) = g_{-n}(x) \quad \text{for all } x \in [x_{-n}, x_{-n+1}]\,.$$

Then $f_{-n} = g_{-n}$, for all $n \geq 0$. This means $f = g$ on $(0, x_0)$. So $f = g$ on $(0, \infty)$. Similarly, if $[a', b'] \subset (-\infty, 0)$, it follows that $f = g$ on $(-\infty, 0)$. Suppose now $I = [0, a']$. We take

$$x_1 = a' \quad \text{and} \quad x_0 = \frac{a'}{r_2}.$$

Obviously, it follows that $x_0, x_1 \in I$. The proof is similar to the previous one. The same proof for $I = [a', 0]$. Then $f = g$ on $(-\infty, 0)$.

b) If $r_2 < r_1 < -1$, let us consider the equation

$$f \circ f(x) + af(x) + bx = 0$$

with continuous and decreasing solution which fulfils the condition

$$(\beta) \qquad -r_1(x - y) \leq f(y) - f(x) \leq -r_2(x - y)$$

(for $x > y$). The solutions are given by Theorem 1.7.

Let f, g two solutions that coincide on $[a', b'] \subset (0, \infty)$. We shall prove that there exist

$$x_1, x_2 \in [a', b'] \text{ such that } r_2^2 x_0 \leq x_2 \leq r_1^2 x_0 \text{ and } x_1 \in [r_2 x_0, r_1 x_0]$$

such that

$$x_2 + ax_1 + bx_0 = 0.$$

We choose $x_0 = a'$. But $b' \geq a' r_2^2 \Rightarrow b' \geq x_0 r_2^2$. Because

$$\left[x_0 r_1^2, x_0 r_2^2\right] \subset [a', b'],$$

we can choose

$$x_2 \in \left[x_0 r_1^2, x_0 r_2^2\right] \text{ such that } x_2 \in [a', b'],$$

$$x_1 = -\frac{bx_0 - x_2}{a} \leq \frac{-r_1 r_2 x_0 - x_0 r_1^2}{-(r_1 + r_2)} = \frac{-x_0 r_1 (r_1 + r_2)}{-(r_1 + r_2)} = x_0 r_1.$$

Similar proof for $x_1 \geq x_0 r_2$, hence the condition $x_1 \in [x_0 r_2, x_0 r_1]$ is fulfilled.

Then we can define $(x_n)_{n \in \mathbb{Z}}$ as in Theorem 1.7. We can consider

$$f_0|_{[x_0, x_2]} = g_0|_{[x_0, x_2]} = g_0$$

and

$$f_0,\ g_0 : [x_0, x_2] \to [x_3, x_1]$$

fulfil the conditions from Theorem 1.7. Then we can define $(f_n)_{n\in\mathbb{Z}}$ and $(g_n)_{n\in\mathbb{Z}}$ as in Theorem 1.7. They are decreasing, continuous, bijective and satisfy (β).

The functions $f_1 : [x_3, x_1] \to [x_2, x_4]$ and $g_1 : [x_3, x_1] \to [x_2, x_4]$ are given by the formulas

$$f_1(x) = -ax - bf_0^{-1}(x), \quad g_1(x) = -ax - bg_0^{-1}(x).$$

Hence $f_1 = g_1$ on $[x_3, x_1]$.

Now suppose inductively that $f_{2n-1} = g_{2n-1}$ on $[x_{2n+1}, x_{2n-1}]$. We shall prove that $f_{2n} = g_{2n}$ on $[x_{2n}, x_{2n+2}]$, where

$$f_{2n-1},\ g_{2n-1} : [x_{2n+1}, x_{2n-1}] \to [x_{2n}, x_{2n+2}]$$

and

$$f_{2n},\ g_{2n} : [x_{2n}, x_{2n+2}] \to [x_{2n+3}, x_{2n+1}]\,.$$

We know that

$$f_{2n}(x) = -ax - b\cdot f_{2n-1}^{-1}(x), \quad g_{2n}(x) = -ax - b\cdot g_{2n-1}^{-1}(x).$$

Then it follows that $f_{2n} = g_{2n}$ on $[x_{2n}, x_{2n+2}]$.

In the same way it can be shown that $f_{2n+1} = g_{2n+1}$, where

$$f_{2n+1},\ g_{2n+1} : [x_{2n+3}, x_{2n+1}] \to [x_{2n+2}, x_{2n+4}]$$

and $f_{2n+2} = g_{2n+2}$, where

$$f_{2n+2},\ g_{2n+2} : [x_{2n+2}, x_{2n+4}] \to [x_{2n+5}, x_{2n+3}]\,.$$

Then $f_n = g_n$ for all $n \geq 0$, i.e., $f = g$ on $(-\infty, x_1] \cup [x_0, \infty)$.

We prove now that $f_{-n} = g_{-n}$ $(n \geq 0)$. First, we have

$$f_{-1}^{-1}(x) = \frac{1}{b}\left(-ax - f_0(x)\right), \quad g_{-1}^{-1}(x) = \frac{1}{b}\left(-ax - g_0(x)\right),$$

where $f_{-1}, g_{-1} : [x_1, x_{-1}] \to [x_0, x_2]$. Because $f_0 = g_0$ on $[x_0, x_2]$ it follows that $f_{-1}^{-1} = g_{-1}^{-1}$ on $[x_0, x_2]$, so $f_{-1} = g_{-1}$ on $[x_1, x_{-1}]$. Now, suppose inductively that $f_{-2n+1} = g_{-2n+1}$ on $[x_{-2n+3}, x_{-2n+1}]$, where

$$f_{-2n+1}, g_{-2n+1} : [x_{-2n+3}, x_{-2n+1}] \to [x_{-2n+2}, x_{-2n+4}]$$

and we shall prove that $f_{-2n} = g_{-2n}$ on $[x_{-2n}, x_{-2n+2}]$. Indeed

$$f_{-2n}^{-1}(x) = \frac{1}{b}\left(-ax - f_{-2n+1}(x)\right), \quad g_{-2n}^{-1}(x) = \frac{1}{b}\left(-ax - g_{-2n+1}(x)\right),$$

where

$$f_{-2n} : [x_{-2n}, x_{-2n+2}] \to [x_{-2n+3}, x_{-2n+1}]$$

and

$$g_{-2n} : [x_{-2n}, x_{-2n+2}] \to [x_{-2n+3}, x_{-2n+1}]\,.$$

Because $f_{-2n+1} = g_{-2n+1}$ on $[x_{-2n+3}, x_{-2n+1}]$ we have

$$f_{-2n}^{-1} = g_{-2n}^{-1} \quad \text{on } [x_{-2n+3}, x_{-2n+1}] .$$

Then $f_{-2n} = g_{-2n}$ on $[x_{-2n}, x_{-2n+2}]$. Similarly, we can prove that $f_{-2n-1} = g_{-2n-1}$, where

$$f_{-2n-1}, g_{-2n-1} : [x_{-2n+1}, x_{-2n-1}] \to [x_{-2n}, x_{-2n+2}]$$

and $f_{-2n-2} = g_{-2n-2}$, where

$$f_{-2n-2}, g_{-2n-2} : [x_{-2n-2}, x_{-2n}] \to [x_{-2n+1}, x_{-2n-1}] .$$

Thus it follows that $f_{-n} = g_{-n}$, for all $n \geq 0$. Then it follows that $f = g$ on $(x_1, 0) \cup (0, x_0)$. Obviously, $f(0) = g(0) = 0$ and so $f = g$.

Let f and g two solutions which coincide on $[a', b'] \subset (-\infty, 0)$. We shall prove that there exist $x_1, x_3 \subset [a', b']$ with $x_3 \in \left[x_1 r_2^2, x_1 r_1^2\right]$ and $x_2 \in [x_1 r_1, x_1 r_2]$, respectively

$$x_0 \in \left[\frac{x_1}{r_2}, \frac{x_1}{r_1}\right]$$

such that $x_3 + a x_2 + b x_1 = 0$ and $x_2 + a x_1 + b x_0 = 0$. We choose $x_1 = b'$. Because

$$\left[b' r_2^2, b' r_1^2\right] \subset [a', b'],$$

we can choose

$$x_3 \in \left[b' r_2^2,\, b' r_1^2\right]$$

and then $x_3 \in [a', b']$. We choose $x_2 = -\frac{x_3 - b x_1}{a}$. We must prove that $x_2 \in [x_1 r_1, x_1 r_2]$. But

$$x_2 \leq \frac{-x_1 r_2^2 - r_1 r_2 x_1}{-(r_1 + r_2)} = \frac{-x_1 r_2 (r_1 + r_2)}{-(r_1 + r_2)} = x_1 r_2.$$

Similarly, we have $x_2 \geq x_1 r_1$. Choose

$$x_0 = \frac{-a x_1 - x_2}{b} \leq \frac{(r_1 + r_2) x_1 - x_1 r_1}{r_1 r_2} = \frac{x_1}{r_1}.$$

Similarly, we have $x_0 \geq \frac{x_1}{r_2}$, hence $x_1 \in [x_0 r_2, x_0 r_1]$. So, we can define $(x_n)_{n \in \mathbb{Z}}$ like in Theorem 1.7. Let us prove that $f = g$ on $[x_0, x_2]$. Denote

$$f_1 = f|_{[x_3, x_1]} \quad \text{and} \quad g_1 = g|_{[x_3, x_1]}.$$

Obviously, $f_1 = g_1$. The functions $f_1, g_1 : [x_3, x_1] \to [x_2, x_4]$ are continuous, bijective and fulfill the relationship (β).

We define the function $h : [x_3, x_1] \to \mathbb{R}$ (we shall see that one can consider $h : [x_3, x_1] \to [x_0, x_2]$)

$$h(x) = -\frac{f_1(x) + a x}{b} = -\frac{g_1(x) + a x}{b}.$$

The function h is continuous; because f_1 and g_1 fulfill (β). It follows that h fulfills the relationship

$$\frac{x-y}{r_1} \le h(x) - h(y) \le \frac{x-y}{r_2}, \quad x > y.$$

Hence h is decreasing on $[x_3, x_1]$ and

$$h(x_3) = -\frac{f_1(x_3) + ax_3}{b} = -\frac{x_4 + ax_3}{b} = x_2,$$
$$h(x_1) = -\frac{f_1(x_1) + ax_1}{b} = -\frac{x_2 + ax_1}{b} = x_0.$$

Therefore $h : [x_3, x_1] \to [x_0, x_2]$ is bijective. Denote

$$f_0 = h^{-1} \quad \text{and} \quad g_0 = h^{-1}, \quad f_0, g_0 : [x_0, x_2] \to [x_3, x_1].$$

It follows that $f_0 = g_0$ and obviously

$$f_1(x) = -ax - bf_0^{-1}(x), \quad g_1(x) = -ax - bg_0^{-1}(x).$$

It is clear that $x_0 r_1^2 \le x_2 \le x_0 r_2^2$. So, f and g coincide on $[x_0, x_2]$, where $(x_n)_{n\in\mathbb{Z}}$ is defined as in Theorem 1.7. Using the same reasoning as in the previous case it will follow that f and g coincide on $\mathbb{R}$.

c) If $r_1 < 1 < r_2$, we consider the equation

$$f \circ f(x) - af(x) + bx = 0$$

with continuous and increasing solutions given by Theorem 1.8 (case $f(x) > x$).

Let f and g two solutions which coincide on $[0, a']$. It follows from the hypothesis that f and g have no fixed points. If $f(x) > x$, for any $x \in \mathbb{R}$ it follows that $g(x) > x$, for any $x \in \mathbb{R}$ (f and g coincide on $[0, a']$). Choose $x_1 = a'$, $x_0 = 0$. We can define $(x_n)_{n\in\mathbb{Z}}$ as in Theorem 1.8; $x_n \underset{n}{\to} \infty$ and $x_{-n} \underset{n}{\to} -\infty$. We can define f_0 and g_0 by $f_0 = f|_{[0,x_1]}$ and $g_0 = g|_{[0,x_1]}$. The functions $f_0, g_0 : [0, x_1] \to [x_1, x_2]$ are continuous, bijective and fulfill (α). Defining $(f_n)_{n\in\mathbb{Z}}$ as in Theorem 1.8, it will follow that f_n are continuous, bijective and fulfill (α). Similarly to a) one can prove inductively that $f_n = g_n$, for all $n \in \mathbb{Z}$. Therefore $f = g$ on $\mathbb{R}$.

We shall prove now that if f and g coincide on $[b', 0]$, they coincide on $\mathbb{R}$ (where $b' < 0$). Let $(x_n)_{n\in\mathbb{Z}}$ be the sequence which appears in the construction of the solution in Theorem 1.8. We choose $x_0 = 0$ and $x_1 = -b \cdot b'$, $x_1 > 0$. Let us prove that $f(x) = g(x)$ for $x \in [0, -b \cdot b']$. We have $x_1 - ax_0 + bx_{-1} = 0$; $x_0 = 0 \Rightarrow x_{-1} = b'$. Hence, considering $(f_n)_{n\in\mathbb{Z}}$ and $(g_n)_{n\in\mathbb{Z}}$ (the functions which appear in the construction of the solution in Theorem 1.8), we have $f_{-1} : [x_{-1}, 0] \to [0, x_1]$ and $g_{-1} : [x_{-1}, 0] \to [0, x_1]$. It is clear that $f_{-1} = g_{-1}$

implies the fact that $f_{-1}^{-1}(x) = g_{-1}^{-1}(x)$ for $x \in [0, x_1]$. But

$$f_{-1}^{-1}(x) = -\frac{1}{b}\left(ax + f_0(x)\right), \quad g_{-1}^{-1}(x) = -\frac{1}{b}\left(ax + g_0(x)\right),$$

where $f_0 : [0, x_1] \to [x_1, x_2]$, $g_0 : [0, x_1] \to [x_1, x_2]$. Hence $f_0(x) = g_0(x)$, for all $x \in [0, x_1]$. According to the fact which has been proved, it follows that $f = g$ on $\mathbb{R}$.

The case $f(x) < x$, for $x \in \mathbb{R}$, is similar. □

Remark. Concerning the uniqueness properties of the solutions of a functional equation, the reader may consult the papers [4], [7] and, especially, [6].

3. QUALITATIVE PROPERTIES OF THE SOLUTIONS

In this paragraph, we shall establish that solutions of equation

$$f \circ f(x) + af(x) + bx = 0, \tag{E1}$$

which are monotonous, are also continuous if $b \neq 0$ and $a = 0$. We shall look for conditions in general case such that a monotonous solution of equation (E1) becomes continuous. At the end of this paragraph, we shall give some topological properties for the set of the solutions of equation (E1) if $a \neq 0$, $b \neq 0$.

A) First, we study the case $a = 0$ and $b \neq 0$. So far, we studied only the continuous solutions of equation (E1). In Theorems 3.1–3.5 one can consider discontinuous solutions (namely, we give conditions such that monotonous solutions becomes continuous).

THEOREM 3.1. *Let us consider equation* (E1). *Then if* $a = 0$, $b \neq 0$, *every monotonous solution is also continuous.*

Proof. a) If $b > 0$ the equation becomes $f \circ f(x) = -bx$, $b > 0$. Since f is monotonous, this equality is absurd.

b) If $b < 0$ the equation becomes $f \circ f(x) = -bx$, $-b > 0$. Obviously, $-bx$ is bijective, so f is bijective. Because f is monotonous and bijective it is continuous. □

B1) Let us consider the case $a \neq 0$, $b \neq 0$. First, we study the case $\Delta < 0$.

THEOREM 3.2. *If the characteristic equation of the functional equation* (E1) *has no real roots* (*so the equation* $r^2 + ar + b = 0$ *has the discriminant* $\Delta < 0$), *then equation* (E1) *has no monotonous solutions.*

Proof. Let us suppose for n contradiction that there exists f monotonous which satisfies equation (E1). We have $\Delta < 0 \Rightarrow b > 0$. Hence $-af(x) = f \circ f(x) + bx$. Next, $a < 0 \Leftrightarrow f \uparrow$ (i.e., f is increasing). The equation becomes $f \circ f(x) - af(x) + bx = 0$, $a, b > 0$. Consequently, $bx = af(x) - f \circ f(x) \Rightarrow$

$ax - f(x) \uparrow$ on $A = \operatorname{Im} f$. Then, for $x > y$ we have $ax - fx > ay - f(y)$. Hence $f(x) - f(y) < a(x-y)$ for $x > y$, $x, y \in \operatorname{Im} f$.

Let

$$B_0 = \left\{ \frac{f(x) - f(y)}{x - y} \,\middle|\, (x,y) \in \mathbb{R} \times \mathbb{R},\ x > y \right\}.$$

B_0 is bounded above by a, so there exists $\lambda_0 = \sup B_0$, $\lambda_0 > 0$. Hence

$$\frac{f(x) - f(y)}{x - y} \leq \lambda_0 \quad \text{for all } x > y,\ x, y \in \operatorname{Im} f.$$

Let $x > y$, $x, y \in \operatorname{Im} f$. It follows that $f(x), f(y) \in \operatorname{Im} f$

$$(\beta) \qquad \frac{ff(x) - ff(y)}{f(x) - f(y)} = a - \frac{b}{\frac{f(x)-f(y)}{x-y}} \leq a - \frac{b}{\lambda_0}.$$

Write $B_1 = \left\{ \frac{ff(x)-ff(y)}{f(x)-f(y)} \,\middle|\, (x,y) \in \mathbb{R} \times \mathbb{R},\ x > y \right\}$. Then it is obvious that $B_1 \subset B_0$. It is obvious that there exists $\lambda_1 = \sup B_1$. Since $B_1 \subset B_0 \Rightarrow \lambda_1 \leq \lambda_0$ and $\lambda_1 \geq 0$. Taking into account (β) we get

$$\lambda_1 \leq a - \frac{b}{\lambda_0}.$$

In the same way one can obtain the sets

$$B_n = \left\{ \frac{f^{n+1}(x) - f^{n+1}(y)}{f^n(x) - f^n(y)} \,\middle|\, (x,y) \in \mathbb{R} \times \mathbb{R},\ x > y \right\}.$$

Obviously, $B_n \subset B_{n-1} \subset \cdots \subset B_1 \subset B_0$. Then, there exists $\lambda_n = \sup B_n$. Since $B_n \subset B_{n-1} \Rightarrow \lambda_n \leq \lambda_{n-1}$ and $\lambda_n \geq 0$. But

$$\frac{f^{n+1}(x) - f^{n+1}(y)}{f^n(x) - f^n(y)} = a - \frac{b}{\frac{f^n(x)-f^n(y)}{f^{n-1}(x)-f^{n-1}(y)}} \Rightarrow$$

$$\Rightarrow \frac{f^{n+1}(x) - f^{n+1}(y)}{f^n(x) - f^n(y)} \leq a - \frac{b}{\lambda_{n-1}} \Rightarrow \lambda_n \leq a - \frac{b}{\lambda_{n-1}}.$$

Hence it is obvious that the sequence λ_n is decreasing and all the terms are positive so it is convergent. Let λ the limit of the sequence. Passing to the limit in the previous relation we obtain

$$\lambda \leq a - \frac{b}{\lambda} \Rightarrow \lambda^2 - a\lambda + b \leq 0.$$

Taking into account that $\Delta < 0$, this inequality is absurd. Similar proof if $a > 0 \Leftrightarrow f \downarrow$. $\square$

B2) We shall now deal with the case $\Delta > 0$.

Theorem 3.3 (Case $0 < r_1 \le r_2$). *Let f be an increasing function, which satisfies* (E1). *If, additionally, for*

$$B_0 = \left\{ \frac{f(x) - f(y)}{x - y} \,\middle|\, (x, y) \in \operatorname{Im} f \times \operatorname{Im} f,\ x > y \right\},$$
$$B_1 = \left\{ \frac{f \circ f(x) - f \circ f(y)}{f(x) - f(y)} \,\middle|\, (x, y) \in \operatorname{Im} f \times \operatorname{Im} f,\ x > y \right\}$$

we have $\lambda = \sup B_0 = \sup B_1$ (if this supremum is bounded), then f is continuous.

Proof. The equation can be written $f \circ f(x) - af(x) + bx = 0$, with $a, b > 0$. Hence $af(x) - f \circ f(x) \uparrow$ or $ax - f(x) \uparrow$ on $A = \operatorname{Im} f$. Thus

$$0 \le \frac{f(x) - f(y)}{x - y} \le a \quad \text{for all } x > y,\ x, y \in A.$$

We write $\lambda = \sup B_0$. According to the hypothesis, we have $\lambda = \sup B_1$. Let us suppose that $\lambda > r_2$. We have

$$\frac{f \circ f(x) - f \circ f(y)}{f(x) - f(y)} = a - \frac{b}{\frac{f(x)-f(y)}{x-y}} \quad \text{for all } x, y \in \operatorname{Im} f.$$

Consequently,

$$\frac{f \circ f(x) - f \circ f(y)}{f(x) - f(y)} \le a - \frac{b}{\lambda}.$$

But $a - \frac{b}{\lambda} < \lambda$ ($\lambda^2 - a\lambda + b > 0$, because $\lambda > r_2$). This contradicts the fact that $\lambda = \sup B_1$. It follows that $\lambda \le r_2$. Hence

$$\frac{f(x) - f(y)}{x - y} \le r_2 \quad \text{for all } x > y,\ x, y \in \operatorname{Im} f.$$

Let us suppose for a contradiction that there exists $x_0, y_0 \in \mathbb{R}$ with $x_0 > y_0$, such that

$$\frac{f(x_0) - f(y_0)}{x_0 - y_0} > r_2.$$

Then $f(x_0), f(y_0) \in \operatorname{Im} f$ and

$$\frac{f \circ f(x_0) - f \circ f(y_0)}{f(x_0) - f(y_0)} = a - \frac{b}{\frac{f(x_0)-f(y_0)}{x_0-y_0}} > a - \frac{b}{r_2}.$$

But $a - \frac{b}{r_2} = r_1 + r_2 - \frac{r_1 r_2}{r_2} = r_2$. Contradiction. Hence

$$\frac{f(x) - f(y)}{x - y} \le r_2 \quad \text{for } x > y,\ x, y \in \mathbb{R}.$$

This means that f is continuous. □

Theorem 3.4 (Case $r_2 \leq r_1 < 0$). *Let f be a function which satisfies* (E1). *If, additionally the sets B_0 and B_1 from Theorem* 3.3 *have the same infimum* (*if this infimum is bounded*) *then f is continuous.*

Proof. The functional equation becomes

$$f \circ f(x) + af(x) + bx = 0, \tag{E2}$$

$a, b > 0$. Obviously, f is decreasing ($f \downarrow$). Since $-bx = f \circ f(x) + a \cdot f(x) \Rightarrow f \circ f(x) + af(x) \downarrow$. But $f \downarrow \Rightarrow f(x) + ax \uparrow$ on $\operatorname{Im} f$

$$\Rightarrow \frac{f(y) - f(x)}{x - y} < a, \quad x, y \in \operatorname{Im} f,\ x > y.$$

Consequently, there exists

$$\lambda = \sup \left\{ \frac{f(y) - f(y)}{x - y} \,\middle|\, (x, y) \in \operatorname{Im} f \times \operatorname{Im} f,\ x > y \right\}.$$

According to the hypothesis

$$\lambda = \sup \left\{ \frac{f \circ f(x) - f \circ f(y)}{f(y) - f(x)} \,\middle|\, (x, y) \in \operatorname{Im} f \times \operatorname{Im} f,\ x > y \right\}.$$

Hence $\lambda \leq a$ and $\lambda > 0$.

Let us suppose for a contradiction that $\lambda > -r_2$. For $x, y \in \operatorname{Im} f$,

$$\frac{f \circ f(x) - f \circ f(y)}{f(x) - f(y)} = -a - \frac{b}{\frac{f(x)-f(y)}{x-y}}, \quad x > y.$$

We write $f(x) = u$ and $f(y) = v \Rightarrow u < v$. Hence

$$\frac{f(u) - f(v)}{u - v} = -a - \frac{b}{\frac{f(x)-f(y)}{x-y}} \Rightarrow \frac{f(u) - f(v)}{v - u} =$$

$$= a - \frac{b}{\frac{f(y)-f(x)}{x-y}} \Rightarrow \frac{f(u) - f(v)}{v - u} < a - \frac{b}{\lambda}.$$

We have

$$a - \frac{b}{\lambda} < \lambda,$$

because $\lambda^2 - a\lambda + b > 0$. The inequality

$$a - \frac{b}{\lambda} < \lambda$$

contradicts the definition of λ. Hence $\lambda \leq -r_2$. Then it follows that

$$\frac{f(y) - f(x)}{x - y} \leq -r_2 \quad \text{for any } x, y \in \operatorname{Im} f,\ x > y.$$

If there exist $x_0 > y_0$, $x_0, y_0 \in \mathbb{R}$, such that

$$\frac{f(y_0) - f(x_0)}{x_0 - y_0} > -r_2,$$

then obviously $f(x_0), f(y_0) \in \operatorname{Im} f$, $f(y_0) > f(x_0)$. However, we have

$$\frac{f \circ f(x_0) - f \circ f(y_0)}{f(y_0) - f(x_0)} = a - \frac{b}{\frac{f(y_0)-f(x_0)}{x_0-y_0}} > a - \frac{b}{-r_2}.$$

But $a - \frac{b}{-r_2} = -r_2$. Contradiction. Hence

$$\frac{f(y) - f(x)}{x - y} \leq -r_2, \quad \text{for all } x, y \in \mathbb{R},\ x > y.$$

This means that f is continuous on $\mathbb{R}$. □

THEOREM 3.5 (Case $r_1 < 0 < r_2$). *Let us consider the functional equation*

$$\text{(E3)} \qquad f \circ f(x) - af(x) - bx = 0,$$

with $a, b > 0$. Then

a) *Every decreasing solution is continuous.*

b) *Let us suppose that there exists an increasing solution f of equation* (E3).

Consider the sets

$$C_0 = \left\{ \frac{f(x) - f(y)}{x - y} \,\middle|\, (x, y) \in B \times B,\ x > y \right\} \text{ where } B = f(\operatorname{Im} f),$$

and

$$C_1 = \left\{ \frac{f \circ f(x) - f \circ f(y)}{f(x) - f(y)} \,\middle|\, (x, y) \in B \times B \right\}.$$

Let us additionally suppose that if $\lambda_1 = \inf C_0 > -\infty$ then $\lambda_1 = \inf C_1$ and if $\lambda_2 = \sup C_0 < \infty$ then $\lambda_2 = \sup C_1$. Then f is continuous.

Proof. a) Let us suppose that f is decreasing. Since $f \circ f$ is increasing it follows that $af(x) + bx$ is increasing, so, if $x > y$, we have

$$af(x) + bx > af(y) + by \Rightarrow 0 < \frac{f(y) - f(x)}{x - y} < \frac{b}{a}.$$

This shows that f is continuous.

b) Let us suppose that f is increasing. Firstly we shall prove that $f(x) - f(y) = r_2(x - y)$ for all $x > y$, $x, y \in f(\operatorname{Im} f)$. Denote $A = \operatorname{Im} f$. Since

$f \circ f(x) - af(x) \uparrow$, we have $f(x) - ax \uparrow$ on A. Hence for all $x > y$,

$$x, y \in \operatorname{Im} f \Rightarrow \frac{f(x) - f(y)}{x - y} \geq a.$$

$$\frac{f \circ f(x) - f \circ f(y)}{f(x) - f(y)} = a + \frac{b}{\frac{f(x)-f(y)}{x-y}} \Rightarrow \frac{f \circ f(x) - f \circ f(y)}{f(x) - f(y)} \leq a + \frac{b}{a}.$$

Then for all $x > y$, $x, y \in f(A)$, there exist $p_1 < p_2$, with $p_1 > 0$, such that

$$p_1 < \frac{f(x) - f(y)}{x - y} < p_2.$$

Then $\lambda_1 = \inf C_0$ and $\lambda_2 = \sup C_0$, $0 < \lambda_1 \leq \lambda_2$. Let $x, y \in B$, $x > y$

$$\frac{f \circ f(x) - f \circ f(y)}{f(x) - f(y)} = a + \frac{b}{\frac{f(x)-f(y)}{x-y}} \leq a + \frac{b}{\lambda_1}.$$

According to the choice of λ_2 it is obvious that $a + \frac{b}{\lambda_1} \geq \lambda_2$. Anyway we have

$$\frac{f \circ f(x) - f \circ f(y)}{f(x) - f(y)} = a + \frac{b}{\frac{f(x)-f(y)}{x-y}} \geq a + \frac{b}{\lambda_2}.$$

According to the choice of λ_1 it follows that $a + \frac{b}{\lambda_2} \leq \lambda_1$. Because

$$a + \frac{b}{\lambda_1} \geq \lambda_2$$

we have $a\lambda_1 + b \geq \lambda_1\lambda_2$. Because

$$a + \frac{b}{\lambda_2} \leq \lambda_1$$

we have $a\lambda_2 + b \leq \lambda_1\lambda_2$. Consequently, $a\lambda_2 + b \leq a\lambda_1 + b$ and thus it follows that $\lambda_2 \leq \lambda_1$. It follows from these facts and from definition of λ_1 and λ_2 that $\lambda_1 = \lambda_2$. Then we have

$$a + \frac{b}{\lambda} = \lambda.$$

Hence $\lambda = r_2$. Then we have

$$\frac{f(x) - f(y)}{x - y} \leq r_2,$$

and

$$\frac{f(x) - f(y)}{x - y} \geq r_2 \quad \text{for all } x > y,\ x, y \in f(A).$$

This means

$$\frac{f(x) - f(y)}{x - y} = r_2, \quad x, y \in f(A).$$

Then, for all $x, y \in A$, $x > y$,

$$\frac{f(x)-f(y)}{x-y} = \frac{b}{-a+\left(\frac{f\circ f(x)-f\circ f(y)}{f(x)-f(y)}\right)} = \frac{b}{-a+r_2}.$$

Hence

$$\frac{f(x)-f(y)}{x-y} = r_2 \quad \text{for all } x, y \in A,\ x > y.$$

Let $x, y \in \mathbb{R}$ be such that $x > y$ (obviously $f(x) > f(y)$ and $f(x)$, $f(y) \in A$). We write $f(x) = u$ and $f(y) = v \Rightarrow$

$$\Rightarrow \frac{f(x)-f(y)}{x-y} = \frac{b}{-a+\left(\dfrac{f\circ f(x)-f\circ f(y)}{f(x)-f(y)}\right)} = \frac{b}{-a+\dfrac{f(u)-f(v)}{u-v}},$$

for $u, v \in A$, $u > v$,

$$\Rightarrow \frac{f(x)-f(y)}{x-y} = \frac{b}{-a+r_2} = r_2 \quad \text{for all } x, y \in \mathbb{R},\ x > y.$$

Hence $f(x) = r_2 x + c$, for all $x \in \mathbb{R}$. Obviously, f is continuous. □

Remark. We have a similar result in the other case, i.e.,

$$f \circ f(x) + af(x) - bx = 0, \quad a, b > 0.$$

Remark. We are mainly concerned with continuous solutions of our functional equation. Of course, discontinuous solutions do exist.

We shall present an **example of discontinuous solutions of equation (E1)**.

Assume $a, b \in \mathbb{Q}$ are such that $\Delta > 0$ and $\sqrt{\Delta} \notin \mathbb{Q}$. Then r_1 and r_2 belong to the field $\mathbb{Q}[\sqrt{\Delta}]$. Let us define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} r_1 x & \text{if } x \in \mathbb{Q}[\sqrt{\Delta}], \\ r_2 x & \text{if } x \notin \mathbb{Q}[\sqrt{\Delta}]. \end{cases}$$

a) The function f is continuous only at $x = 0$.

Indeed, $\mathbb{Q}[\sqrt{\Delta}]$ and $\mathbb{R} \setminus \mathbb{Q}[\sqrt{\Delta}]$ are dense in $\mathbb{R}$ and the continuity at some point $x \neq 0$ would lead to a contradiction.

b) The function f satisfies equation (E1) because $x \in \mathbb{Q}[\sqrt{\Delta}] \Rightarrow r_1 x \in \mathbb{Q}[\sqrt{\Delta}]$ and $x \notin \mathbb{Q}[\sqrt{\Delta}] \Rightarrow r_2 x \notin \mathbb{Q}[\sqrt{\Delta}]$.

We end with some topological properties for the set of solutions of the equation (E1). As usual, we shall write

$$C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous}\}.$$

It is well known that $C(\mathbb{R})$ becomes a Fréchet space with the metric

$$d(f,g)=\sum_{n\geq 1}\frac{1}{2^n}\frac{\sup\limits_{x\in[-n,n]}|f(x)-g(x)|}{1+\sup\limits_{x\in[-n,n]}|f(x)-g(x)|},$$

which generates the topology of uniform convergence on compact sets. If $f_n \to f$ in this topology, we shall write $f_n \xrightarrow{u.c.} f$. From now on, we understand by solution of equation (E1) a *continuous* function, which satisfies this functional equation.

We obtain a general result which holds in all situations:

Theorem 3.6. *The set of solutions of the equation*

$$f\circ f+af(x)+bx=0$$

is closed in the space $C(\mathbb{R})$.

Proof. First, let us prove that the function

$$H:C(\mathbb{R})\times C(\mathbb{R})\to C(\mathbb{R}),\quad H(f,g)=f\circ g$$

is a continuous function in $C(\mathbb{R})$. More precisely, we shall prove that if $f_n \xrightarrow{u.c} f$ and $g_n \xrightarrow{u.c} g$ then $f_n\circ g_n \xrightarrow{u.c} f\circ g$.

Let K be a compact. We must show that $f_n\circ g_n \to f\circ g$ uniformly on K. Let $\varepsilon>0$. We must find $n(\varepsilon)$ such that for $n\geq n(\varepsilon)$ and $x\in K$ one has

$$|(f\circ g)(x)-(f_n\circ g_n)(x)|<\varepsilon. \tag{3.1}$$

We denote $A(x)=|(f\circ g)(x)-(f_n\circ g_n)(x)|$. Then for any $x\in\mathbb{R}$ and $n\in\mathbb{N}$ one has

$$A(x)\leq|(f\circ g)(x)-(f\circ g_n)(x)|+|(f\circ g_n)(x)-(f_n\circ g_n)(x)|\,. \tag{3.2}$$

First, we shall prove that there exists a compact K_0 such that for any n one has $g_n(K)\subset K_0$, $g(K)\subset K_0$. Because $g_n \xrightarrow{u} g$ on K, there exists n_1 such that if $n\geq n_1$ and $x\in K$ one has

$$|g(x)-g_n(x)|<1. \tag{3.3}$$

It follows from (3.1) that $-1+g(x)<g_n(x)<1+g(x)$. We write $K_2=g(K)$ and we notice that K_2 is also compact.

$\Rightarrow g_n(x)\in[-1+\inf K_2,\,1+\sup K_2]$ for all $n\geq n_1$ and thus we have

$$g_n(K)\subset[-1+\inf K_2,\,1+\sup K_2]=[a,\,b]\quad\text{for all } n\geq n_1.$$

With this notation we have

$$g_n(K)\subset[a,b]\,,\quad n\geq n_1. \tag{α}$$

Obviously, if $n \leq n_1$ there exist $a_n, b_n \in \mathbb{R}$ such that $g_n(K) \subset [a_n, b_n]$. Then it follows that

$$g_n(K) \subset [\min(a_1, a_2, a_3, \ldots, a_{n_1}, a), \max(b_1, b_2, b_3, \ldots, b_{n_1}, b)] = K_0.$$

Finally, we have

$$g_n(K) \subset K_0, \quad \forall n \geq 1. \tag{3.4}$$

Next, we shall prove that for $\varepsilon > 0$ there exists $n_2(\varepsilon)$ such that if $n \geq n_2(\varepsilon)$ and $x \in K$ one has

$$|f(g(x)) - f(g_n(x))| < \frac{\varepsilon}{2}. \tag{3.5}$$

Let $\varepsilon > 0$. Because f is uniformly continuous on K_0 there exists $\delta > 0$ such that for $u, v \in K_0$, $|u - v| < \delta$ it follows that

$$|f(u) - f(v)| < \frac{\varepsilon}{2}. \tag{3.6}$$

Because $g_n \xrightarrow{u.c} g$, for an arbitrary $\delta > 0$ there exists $n_2(\varepsilon)$ such that for all $n \geq n_2(\varepsilon)$ and $x \in K$ one has $|g(x) - g_n(x)| < \delta$. From (α) it follows that $g(x)$, $g_n(x) \in K_0$ and then from (3.6) for all $n \geq n_2(\varepsilon)$ and $x \in K$ we have

$$|f(g(x)) - f(g_n(x))| < \frac{\varepsilon}{2}.$$

Now, we shall prove that for $\varepsilon > 0$ there exists $n_3(\varepsilon)$ such that for all $n \geq n_3(\varepsilon)$ and $x \in K$ one has

$$|f(g_n(x)) - f_n(g_n(x))| < \frac{\varepsilon}{2}. \tag{3.7}$$

Because $f_n \xrightarrow{u.c} f$, for an arbitrary $\varepsilon > 0$ there exists $n_3(\varepsilon)$ such that for all $n \geq n_3(\varepsilon)$ and $y \in K_0$ one has

$$|f(y) - f_n(y)| < \frac{\varepsilon}{2}.$$

Obviously for $x \in K$, we have $g_n(x) \in K_0$ and so we have (3.7). It follows from (3.2), (3.5) and (3.7) that for an arbitrary $\varepsilon > 0$ there exists $n(\varepsilon) = \max[n_2(\varepsilon), n_3(\varepsilon)]$ such that for all $n \geq n(\varepsilon)$ and $x \in K$ one has

$$|(f \circ g)(x) - (f_n \circ g_n)(x)| < \varepsilon.$$

Thus it follows that $f_n \circ g_n \to f \circ g$ uniformly on K. Therefore,

$$f_n \circ g_n \xrightarrow{u.c} f \circ g,$$

so H is continuous.

From this one can see that the function $F : C(\mathbb{R}) \to C(\mathbb{R})$, defined by $F(f) = f \circ f + af + b1_{\mathbb{R}}$ is continuous.

Then $F^{-1}(\{0\})$ is a closed set, i.e., the set of solutions of equation (E1), is a closed set. $\square$

Next, we shall establish some topological properties of the set of solutions for each significantly case. We shall discuss these properties, depending on the roots of the characteristic equation associated to the functional equation (E1).

A) *The case* $1 < r_1 < r_2$.

Let $K \subset \mathbb{R}$ an arbitrary compact.

Notation. a) S is the set of solutions of the equation $f \circ f(x) - af(x) + bx = 0$ with $a, b > 0$ (equation (E1)).

b) Let $C(K)$ the space of continuous real functions on a compact K. Then $C(K)$ is a Banach space with the norm

$$\|f\| = \sup_{x \in K} |f(x)|.$$

c) $S_K = \{f : K \to \mathbb{R} \mid \text{there exists } f_0 \in S \text{ such that } f(x) = f_0(x) \text{ for all } x \in K\}$.

THEOREM 3.7. *The set* S_K *has the following topological properties*:

a) S_K *is equicontinuous*;

b) S_K *is relatively compact in* $C(K)$. (The closure is compact.)

Proof. a) We have to prove that for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$, such that, if $|x - y| < \delta_\varepsilon \Rightarrow |f(x) - f(y)| < \varepsilon$, for every $f \in S_K$.

Since $f = f_0$ on K and $f_0 \in S$, according to Theorem 1.a, we have for every $x, y \in K$

$$|f_0(x) - f_0(y)| \le r_2 |x - y|. \tag{3.8}$$

Hence $|f(x) - f(y)| \le r_2|x - y|$, for every $x, y \in K$. Then $|x - y| < \delta_\varepsilon \Rightarrow |f(x) - f(y)| < r_2 \cdot \delta_\varepsilon < \varepsilon$ (see (3.8)). The sufficient condition is

$$\delta_\varepsilon < \frac{\varepsilon}{r_2}.$$

Obviously, δ_ε does not depend on f, so S_K is equicontinuous.

b) Now we shall prove that S_K is a bounded set. Let $f_0 \in S \Rightarrow |f_0(x) - f_0(0)| \le r_2|x|$ (see (3.8). Since $f_0(0) = 0$ we have

$$|f_0(x)| \le r_2|x|. \tag{3.9}$$

It is obvious that for $f \in S_K$ there exists $f_0 \in S_K$ with $f_0(x) = f(x)$. Therefore,

$$|f(x)| \le r_2|x|, \tag{3.10}$$

for all $x \in K$. It is obvious that we can find M such that $|x| < M$ for all $x \in K$ (K is compact in $\mathbb{R}$, so it is bounded) $\Rightarrow f(x) \le r_2 M$ (see (3.10). This bound is the same for all the functions $f \in S_K$. So, according to Arzela-Ascoli Theorem, S_K is relatively compact. □

Next, we shall establish some topological properties of S (in $C(\mathbb{R})$).

Theorem 3.8. *The set S is a compact set in $C(\mathbb{R})$.*

Proof. According to Theorem 3.6 it is enough to prove that S is relatively compact in $C(\mathbb{R})$. Because $C(\mathbb{R})$ is a metric space, it is enough to prove that every sequence in S has a Cauchy subsequence. Let $K_n = [-n, n]$. Obviously

$$\mathbb{R} = \bigcup_{n\geq 1} K_n$$

and $K_n \subset \overset{\circ}{K}_{n+1}$. According to Theorem 3.7 we have

$$S_{K_n} \text{ is relatively compact in } C(K_n). \tag{3.11}$$

Let $(f_p)_p$ be a sequence in S. According to (3.11), $(f_p)_p$ furnishes the sequence $\left(f_p|_{K_n}\right)_p$ which has a Cauchy subsequence with respect to the norm on $C(K_n)$.

Write this subsequence in the form $(u_p^1)_p \subset C(\mathbb{R})$, hence $(u_p^1)_p$ is Cauchy in the seminorm $\| \ \|_{K_1}$ on $C(\mathbb{R})$ given by the formula

$$\|f\|_{K_1} = \sup_{x\in K_1} |f(x)| .$$

Continuing, $(u_p^1)_p$ furnishes a subsequence $(u_p^2)_p \subset (u_p^1)_p$ which is Cauchy in the seminorm $\| \ \|_{K_2}$ on $C(\mathbb{R})$, given by the formula

$$\|f\|_{K_2} = \sup_{x\in K_2} |f(x)| .$$

Continuing this process, we obtain at step h, the subsequence $(u_p^h)_p$ of the preceding sequence which is Cauchy in the seminorm $\| \ \|_{K_h}$ given by

$$\|f\|_{K_h} = \sup_{x\in K_h} |f(x)| .$$

The final step consists in considering the sequence $(v_p)_p$ given by $v_p = u_p^p$, which is Cauchy in any $\| \ \|_{K_p}$ for $p \geq 1$, and is a subsequence of all $(u_p^i)_p$, $i \geq 1$. Then $(v_p)_p$ is a Cauchy sequence, with respect to the distance d which exists on $C(\mathbb{R})$. Therefore, S is relatively compact in $C(\mathbb{R})$. Thus it follows that S is compact in $C(\mathbb{R})$. □

Remark 1. We take a sequence of compact sets $K_n \subset \overset{\circ}{K}_{n+1}$, $K_n \subset \mathbb{R}$, $n \geq 1$ and

$$K = \bigcup_{n\geq 1} K_n.$$

Let us denote by $C(K)$ the space of continuous functions defined on K. Then $C(K)$ is a Fréchet space with the distance

$$d = \sum_{n\geq 1} \frac{1}{2^n} \frac{\sup\limits_{x\in K_n} |f(x) - g(x)|}{1 + \sup\limits_{x\in K_n} |f(x) - g(x)|}.$$

Let the topology on $C(K)$ be the topology of uniform convergence on compact sets. Then, we denote

$S_K = \{f : K \to \mathbb{R} \mid \text{there exists } f_0 \in S \text{ such that } f(x) = f_0(x) \text{ for all } x \in K\}$.

It can be proved in the same way that S_K is relatively compact in $C(K)$.

Remark 2. The case $r_2 < r_1 < -1$ is similar to the previous one.

B) *The case* $r_1 < 1 < r_2$.
Let

$$K = \bigcup_{n\geq 1} K_n$$

with $K_n \subset \overset{\circ}{K}_{n+1}$, K_n compacts, $K_n \subset \mathbb{R}$.

Let us consider $S \subset C(\mathbb{R})$, $S_{K_n} \subset C(K_n)$ and $S_K \subset C(K)$ with the same significance as in case A. The difference with respect to the previous case is that $f(0)$ can take any real value. Therefore S_K is no longer relatively compact in $C(K)$ because S_K is no longer a bounded set. (In fact, neither S_{K_n} are no longer bounded sets.)

We shall try to find similar topological properties in this case.

Notations. d) $S(K,m) = \{f : K \to \mathbb{R} \mid$ there exists $f_0 \in S$ such that $f(x) = f_0(x)$ for all $x \in K$ and $f_0(0) \in [-m, m]\}$, $m \in \mathbb{N}$ ($S(K,m) \subset C(K)$ for $m \in \mathbb{N}$).

e) $S(K_n, m) = \{f : K_n \to \mathbb{R} \mid$ there exists $f_0 \in S$ such that $f(x) = f_0(x)$ for all $x \in K_n,\ f_0(0) \in [-m, m]\}$, $m \in \mathbb{N}$ ($S(K,m) \subset C(K_n)$ for all $m \in \mathbb{N}$).

f) $S(m) = \{f : \mathbb{R} \to \mathbb{R} \mid f \in S$ and $f(0) \in [-m, m]\}$, $m \in \mathbb{N}$.

With this notation we shall prove that S_K is the limit of an increasing sequence of relatively compact sets of $C(K)$. Also we shall prove that S_K is the limit of an increasing sequence of relatively compact sets of $C(K)$.

Lemma 3.9. *The sets $S(K_n, m)$ have the following topological properties*:
a) $S(K_n, m)$ *are bounded sets in* $C(K_n)$;
b) $S(K_n, m)$ *are relatively compact sets in* $C(K_n)$.

Proof. a) Let $f \in S(K_n, m)$. Hence there exists $f_0 \in S$ such that $f(x) = f_0(x)$ and $f_0(0) \in [-m, m] \Rightarrow r_1(x - 0) \leq f_0(x) - f_0(0) \leq r_2(x - 0)$, for every

$x \in K_n$ (see Theorem 1.a)

$$\Rightarrow f_0(x) \leq r_2 x + f_0(0) \leq r_2 x + m, \tag{3.12}$$

$$\Rightarrow f_0(x) \geq r_1 x + f(0) \geq r_1 x - m. \tag{3.13}$$

Since $x \in K_n$ we have

$$\text{there exists } M_n = \sup K_n \text{ and } m_n = \inf K_n. \tag{3.14}$$

It follows from (3.12) and (3.14) that $f_0(x) \leq r_2 M_n + m$. It follows from (3.13) and (3.14) that $f_0(x) \geq r_1 m_n - m$.

These bounds do not depend on f, so $S(K_n, m)$ is a bounded set in $C(K_n)$.

b) The proof of the fact that $S(K_n, m)$ is equicontinuous is the same as the proof of Theorem 3.7 point a). According to the Theorem of Arzela-Ascoli, $S(K_n, m)$ is relatively compact in $C(K_n)$. □

THEOREM 3.10. *Let us consider the sets $S(K, m)$ and S_K. Then they have the following topological properties*:

a) *The sets $S(K, m)$ are relatively compact in $C(K)$.*

b) *The set S_K is the limit* (*in the sense of set theory*) *of an increasing sequence* (*with respect to the inclusion relation*) *of relatively compact sets from $C(K)$.*

Proof. a) The proof is analogous to the proof of Theorem 3.8 (see Remark 1).

b) It is obvious that

$$S(K, m) \subset S_K, \tag{3.15}$$

for all $m \in \mathbb{N}$. Also, it is clear that $S(K, m) \subset S(K, m+1)$. Hence, the sequence of sets $S(K, m)$ is increasing (with respect to the inclusion relation). From (3.15) we have

$$\bigcup_{m \geq 1} S(K, m) \subset S_K. \tag{3.16}$$

We shall prove that

$$S_K \subset \bigcup_{m \geq 1} S(K, m).$$

Let $f_0 \in S$ such that $f = f_0$ on K with $f_0(0) \in \mathbb{R}$. There exists m such that $f_0(0) \in [-m, m]$. Then there exists m such that $f \in S(K, m)$, so

$$S_K \subset \bigcup_{m \geq 1} S(K, m). \tag{3.17}$$

It follows from (3.16) and (3.17) that

$$S_K = \bigcup_{m \geq 1} S(K, m).$$

According to Lemma 3.9, the sets $S(K,m)$ are relatively compact in $C(K)$, so S_K is the limit of an increasing sequence of relatively compact sets from $C(K)$. □

CONSEQUENCE 3.11. *The set $S \subset C(\mathbb{R})$ is the limit (in the sense of set theory) of an increasing sequence (with respect to the inclusion relation) of compact sets of $C(\mathbb{R})$.*

Proof. We choose $K_n = [-n, n]$. Obviously,

$$\mathbb{R} = \bigcup_{n \geq 1} K_n.$$

According to Theorem 3.10,

$$S = \bigcup_{m \geq 1} S(m),$$

and the sets $S(m)$ are relatively compact in $C(\mathbb{R})$. (Obviously, $S(m \subset S(m+1)$.) It remains to prove that $S(m)$ are closed in $C(\mathbb{R})$.

Let $m \in \mathbb{N}$ arbitrary taken. Let $f_p \in S(m)$, $f_p \xrightarrow{u.c} f$. We shall prove that $f \in S(m)$. Since $f_p \in S(m)$ we have $f_p \in S$, so

$$f \in S \tag{3.18}$$

(see Theorem 3.6). Let us prove that $f(0) \in [-m, m]$. Since $f_p \in S(m)$. We have

$$-m \leq f_p(0) \leq m. \tag{3.19}$$

Letting $p \to \infty$ in (3.19), we get

$$-m \leq f(0) \leq m. \tag{3.20}$$

It follows from (3.18) and (3.20) that $S(m)$ are closed sets in $C(\mathbb{R})$. Since $S(m)$ are also relatively compact it follows that $S(m)$ are compact in $C(\mathbb{R})$. □

C) *The case* $0 < r_1 < r_2 < 1$.

Let us consider the set $S \subset C(\mathbb{R})$, where S has the same meaning as in previous case. First, we shall give a theorem of uniqueness for this case.

THEOREM 3.12. *Let two solutions of equation* (E1) *which coincide on* $[a', b']$, $[a', b'] \subset (0, \infty)$ *such that*

$$\frac{b'}{a'} \geq \frac{r_2 - r_1 r_2 + r_1^2}{r_1^2}.$$

Then they coincide on $(0, \infty)$.

Proof. Let f and g two solutions which coincide on $[a', b']$. Denote

$$f_1 = f|_{[a',b']}, \quad g_1 = g|_{[a',b']}.$$

Let $f_1(a') = c'$ and $f_1(b') = d'$. Then $f_1 : [a', b'] \to [c', d']$ and $g_1 : [a', b'] \to [c', d']$ are bijective, continuous and increasing. It follows from hypothesis that

$$f_1^{-1} \text{ and } g_1^{-1} \text{coincide on } [c', d'].$$

We shall prove that

$$d' \geq \frac{c'}{r_1}.$$

We know (see Theorem 1.a) that

$$r_2(x - y) \geq f_1(x) - f_1(y) \geq r_1(x - y), \tag{3.21}$$

for $x, y \in [a', b']$; $x > y$. From hypothesis we have $b' \geq a' \left(\frac{r_2 - r_1 r_2 + r_1^2}{r_1^2}\right)$. Hence

$$\frac{r_1 b'}{1 - r_1} \geq a' \left(r_2 + \frac{r_1^2}{1 - r_1}\right) \Rightarrow a' r_2 \leq \frac{r_1^2}{1 - r_1}(b' - a'). \tag{3.22}$$

We put in (3.21) $x = a'$ and $y = 0$. Thus, we obtain

$$f_1(a') \leq a' r_2. \tag{3.23}$$

From (3.22) and (3.23) we have

$$f_1(a') \leq \frac{r_1^2}{1 - r_1}(b' - a'). \tag{3.24}$$

We put in (3.21) $x = b'$ and $y = a'$. Thus, we obtain

$$f_1(b') - f_1(a') \geq r_1(b' - a') \Rightarrow f_1(b') \geq f_1(a') + r_1(b' - a'). \tag{3.25}$$

From (3.24) we obtain

$$f_1(a') + r_1(b' - a') \geq \frac{f_1(a')}{r_1}.$$

From (3.25) we have

$$f_1(b') \geq \frac{f_1(a')}{r_1}, \quad \text{i.e., } d' \geq \frac{c'}{r_1}.$$

Obviously, f^{-1} and g^{-1} fulfill the equation

$$bf^{-1} \circ f^{-1}(x) - af^{-1}(x) + x = 0.$$

The second degree equation associated with our functional equation has the solutions

$$\frac{1}{r_1} > \frac{1}{r_2} > 1$$

and f^{-1} , g^{-1} coincide on $[c', d']$, with

$$\frac{d'}{c'} \geq \frac{1}{r_1}.$$

According to Theorem 3.1 applied for f^{-1} and g^{-1}, f^{-1} and g^{-1} coincide on $(0,\infty)$. Then f and g coincide on $(0,\infty)$. $\square$

Remark 3. We have a similar result if $[b',a'] \subset (-\infty,0)$ such that

$$\frac{b'}{a'} \geq \frac{r_1^2 - r_1 r_2 + r_2}{r_1^2}.$$

Consequence 3.13. *Let two solutions of equation* (E1) *which coincide on $I=[p,q]$, where $p<0$ and $q>0$. Then they coincide on $\mathbb{R}$.*

In the sequel, we shall make some topological considerations. Let $f \in S$. According to Theorem 1.a, f is a contraction. Hence, it appears a new phenomenon over the previous cases.

Notation. g) Let $J_1 \subset J_2 \subset J_3 \subset \cdots \subset J_n$ a sequence of compact intervals, $J_n \subset \mathbb{R}$ such that $0 \in \overset{\circ}{J_1}$ and $J_i \subset \overset{\circ}{J}_{i+1}$. Write

$$J=(a_1,b_1)=\bigcup_{n\geq 1} J_n.$$

Denote by $C(J)$ the space of continuous real functions defined on J.

Next, we take the distance and the topology in the Fréchet space $C(J)$ as in Remark 1. For this reason we put $K_n=J_n$ and $K=J$.

$C(J_n)$ has the same meaning as in notation b) by taking $K=J_n$. On $C(J_n)$ we consider the norm

$$\|f\| = \sup_{x\in J_n} |f(x)|.$$

The set $C(J_n)$ with this norm is, obviously, a Banach space. We write also:

h) $S(J)=\{f: J\to\mathbb{R} \mid \text{there exists } f_0\in S \text{ such that } f(x)=f_0(x) \text{ for all } x\in J\}$, $S(J)\subset C(J)$;

i) $S(J_n)=\{f: J_n\to\mathbb{R} \mid \text{there exists } f_0\in S \text{ such that } f(x)=f_0(x) \text{ for all } x\in J_n\}$, $S(J_n)\subset C(J_n)$.

With this notations we shall prove that $S(J)$ is compact in $C(J)$.

Lemma 3.14. *The set $S(J_n)$ has the following properties*:

a) *$S(J_n)$ is a bounded set in $C(J_n)$.*

b) *$S(J_n)$ is a relatively compact set in $C(J_n)$.*

The *proof* of this lemma is analogous with the proof of Theorem 3.7.

Theorem 3.15. *The set $S(J)$ has the following properties.*

a) *$S(J)$ is relatively compact in $C(J)$.*

b) *$S(J)$ is closed in $C(J)$.*

Proof. a) The proof is similar to the proof of Theorem 3.8 (also see Remark 1).

b) It is obvious that $0 \in \overset{\circ}{J_n}$ for all $n \geq 1$ and $0 \in \overset{\circ}{J}$. First, we shall prove that $S(J_n)$ is closed in $C(J_n)$. Now, let us take an arbitrary $n \geq 1$. Let $f_p \in S(J_n)$ such that $f_p \xrightarrow[p]{u} f$ on J_n. Let us prove that $f \in S(J_n)$, i.e., one has to prove that there exists $f_0 \in S$ such that $f = f_0$ on J_n. In any case there exists $f_{\circ p} \in S$ such that $f_{\circ p} = f_p$ on J_n. We shall prove that for all $\alpha \in J_n$, $f_{\circ p}(\alpha) \in J_n$. According to Theorem 1.a we have

$$r_1(x-0) \leq f_{\circ p}(x) - f_{\circ p}(0) \leq r_2(x-0).$$

Thus, if $x > 0$ we have $f_{\circ p}(x) \leq r_2 x \leq x \leq b_n$, and if $x < 0$ we have $f_{\circ p}(x) \geq r_2 x \geq x \geq a_n$, so for all $\alpha \in J_n \Rightarrow f_{\circ p}(\alpha) \in J_n$. Because $f_{\circ p}(x) \in J_n$ for all $x \in J_n \Rightarrow f(x) \in J_n$ for $x \in J_n$ (J_n is compact). Then, it is possible to speak about $f \circ f(x)$ for all $x \in J_n$. Next

$$f_{\circ p} \circ f_{\circ p} \xrightarrow{u} f \circ f \quad \text{on } J_n. \tag{3.26}$$

(See also the proof of Theorem 3.6.) Since $f_{\circ p}$ is a solution on $\mathbb{R}$ we have $f_{\circ p} \circ f_{\circ p}(x) - a f_{\circ p}(x) + bx = 0$ for all $x \in \mathbb{R}$. Hence

$$\lim_{p \to \infty} f_{\circ p} \circ f_{\circ p}(x) - a f_{\circ p}(x) + bx = 0 \quad \text{for all } x \in J_n. \tag{3.27}$$

According to (3.26) and (3.27), $f \circ f(x) - af(x) + bx = 0$, for all $x \in J_n$. Thus it follows that there exists $f_0 \in S$, such that $f = f_0$ on J_n. Hence $S(J_n)$ is closed in $C(J_n)$.

Let us prove now that $S(J)$ is closed in $C(J)$. We have to prove that if $f_p \in S(J)$ such that $f_p \xrightarrow{u.c} f$ on J, then $f \in S(J)$. Since $f_p \in S(J) \Rightarrow f_p \in S(J_n)$ for all $n \geq 1$. Because $S(J_n)$ is closed and $f_p \xrightarrow{u} f$ on J_n, there exists $f_{0n} \in S$ such that $f = f_{\circ n}$ on J_n, for all $n \in \mathbb{N}^*$. Then $f_{\circ 1} = f_{\circ 2} = \cdots = f_{\circ n}$ on J_1, for all $n \in \mathbb{N}^*$.

Now, we take into account that two solutions which coincide on J_1 coincide everywhere according to Consequence 3.13. (Obviously we have $0 \in \overset{\circ}{J_1}$.) Consequently, $f_{\circ 1} = f_{\circ 2} = \cdots = f_{\circ n}$ on J_1 for all $x \in \mathbb{R}$. Let $x_0 \in J$ arbitrary. There exists n such that $x_0 \in J_n$. Hence $f(x_0) = f_{\circ n}(x_0) = f_{\circ 1}(x_0)$. Then $f(x) = f_{\circ 1}(x)$ for all $x \in J \Rightarrow f \in S(J)$, so $S(J)$ is closed in $C(J)$. □

Finally, we have the following theorems.

THEOREM 3.16. *$S(J)$ is compact in $C(J)$.*

THEOREM 3.17. *S is compact in $C(\mathbb{R})$.*

REFERENCES

[1] K. Baron and W. Jarczyk, *Recent results on functional equations in a single variable perspectives and open problems*. Aequationes Math. **61** (2001), *2*, 1–48.
[2] I. Chiţescu and T. Gâdea, *The functional equation* $f \circ f + af + b1_{\mathbb{R}} = 0$. An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. VI (2010), 279–310.
[3] T. Gîdea, *Contributions to the theory of functional equations*. Ph. D. Thesis.
[4] S. Marcus, *Sur une généralisation des functiones de G. Hamel*. Atti Acad. Naz. Lincei **20** (1956), 584–589.
[5] S. Marcus, *La superposition des functions et l'isométrie de certaines classes de fonctions*. Bull. Math. Soc. Sci. Math. Phys. R.P. Roumaine **1** (1957), 69–76.
[6] S. Marcus, *Sur une classe de fonctions définiès par des inégalités, introduite par M.A. Császár*. Acta Sci. Math. Szeged **19** (1958), 192–218.
[7] S. Marcus, *Généralisation aux fonctions de plusièurs variables, des théorèmes de Alexander Ostrowski et Masuo Hukuhara, concernant les fonctions convexes*. J. Math. Soc. Japan **11** (1959), 171–176.
[8] S. Marcus, *Functions with the Darboux property and functions with connected graphs*. Math. Ann. **101** (1960), 311–317.
[9] L.R. Zhao, *Existence and uniqueness theorem for the solution of the functional equation* $\lambda_1 f(x) + \lambda_2 f^2(x) = F(x)$. J. China Univ. Sci. Tech. **13** (1983), 21–27.

Received 3 June 2010

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