AN INVERSE PROBLEM FOR A NONLINEAR PARABOLIC MODEL

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The purpose of this paper is to study an inverse problem in relation with a nonlinear parabolic model.

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1. PROBLEM SETTLEMENT

The purpose of this paper is to study an inverse problem in relation with a nonlinear parabolic model with application to water infiltration in soils. Namely, we want to determine the initial condition from the available data observed for the solution (soil moisture) in a subset of the flow domain.

The mathematical model we consider is that corresponding to the nonlinear saturated-unsaturated infiltration in a porous medium (see [8]), i.e.,

$$\begin{gathered} \frac{\partial \theta}{\partial t} - \Delta\beta^*(\theta) + \nabla \cdot K(\theta) \ni f \text{ in } Q, \\ \theta(0,x) = u(x) \text{ in } \Omega, \\ (K(\theta) - \nabla\beta^*(\theta)) \cdot \nu - \alpha\beta^*(\theta) \ni 0 \text{ on } \Sigma := (0,T) \times \Gamma, \end{gathered} \tag{1}$$

which is called the *original state system*. The domain $Q = (0,T) \times \Omega$, with Ω an open bounded subset of $\mathbb{R}^N$, $N = 1, 2, 3$.

The function β^* is defined as

$$\beta^*(\theta) := \begin{cases} \int_0^\theta \beta(\zeta)\mathrm{d}\zeta, & \theta < \theta_s, \\ [K_s^*, \infty), & \theta = \theta_s, \end{cases}$$

where β has the following properties

$$\lim_{\theta \nearrow \theta_s} \beta(\theta) = \infty, \quad \lim_{\theta \nearrow \theta_s} \int_0^\theta \beta(\zeta)\mathrm{d}\zeta = K_s^* < \infty.$$

Here θ is the soil moisture, β represents the diffusion coefficient, K the water conductivity and θ_s the saturation value.

The original state system can be written under the abstract form

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}(t) + A\theta(t) = f(t), \text{ a.e. } t \in (0,T), \qquad \theta(0) = u, \tag{2}$$

where $A : D(A) \subset V' \to V'$ is the multivalued operator defined by

$$\langle A\theta, \psi\rangle_{V',V} = \int_\Omega (\nabla\eta - K(\theta)) \cdot \nabla\psi \mathrm{d}x + \int_\Gamma \alpha\eta\psi \mathrm{d}\sigma, \tag{3}$$

for any $\psi \in V$, where $\eta(x) \in \beta^*(\theta(x))$ a.e. $x \in \Omega$ and the domain operator is defined by

$$D(A) = \{\theta \in L^2(\Omega);\ \text{there exists } \eta \in V,\ \eta(x) \in \beta^*(\theta(x)) \text{ a.e. } x \in \Omega\}. \tag{4}$$

The existence and uniqueness of the solution to problem (2) was proved in [3].

We consider a given set of observations for $\theta(t,x)$ denoted by $\theta^{obs}(t,x)$, on the subset $Q_0 = (0,T) \times \Omega_0 \subseteq Q$ (where $\Omega_0 \subseteq \Omega$), $\theta^{obs} \in L^2(Q_0)$.

The problem of the initial data identification is expressed in mathematical terms as the cost functional minimization

$$\min_{u \in U} J(u), \tag{P}$$

for

$$J(u) = \int_{\Omega_0} \int_0^T \left(\theta(t,x) - \theta^{obs}(t,x)\right)^2 \mathrm{d}x\mathrm{d}t + \mu \int_\Omega u^2(x)\mathrm{d}x,$$

subject to (1) where the admissible set U consists of

$$U = \{u \in L^\infty(\Omega);\ 0 \le u \le \theta_s\}. \tag{5}$$

The weight μ is used for a greater influence of one of the terms in the functional cost (e.g., $\mu > 1$ implies that the control term has a higher importance).

The steps we follow are the setting of the cost functional, proof of the existence of an optimal pair and determination of the optimality conditions for the approximating problem.

Theorem 1. *Let $f \in L^2(0,T;V')$. Then (P) has at least one solution.*

Proof. We see that $J(u) \ge 0$ for any $u \in U$. Therefore, there exists $d = \inf\limits_{u \in U} J(u) \ge 0$. We take a minimizing sequence $\{u_n\}_{n\ge1} \subset U$, i.e.,

$$d \le J(u_n) \le d + \frac{1}{n}, \quad \forall n \ge 1. \tag{6}$$

The above relationship gives

$$d \le \int_{Q_0} \Big(\theta_n(t,x) - \theta^{obs}(t,x)\Big)^2 \mathrm{d}x\mathrm{d}t + \mu \int_\Omega u_n^2 \mathrm{d}x \le d + \frac{1}{n}, \quad \forall n \ge 1, \tag{7}$$

where θ_n is the solution to the Cauchy problem (2) corresponding to u_n, i.e.,

$$\begin{aligned} \frac{\mathrm{d}\theta_n}{\mathrm{d}t}(t) + A\theta_n(t) &= f(t), \text{ a.e. } t \in (0,T), \\ \theta_n(0) &= u_n. \end{aligned} \tag{8}$$

As proved in [8], system (8) admits an unique solution $\theta_n \in W^{1,2}(0,T;V') \cap L^2(0,T;V)$, $0 \le \theta_n \le \theta_s$ which satisfies the estimate

$$\int_0^T \|\theta_n(t)\|_V^2 \,\mathrm{d}t + \int_\Omega j(\theta_n)\mathrm{d}x + \int_0^T \|\eta_n(t)\|_V^2 \,\mathrm{d}t + \int_0^T \left\|\frac{\mathrm{d}\theta_n}{\mathrm{d}t}(t)\right\|_{V'}^2 \mathrm{d}t \le C, \tag{9}$$

where $\eta_n \in \beta^*(\theta_n)$ a.e. on Q.

Therefore, we can extract a subsequence still denoted $\{\theta_n\}_{n\ge1}$ such that

$$\theta_n \to \theta^* \text{ weakly in } W^{1,2}(0,T;V') \cap L^2(0,T;V), \tag{10}$$

$$\frac{\mathrm{d}\theta_n}{\mathrm{d}t} \to \frac{\mathrm{d}\theta^*}{\mathrm{d}t} \text{ weakly in } L^2(0,T;V'). \tag{11}$$

These two weakly convergences lead us to

$$\theta_n \to \theta^* \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{12}$$

We also have

$$K(\theta_n) \to K(\theta^*) \text{ strongly in } L^2(0,T;V), \tag{13}$$

$$\eta_n \to \eta^* \text{ weakly in } L^2(0,T;V), \tag{14}$$

where $\eta_n \in \beta^*(\theta_n)$ a.e. on Q and $\eta^* \in \beta^*(\theta^*)$ a.e. on Q (see [5]).

Since $u_n \in U$, then $0 \le u_n \le \theta_s$ and

$$u_n \to u^* \text{ weak-star in } L^\infty(\Omega). \tag{15}$$

Using (9) we prove that the sequence $\{\theta_n(t)\}_{n\ge1}$ is equicontinuous. Let $\varepsilon > 0$ and consider that $\delta(\varepsilon) > 0$ exists such that $|t - t'| < \delta(\varepsilon)$, for any $0 \le t' < t \le T$. We compute

$$\big\|\theta_n(t) - \theta_n(t')\big\|_{V'} = \left\|\int_{t'}^t \frac{\mathrm{d}\theta_n}{\mathrm{d}\tau}(\tau)\mathrm{d}\tau\right\|_{V'} \le \int_{t'}^t \left\|\frac{\mathrm{d}\theta_n}{\mathrm{d}\tau}(\tau)\right\|_{V'} \mathrm{d}\tau \tag{16}$$

$$\le (t-t')^{1/2} \left\|\frac{\mathrm{d}\theta_n}{\mathrm{d}\tau}(\tau)\mathrm{d}\tau\right\|_{L^2(0,T;V')} < \varepsilon, \tag{17}$$

if $\delta(\varepsilon) \le \frac{\varepsilon}{C}$.

By (9) we also get that $\|\theta_n(t)\|_{L^2(\Omega)} \le C$, $\forall t \in [0,T]$.

Since $L^2(\Omega)$ is compact in V', it follows that the sequence $\{\theta_n(t)\}_{n\geq 1}$ is compact in V', for each $t \in [0,T]$. We apply Ascoli-Arzelà theorem and obtain the convergence

$$\theta_n(t) \to \theta(t) \text{ strongly in } V',\ \forall t \in [0,T]. \tag{18}$$

Hence, this relation is also true for $t = 0$, i.e., $\theta_n(0) \to \theta(0)$ strongly in V'.

Relation (15) implies $\theta_n(0) = u_n \to u^*$ weakly in $L^2(\Omega)$.

The uniqueness of the limit yields to

$$\theta(0) = u^* \text{ a.e. in } \Omega. \tag{19}$$

Now we pass to show that θ^* is a solution to problem (2) for u replaced by u^*. This result is obtained by passing to the limit in the following equivalent form of (2)

$$\int_Q \left(\frac{\partial \theta_n}{\partial t}\phi + \nabla \eta_n \cdot \nabla \phi - K(\theta_n)\nabla\phi \right) \mathrm{d}x\mathrm{d}t =$$
$$= \int_Q f\phi \mathrm{d}x\mathrm{d}t - \int_\Sigma \alpha\eta_n\phi \mathrm{d}\sigma\mathrm{d}t, \quad \forall \phi \in L^2(0,T;V)$$

and we get

$$\int_Q \left(\frac{\partial \theta^*}{\partial t}\phi + \nabla \eta^* \cdot \nabla \phi - K(\theta^*)\nabla\phi \right) \mathrm{d}x\mathrm{d}t = \tag{20}$$
$$= \int_Q f\phi \mathrm{d}x\mathrm{d}t - \int_\Sigma \alpha\eta^*\phi \mathrm{d}\sigma\mathrm{d}t, \quad \forall \phi \in L^2(0,T;V),$$

i.e., θ^* is the solution to (2) for $u = u^*$.

We pass to the limit in (7) as $n \to \infty$, using the weakly lower semicontinuity, and we obtain

$$d \leq \int_{Q_0} \left(\theta^*(t,x) - \theta^{obs}(t,x) \right)^2 \mathrm{d}x\mathrm{d}t + \mu \int_\Omega (u^*)^2 \,\mathrm{d}x \leq d. \tag{21}$$

Hence $J(u^*) = d$, which means that the pair (u^*, θ^*) realizes the minimum of the cost functional J. $\square$

2. THE APPROXIMATING CONTROL PROBLEM

Now we introduce the following approximating identification problem

$$\min_{u \in U} J_\varepsilon(u), \tag{P_ε}$$

for

$$J_\varepsilon(u) = \int_{Q_0} \left(\theta(t,x) - \theta^{obs}(t,x) \right)^2 \mathrm{d}x\mathrm{d}t + \mu \int_\Omega u^2 \mathrm{d}x$$

subject to the approximating system

$$(22)\qquad \begin{gathered} \frac{\partial \theta}{\partial t} - \Delta\beta_\varepsilon^*(\theta) + \nabla\cdot K(\theta) = f \text{ in } Q,\\ \theta(0,x) = \omega_\varepsilon(x) \text{ in } \Omega,\\ (K(\theta) - \nabla\beta_\varepsilon^*(\theta))\cdot\nu - \alpha\beta_\varepsilon^*(\theta) = 0 \text{ on } \Sigma, \end{gathered}$$

where β_ε^* is a smooth function (e.g., in $C^3(\mathbb{R})$) which approximates β^* and ω_ε is a sequence which approximates u. In particular, it can be constructed using a mollifier ρ_ε

$$(23)\qquad \omega_\varepsilon(x) := u(x) * \rho_\varepsilon(x) = \int_{\mathbb{R}^N} u(x)\rho_\varepsilon(x-s)\mathrm{d}s.$$

We have that $\omega_\varepsilon \in C^\infty(\Omega)$ and $\omega_\varepsilon \to u$ strongly in $L^2(\Omega)$ (see [6]).

Consequently, we introduce the associated Cauchy problem

$$(24)\qquad \begin{gathered} \frac{\mathrm{d}\theta_\varepsilon}{\mathrm{d}t}(t) + A_\varepsilon\theta_\varepsilon(t) = f(t), \text{ a.e. } t\in(0,T),\\ \theta_\varepsilon(0) = u, \end{gathered}$$

where $A_\varepsilon : D(A) \subset V' \to V'$ is the single-valued operator defined by

$$(25)\qquad \langle A_\varepsilon\theta_\varepsilon, \psi\rangle_{V',V} = \int_\Omega (\nabla\beta_\varepsilon^*(\theta_\varepsilon) - K(\theta_\varepsilon))\cdot\nabla\psi\mathrm{d}x + \int_\Gamma \alpha\beta_\varepsilon^*(\theta_\varepsilon)\psi\mathrm{d}\sigma,$$

for any $\psi \in V$ and the domain operator

$$(26)\qquad D(A_\varepsilon) = \{\theta_\varepsilon \in L^2(\Omega);\ \beta_\varepsilon^*(\theta_\varepsilon) \in V\}.$$

THEOREM 2 (existence theorem, see [5]). *Let $f \in W^{1,2}(0,T;L^2(\Omega))$ and $u_\varepsilon \in H^2(\Omega)$. Then, for each $\varepsilon > 0$, problem (24) has a unique solution*

$$(27)\qquad \theta_\varepsilon \in W^{1,\infty}(0,T;L^2(\Omega)) \cap W^{1,2}(0,T;V) \cap L^\infty(0,T;H^2(\Omega)),$$

$$(28)\qquad \beta_\varepsilon^*(\theta_\varepsilon) \in W^{1,\infty}(0,T;L^2(\Omega)) \cap W^{1,2}(0,T;V) \cap L^\infty(0,T;H^2(\Omega))$$

which satisfies the estimates

$$(29)\qquad \|\theta_\varepsilon\|_{W^{1,\infty}(0,T;L^2(\Omega))} + \|\theta_\varepsilon\|_{W^{1,2}(0,T;V)} + \|\theta_\varepsilon\|_{L^\infty(0,T;H^2(\Omega))} \le C,$$

$$(30)\qquad \|\beta_\varepsilon^*(\theta_\varepsilon)\|_{W^{1,\infty}(0,T;L^2(\Omega))} + \|\beta_\varepsilon^*(\theta_\varepsilon)\|_{W^{1,2}(0,T;V)} + \|\beta_\varepsilon^*(\theta_\varepsilon)\|_{L^\infty(0,T;H^2(\Omega))} \le C.$$

THEOREM 3. *Let $f \in L^2(0,T;V')$. Then problem (P_ε) has at least one solution $(u_\varepsilon^*, \theta_\varepsilon^*)$.*

The proof is identical to the one of Theorem 1, just that β^* is replaced by the smoother function β_ε^*.

LEMMA 4. *Let θ_ε be a solution to (24) corresponding to $\omega_\varepsilon = u * \rho_\varepsilon$, $u \in U$. Then,*

$$\omega_\varepsilon \to u \text{ strongly in } L^2(\Omega) \text{ as } \varepsilon \to 0$$

and there exists a subsequence of $\{\theta_\varepsilon\}$ such that

$$\theta_\varepsilon \to \theta \text{ weakly in } W^{1,2}(0,T;V') \cap L^2(0,T;V), \text{ strongly in } L^2(Q) \tag{31}$$

and θ is the solution to system (2) corresponding to u.

Proof. We recall that a solution to (24) is a solution in the generalized sense to (22). Since u_ε is the function defined by (23) and $\omega_\varepsilon \in L^\infty(0,T;H^2(\Omega))$, it follows that the approximating problem has a unique strong solution denoted θ_ε. Then, the proof of (31) follows like in [7]. □

THEOREM 5. *Let $f \in W^{1,2}(0,T;L^2(\Omega))$ and the pair $(u_\varepsilon, \theta_\varepsilon)$ be a solution to the approximating problem (P_ε). Then,*

$$u_\varepsilon \to u^* \text{ weak-star in } L^\infty(\Omega), \tag{32}$$

$$\theta_\varepsilon \to \theta \text{ weakly in } W^{1,2}(0,T;V') \cap L^2(0,T;V), \tag{33}$$

where $u^* \in U$ and θ^* is the solution to the original problem (1) for $u = u^*$. Moreover, u^* is a solution to (P) and

$$\lim_{\varepsilon \to 0} (P_\varepsilon) = P. \tag{34}$$

Proof. Let $u^* \in U$ be a solution to (P) and θ^ε a solution to the approximating problem (22) for $u = u^*$. By the optimality pair $(u_\varepsilon, \theta_\varepsilon)$ in (P_ε), we have

$$\int_{Q_0} \left(\theta_\varepsilon - \theta^{obs}\right)^2 \mathrm{d}x\mathrm{d}t + \mu \int_\Omega u_\varepsilon^2 \mathrm{d}x \le \int_{Q_0} \left(\theta^\varepsilon - \theta^{obs}\right)^2 \mathrm{d}x\mathrm{d}t + \mu \int_\Omega (u^*)^2 \mathrm{d}x.$$

As $\varepsilon \to 0$, by Lemma 4 we have that $\theta^\varepsilon \to \theta^*$ strongly in $L^2(Q_0)$, where θ^* is the solution to system (2) for $u = u^*$. Hence

$$\left\|\theta_\varepsilon^* - \theta^{obs}\right\|_{L^2(Q_0)} \to \left\|\theta - \theta^{obs}\right\|_{L^2(Q_0)}.$$

For this reason, we have the following inequalities

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \left[\int_{Q_0} \left(\theta_\varepsilon^* - \theta^{obs}\right)^2 \mathrm{d}x\mathrm{d}t + \mu \int_\Omega (u_\varepsilon^*)^2 \mathrm{d}x\right] \le \\ &\le \limsup_{\varepsilon \to 0} \left[\int_{Q_0} \left(\theta^\varepsilon - \theta^{obs}\right)^2 \mathrm{d}x\mathrm{d}t + \mu \int_\Omega (u^*)^2 \mathrm{d}x\right] \le \\ &\le \int_{Q_0} \left(\theta^* - \theta^{obs}\right)^2 \mathrm{d}x\mathrm{d}t + \mu \int_\Omega (u^*)^2 \mathrm{d}x. \end{aligned} \tag{35}$$

Since $\{u_\varepsilon^*\}_{\varepsilon\geq 1} \subset U$, there exists a subsequence denoted the same way such that

$$u_\varepsilon^* \to u^* \text{ weak-star in } L^\infty(\Omega).$$

As shown before, on a subsequence we have that

$$\theta_\varepsilon^* \to \theta^* \text{ strongly in } L^2(Q_0) \text{ and weakly in } W^{1,2}(0,T;V) \cap L^2(0,T,V'),$$

where θ^* is the solution to (2) for $u = u^*$. This yields to

$$\begin{aligned}(36)\qquad &\int_{Q_0} \left(\theta^* - \theta^{obs}\right)^2 \mathrm{d}x\mathrm{d}t + \mu \int_\Omega (u^*)^2 \,\mathrm{d}x \leq \\ &\leq \liminf_{\varepsilon\to 0} \left[\int_{Q_0} \left(\theta_\varepsilon^* - \theta^{obs}\right)^2 \mathrm{d}x\mathrm{d}t + \mu \int_\Omega (u_\varepsilon^*)^2 \mathrm{d}x\right] \leq \\ &\leq \limsup_{\varepsilon\to 0} \left[\int_{Q_0} \left(\theta_\varepsilon - \theta^{obs}\right)^2 \mathrm{d}x\mathrm{d}t + \mu \int_\Omega (u_\varepsilon^*)^2 \mathrm{d}x\right] \leq \\ &\leq \int_{Q_0} \left(\theta^* - \theta^{obs}\right)^2 \mathrm{d}x\mathrm{d}t + \mu \int_\Omega (u^*)^2 \,\mathrm{d}x.\end{aligned}$$

By (35) and (36) we get

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \left[\int_{Q_0} \left(\theta_\varepsilon^* - \theta^{obs}\right)^2 \mathrm{d}x\mathrm{d}t + \mu \int_\Omega (u_\varepsilon^*)^2 \mathrm{d}x\right] = \\ &= \int_{Q_0} \left(\theta^* - \theta^{obs}\right)^2 \mathrm{d}x\mathrm{d}t + \mu \int_\Omega (u^*)^2 \,\mathrm{d}x = \min(P).\end{aligned}$$

This completes the proof. □

3. NECESSARY CONDITIONS OF OPTIMALITY FOR THE APPROXIMATING PROBLEM

The next intermediate step is to determine the necessary conditions of optimality for the problem (P_ε) subject to (22). The approximating optimality conditions are required for the numerical computations, because we cannot work with the multivalued function β^* and we use the single-valued β_ε^*.

We assume that $(u_\varepsilon^*, \theta_\varepsilon^*)$ is an optimal pair for (P_ε). We introduce the variation of u_ε^*

$$(37)\qquad u_\varepsilon^\lambda = u_\varepsilon^* + \lambda v_\varepsilon \text{ for } v_\varepsilon = w - u_\varepsilon^*,\ \forall w \in U,\ \lambda > 0,$$

and define $Y_\varepsilon = \lim\limits_{\lambda\to 0} \frac{\theta_\varepsilon^\lambda - \theta_\varepsilon^*}{\lambda}$.

Having introduced these notations, the system in variation reads

$$
\begin{aligned}
&\frac{\partial Y_\varepsilon}{\partial t} - \Delta\left(\beta_\varepsilon(\theta_\varepsilon^*)Y_\varepsilon\right) + \nabla\cdot\left(K'(\theta_\varepsilon^*)Y_\varepsilon\right) = 0 && \text{in } Q,\\
&Y_\varepsilon(0,x) = \omega_\varepsilon^{var}(x) := v_\varepsilon(x) * \rho_\varepsilon(x) && \text{in } \Omega,\\
&\left(K'(\theta_\varepsilon^*)Y_\varepsilon - \nabla\left(\beta_\varepsilon(\theta_\varepsilon^*)Y_\varepsilon\right)\right)\cdot\nu - \alpha\beta_\varepsilon(\theta_\varepsilon^*)Y_\varepsilon = 0 && \text{on } \Sigma.
\end{aligned} \tag{38}
$$

Now we shall give an existence and uniqueness result for the solution to the system in variations.

Proposition 6. *Assume that* $f \in W^{1,2}(0,T;L^2(\Omega))$. *Then system* (38) *has, for each* $\varepsilon > 0$, *a unique solution*

$$
Y_\varepsilon \in C([0,T];L^2(\Omega)) \cap L^2(0,T;V), \quad \frac{\mathrm{d}Y_\varepsilon}{\mathrm{d}t} \in L^2(0,T;V').
$$

Proof. We introduce the linear operator $A_{Y,\varepsilon}(t) : V \to V'$ by

$$
\langle A_{Y,\varepsilon}(t)\phi,\psi\rangle_{V',V} = \tag{39}
$$

$$
= \int_\Omega \left(\nabla\left(\beta_\varepsilon(\theta_\varepsilon^*)\phi\right)\cdot\nabla\psi - K'(\theta_\varepsilon^*)\phi\nabla\psi\right)\mathrm{d}x + \int_\Sigma \alpha\beta_\varepsilon(\theta_\varepsilon^*)\phi\psi\mathrm{d}\sigma, \quad \forall\psi\in V
$$

and write the Cauchy problem

$$
\begin{gathered}
\frac{\mathrm{d}Y_\varepsilon}{\mathrm{d}t}(t) + A_{Y,\varepsilon}(t)Y_\varepsilon(t) = 0 \text{ a.e. } t\in(0,T),\\
Y_\varepsilon(0) = v_\varepsilon.
\end{gathered} \tag{40}
$$

The proof of Proposition 6 is based on Lions' theorem (see [2]) and follows the same steps of Proposition 3.8 in [7]. □

We write the dual system as

$$
\begin{aligned}
&\frac{\partial p_\varepsilon}{\partial t} + \beta_\varepsilon(\theta_\varepsilon^*)\Delta p_\varepsilon + K'(\theta_\varepsilon^*)\nabla p_\varepsilon = -(\theta_\varepsilon^* - \theta^{obs})\chi_{\Omega_0} && \text{in } Q,\\
&p_\varepsilon(T,x) = 0 && \text{in } \Omega,\\
&\alpha p_\varepsilon + \frac{\partial p_\varepsilon}{\partial\nu} = 0 && \text{on } \Sigma,
\end{aligned} \tag{41}
$$

where χ_{Ω_0} is the characteristic function of the set $\Omega_{0.}$

Proposition 7. *Assume that* $f \in W^{1,2}(0,T;L^2(\Omega))$. *Then system* (41) *has a unique solution*

$$
p_\varepsilon \in C([0,T];L^2(\Omega)) \cap L^2(0,T;V), \quad \frac{\mathrm{d}p_\varepsilon}{\mathrm{d}t} \in L^2(0,T;V').
$$

The results of Proposition 7 are obtained by applying Lions' theorem (see Proposition 3.9 in [7]).

The following step consists of determining the approximating dual system. For this purpose we multiply the first equation of system (38) by p_ε, the solution to the approximating dual system, and integrate the result over Q. We obtain

$$(42)\quad -\int_Q\left(\frac{\partial p_\varepsilon}{\partial t}+\beta_\varepsilon(\theta^*_\varepsilon)\Delta p_\varepsilon+K'(\theta^*_\varepsilon)\nabla p_\varepsilon\right)Y_\varepsilon\,\mathrm{d}x\mathrm{d}t+\int_\Omega p_\varepsilon(T,x)Y_\varepsilon(T,x)\mathrm{d}x-$$
$$-\int_\Omega p_\varepsilon(0,x)\omega^{var}_\varepsilon(x)\mathrm{d}x+\int_\Sigma\left(\alpha\beta_\varepsilon(\theta^*_\varepsilon)p_\varepsilon+\beta_\varepsilon(\theta^*_\varepsilon)\frac{\partial p_\varepsilon}{\partial\nu}\right)\mathrm{d}\sigma\mathrm{d}t=0.$$

By (42) we get

$$(43)\quad \int_Q Y_\varepsilon(\theta^*_\varepsilon-\theta^{obs})\chi_{Q_0}\mathrm{d}x\mathrm{d}t=\int_\Omega p_\varepsilon(0,x)\omega^{var}_\varepsilon(x)\mathrm{d}x.$$

The assumption that $(\theta^*_\varepsilon,u^*_\varepsilon)$ is optimal implies $J(u^*_\varepsilon)\le J(u^\lambda_\varepsilon)$. Thus we have

$$J(u^\lambda_\varepsilon)-J(u^*_\varepsilon)=\int_{Q_0}\big(\theta^\lambda_\varepsilon-\theta^*_\varepsilon\big)\big(\theta^\lambda_\varepsilon+\theta^*_\varepsilon-2\theta^{obs}\big)\mathrm{d}x\mathrm{d}t+$$
$$+\int_\Omega\mu\big(u^\lambda_\varepsilon-u^*_\varepsilon\big)\big(u^\lambda_\varepsilon+u^*_\varepsilon\big)\mathrm{d}x.$$

We divide by λ and pass to the limit for $\lambda\to 0$ and taking into account (37), we obtain

$$\lim_{\lambda\to 0}\frac{J(u^\lambda_\varepsilon)-J(u^*_\varepsilon)}{\lambda}=2\int_{Q_0}Y_\varepsilon\big(\theta^*_\varepsilon-\theta^{obs}\big)\mathrm{d}s\mathrm{d}t+\int_\Omega 2\mu v_\varepsilon u^*_\varepsilon\mathrm{d}s.$$

Therefore, by (43) we obtain the condition

$$(44)\quad \int_\Omega v_\varepsilon(s)\left(\int_\Omega\rho_\varepsilon(x-s)p_\varepsilon(0,x)\mathrm{d}x\right)\mathrm{d}s+\mu\int_\Omega\left(u^*_\varepsilon v_\varepsilon\right)(s)\mathrm{d}s\ge 0,$$

that can be still written

$$(45)\quad \int_\Omega v_\varepsilon(s)(F_\varepsilon+\mu u^*_\varepsilon)(s)\mathrm{d}s\ge 0,$$

where $F_\varepsilon(s):=\int_\Omega\rho_\varepsilon(x-s)p_\varepsilon(0,x)\mathrm{d}x$. Thus, $-(F_\varepsilon+\mu u^*_\varepsilon)\in\partial_{I_U}(u^*_\varepsilon)=N_{[0,\theta_s]}(u^*_\varepsilon)$, which is the optimality condition.

4. NUMERICAL RESULTS

For solving (1) we used an optimization algorithm based on the gradient projection method for a restricted minimization problem. For a nonempty convex, closed set U and a problem of minimizing the functional $J:U\to\mathbf{R}$ over U, we have problem (P) which can be written as:

Find $u \in U$ such that

$$(P') \qquad J(u) \leq J(v), \quad \forall v \in U.$$

An existence result is given by the following theorem.

THEOREM 8 (see [1]). *Let J be weakly differentiable.*
(i) *If $u^* \in U$ is optimal, then for any $\alpha > 0$*

$$P_U\left(u^* - \alpha \nabla J(u^*)\right) = u^*. \tag{46}$$

(ii) *If J is also convex and if there exists $\alpha > 0$ such that* (46) *holds, then u^* is a solution to problem (P').*

We performed the numerical results using Rosen's algorithm (see [1]):

S0. Choose $u_0 \in U$ and $\alpha > 0$;
Set $k = 0$;
S1. $\overline{u}_k := P_U\left(u_k - \alpha \nabla J(u_k)\right);$
$v_k := \overline{u}_k - u_k;$
S2. Compute $\lambda_k \in (0, 1]$ such that
$J(u_k + \lambda_k v_k) = \min\left\{J(u_k + \lambda v_k);\ 0 < \lambda \leq 1\right\};$
S3. $u_{k+1} := u_k + \lambda_k v_k;$
S4. The stopping criterion:
If $\|u_{k+1} - u_k\| < \varepsilon$
then stop (u_{k+1} is the solution)
else $k := k + 1$; go to S1.

Numerical results are performed for the 2D case with the domain a square defined by $\Omega = \{(x, y);\ x \in [0, 1],\ y \in [0, 1]\}$. Γ is the soil boundary and the other data are

$$f(t, x, y) = 0.2 \exp(-t^2), \quad u(x, y) = 0.5.$$

Other constants used are $T = 1$, $\mu = 1$, $\theta_s = 1$, $\alpha = 10^{-8}$, $\varepsilon_1 = 0.01$ and $\varepsilon = 0.1$.

Assuming that the model (1) is already written in a dimensionless form, we shall perform numerical tests for

$$\beta(\theta) = \begin{cases} \dfrac{1}{2\sqrt{1-\theta}}, & 0 \leq \theta < \theta_s - \varepsilon_1, \\ \dfrac{1}{2\sqrt{1-\theta_s+\varepsilon_1}}, & \theta_s - \varepsilon_1 \leq \theta \leq \theta_s, \end{cases} \tag{47}$$

$$\beta^*(\theta) = \begin{cases} 1 - \sqrt{1-\theta}, & 0 \leq \theta < \theta_s - \varepsilon_1, \\ \dfrac{1}{2\sqrt{1-\theta}}\left(\theta - \theta_s + \varepsilon_1\right) + 1 - \sqrt{1-\theta_s+\varepsilon_1}, & \theta_s - \varepsilon_1 \leq \theta \leq \theta_s. \end{cases} \tag{48}$$

In what concerns K we consider it of the form $K(\theta) = \theta^2$.

Systems (1) and (41) were solved with Comsol Multiphysyics 3.5a (see [4]) and Matlab (see [9]) for $\alpha = 0.1$ and 1.

The corresponding values for the solution to problem (P'), the norm of ∇J and the error at each iteration are presented in Tables 1 to 3.

Table 1
$\theta^{obs}(t,x,y) = 0.5 - 0.1y$, $\alpha = 0.1$

Iteration k	$\min_{\lambda\in(0,1]}(J_\lambda^k)$	$\|\nabla J_\lambda^{k+1} - \nabla J_\lambda^k\|$	$\|u^{k+1} - u^k\|$
1	0.23079	39.4934	3.9493
2	0.17478	31.8712	3.1871
3	0.1428	25.0697	2.507
4	0.12587	19.3837	1.9384
5	0.11776	14.8405	1.484
6	0.11451	11.3327	1.1333
7	0.11385	8.7031	0.87031
8	0.11446	6.7231	0.66222
9	0.1156	5.2912	0.51235
10	0.11687	4.2884	0.41171
11	0.11786	3.5145	0.050565

Table 2
$\theta^{obs}(t,x,y) = 0.5 - 0.1y$, $\alpha = 1$

Iteration k	$\min_{\lambda\in(0,1]}(J_\lambda^k)$	$\|\nabla J_\lambda^{k+1} - \nabla J_\lambda^k\|$	$\|u^{k+1} - u^k\|$
1	0.085035	39.4934	21.4498
2	0.12184	16.809	2.5213
3	0.11044	9.4706	1.4206
4	0.10837	5.9968	0.89952
5	0.10935	4.3322	0.64945
6	0.11138	3.4552	0.50869
7	0.11358	2.8845	0.41241
8	0.1156	2.4614	0.34175
9	0.1175	2.1184	0.28666
10	0.11923	1.8252	0.24251
11	0.12074	1.5696	0.20593
12	0.12206	1.3452	0.17503
13	0.12321	1.1493	0.14883
14	0.12421	0.979	0.1265
15	0.12508	0.83188	0.10729
16	0.12582	0.70573	0.090954

The smallest error was obtained for the initial guess $\theta^{obs}(t,x,y) = 0.4 + 0.1y$ and $\alpha = 0.1$ (see Table 3) for a minimum value of the cost functional of 0.10907, while the method converged in 11 iterations. For a small value of α (i.e., $\alpha = 0.1$) we got a decrease in the functional cost (see Table 1), meanwhile in the other cases J diminishes for a few steps to increase afterwards. For an $\alpha > 5$ we couldn't reach convergence anymore.

Table 3

$\theta^{obs}(t,x,y) = 0.4 + 0.1y$, $\alpha = 0.1$

Iteration k	$\min\limits_{\lambda\in(0,1]}(J_\lambda^k)$	$\lVert\nabla J_\lambda^{k+1} - \nabla J_\lambda^k\rVert$	$\lVert u^{k+1} - u^k\rVert$
1	0.21783	39.6873	3.9687
2	0.16368	31.8474	3.1847
3	0.13363	24.9041	2.4904
4	0.11836	19.1396	1.914
5	0.11148	14.5685	1.4568
6	0.10914	11.0583	1.1058
7	0.10907	8.4411	0.84411
8	0.11008	6.4843	0.64558
9	0.1115	5.0516	0.49616
10	0.11293	4.0272	0.39333
11	0.11414	3.3009	0.048447

In Figure 1 we plotted the graphics of $\theta^{obs}(x,y) = 0.5 - 0.1y$, θ computed and the control u (corresponding to the data in Table 1).

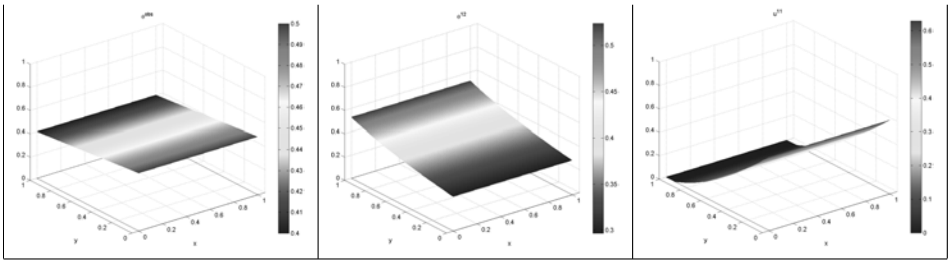

Fig. 1. 3D plots of θ^{obs}, the final solution θ^{11} and the final control u^{11}.

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