Dedicated to Dr. Constantin VÂRSAN
on the occasion of his 70th birthday

SINGULARLY PERTURBED PARABOLIC SYSTEMS LEADING TO TIME-INVARIANT SOLUTIONS

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Singularly perturbed systems of parabolic type which in the perturbed equation the small parameter multiplies only the time partial derivative (and not the entire differential operator as studied in [8]) are studied. Explicit stability conditions (the presence of negative definite matrices in the equations of the system) allow straightforward detailed computations that lead to dynamical limits related to time-invariant measures.

AMS 2000 Subject Classification: 35B25, 35K99.

Key words: singular perturbations, parabolic systems, dynamical limits, reduced system, invariant measure.

1. INTRODUCTION

The main ideas related to the study of singularly perturbed differential systems crystalized by the middle of the 20th century, when A.N. Tihonov established the first fundamental result legitimating the reduced system as being obtained by neglecting the fast variable. The multitude of concrete problems raised since then has led to many approaches and methods, to the extension of the results in the domains of Control Theory and Stochastic processes.

The publication of the monograph [1] encouraged the study of singular perturbations in our country. During the last two decades Professor C. Vârsan initiated and catalyzed the study of singularly perturbed stochastic systems proposing new methods: to use the Lie Algebras generated by the diffusion fields ([9], [10], [11]), consider the dynamical limits, i.e., to rewrite the perturbed system as a dynamical perturbed system admitting generalized controls and to take advantage of the weak convergence of the solutions ([6], [8]) and many other means of encompassing the difficulties related to the complexity of the solutions of such systems.

Original results were obtained for hyperbolic systems ([9], [10]), evolution systems of Cauchy-Kowalevskaia type ([4]), Hamilton-Jacobi systems ([7], [9]–[11]), parabolic systems ([3], [4], [7], [8]), Langevin equations ([4], [6]).

MATH. REPORTS **12(62)**, *1* (2010), 85–98

The present paper contains results not published yet from the author's Ph.D. Thesis [8]; the idea of considering systems leading to invariant measures was suggested by Professor C. Vârsan and the theoretical background was provided by [2], [5] and [12].

2. THE PERTURBED SYSTEM, MAIN HYPOTHESES AND THE STRUCTURE OF ITS SOLUTIONS

We will analyze systems of the form

$$(1)\quad \begin{cases} \partial_t u = \Delta_x u + f(x,u,v), & t \in (0,T],\ u(0) = u_0(x),\ x \in \mathbf{R}^n,\ u \in \mathbf{R}^m, \\ \varepsilon \partial_t v = \Delta_x v + h(x,u,v), & t \in (0,T],\ v(0) = v_0(x),\ x \in \mathbf{R}^n,\ v \in \mathbf{R}^k, \end{cases}$$

where the nonlinear applications $f(x,u,v)$ and $h(x,u,v)$ fulfill the conditions:

(i_1) f and h are continuous and bounded in the domain $D = \mathbf{R}^n \times B(0,\rho_1) \times B(0,\rho_2)$, with $B(0,\rho_1) \subset \mathbf{R}^m$, $B(0,\rho_2) \subset \mathbf{R}^k$;

(i_2)

$$|g(x,u'',v'') - g(x,u',v')| \le L(|u''-u'| + |v''-v'|)$$

for any $(x,u',v'),(x,u'',v'') \in D$, where $g \in \{f,h\}$ and $L > 0$ is a constant.

For any $\varepsilon \in (0,1]$ the system (1) admits a unique local solution

$$(2)\quad (u_\varepsilon(t,x), v_\varepsilon(t,x)) \in B(0,\rho_1) \times B(0,\rho_2),$$

$t \in [0,a_\varepsilon]$, $x \in \mathbf{R}^n$ if $u_0(x) \in B(0,\rho_1/2)$, $v_0(x) \in B(0,\rho_2/2)$ are continuous and the hypotheses (i_1), (i_2) are fulfilled.

Moreover, the solution could be also written with the aid of the associated system of integral equations

$$(3)\quad \begin{cases} u_\varepsilon(t,x) = \displaystyle\int_{R^n} u_0(y) P(t,x,y)\mathrm{d}y + \\ \qquad + \displaystyle\int_0^t \mathrm{d}s \int_{R^n} f(y, u_\varepsilon(s,y), v_\varepsilon(s,y)) P(t-s,x,y)\mathrm{d}y, \\ v_\varepsilon(t,x) = \displaystyle\int_{R^n} v_0(y) P\left(\frac{t}{\varepsilon}, x, y\right)\mathrm{d}y + \\ \qquad + \displaystyle\frac{1}{\varepsilon}\int_0^t \mathrm{d}s \int_{R^n} h(y, u_\varepsilon(s,y), v_\varepsilon(s,y)) P\left(\frac{t-s}{\varepsilon}, x, y\right)\mathrm{d}y, \end{cases}$$

where

$$P(\tau,x,y) = (4\pi\tau)^{-n/2} \exp\left(-\frac{|y-x|^2}{4\tau}\right)$$

is a solution of the fundamental parabolic equation

$$(4)\quad \partial_\tau P(\tau,x,y) = \Delta_x P(\tau,x,y), \quad \tau > 0,\ x,y \in \mathbf{R}^n.$$

In order to find a common interval $t \in [0, a]$, not depending on $\varepsilon \in (0, 1]$, such that the integral equations (3) are verified for any $t \in [0, a]$ and $\varepsilon \in (0, 1]$ it will prove useful to assume that the function h has the form

$$h(x, u, v) = Av + h_0(x), \tag{i$_3$}$$

where the $(k \times k)$ matrix A is stable, i.e., $\langle Av, v\rangle \le -\alpha |v|^2$, $\forall v \in \mathbf{R}^k$, for some constant $\alpha > 0$.

A slightly relaxed hypothesis on the function h will also prove useful

$$h(x, u, v) = Av + h_0(x, u), \tag{i$_3'$}$$

where again $\langle Av, v\rangle \le -\alpha |v|^2$ and $h_0(x, u)$ fulfils conditions similar to (i_1) and (i_2).

By denoting $\Phi(\tau) = \exp(\tau A)$, $\tau \ge 0$ and by using (i_3) we are able to rewrite $v_\varepsilon(t, x)$ as

$$\begin{aligned} v_\varepsilon(t, x) = \Phi(\tau) \int_{R^n} v_0(y) P(\tau, x, y) \mathrm{d}y + \\ + \int_0^\tau \Phi(\tau - s) \mathrm{d}s \int_{R^n} h_0(y) P(\tau - s, x, y) \mathrm{d}y, \end{aligned} \tag{5}$$

where $\tau \triangleq \frac{t}{\varepsilon}$ and $|\Phi(\tau)| \le \exp(-\alpha\tau)$, $\forall \tau \ge 0$.

It is easily seen that

$$v_\varepsilon(t, x) \in B(0, \rho_2) \subseteq \mathbf{R}^k$$

for any $t \in [0, T]$ and $\varepsilon \in (0, 1]$ if $\rho_2 > 0$ is sufficiently large.

Therefore the system (3) could be rewritten in the form

$$\left\{ \begin{aligned} u_\varepsilon(t, x) &= \int_{R^n} u_0(y) P(t, x, y) \mathrm{d}y + \\ &\quad + \int_0^t \mathrm{d}s \int_{R^n} f(y, u_\varepsilon(s, y), v_\varepsilon(s, y)) P(t - s, x, y) \mathrm{d}y, \\ v_\varepsilon(t, x) &= \Phi(\tau) \int_{R^n} v_0(y) P(\tau, x, y) \mathrm{d}y + \\ &\quad + \int_0^\tau \Phi(\tau - s) \int_{R^n} h_0(y) P(\tau - s, x, y) \mathrm{d}y \mathrm{d}s, \quad \tau = \frac{t}{\varepsilon}. \end{aligned} \right. \tag{6}$$

Remark 2.1. Based upon the hypotheses (i_1), (i_2) and the successive approximations method we could find a unique continuous solution

$$(u_\varepsilon(t, x), v_\varepsilon(t, x)) \in B(0, \rho_1) \times B(0, \rho_2), \tag{7}$$

$t \in [0, a]$, $x \in \mathbf{R}^n$ which is uniformly bounded in $\varepsilon \in (0, 1]$ and verifies the system (6) on an interval $t \in [0, a]$ not depending on $\varepsilon \in (0, 1]$.

Further, by using a direct computation involving the improper integrals in (6), we deduce that the solution $(u_\varepsilon(\cdot), v_\varepsilon(\cdot))$ of this latter system verifies also the parabolic system (1).

Let $\hat{v}(x) \in \mathbf{R}^k$ be the continuous function defined by

$$\hat{v}(x) = \int_0^\infty \Phi(\sigma) \left[\int_{R^n} h_0(y) P(\sigma, x, y) \mathrm{d}y \right] \mathrm{d}\sigma. \tag{8}$$

It is not difficult to see that $v_\varepsilon(t,x)$ defined in (6) fulfills

$$\lim_{\varepsilon \downarrow 0} v_\varepsilon(t,x) = \hat{v}(x) \tag{9}$$

for any $t \in [0,a]$, $x \in \mathbf{R}^n$ and that $\hat{v}(\cdot)$ defined by (8) fulfills

$$\Delta_x \hat{v}(x) + A\hat{v}(x) + h_0(x) = 0 \tag{10}$$

for any $x \in \mathbf{R}^n$.

Let $u = \hat{u}(t,x)$ be the unique solution of the first equation in (6) corresponding to $v = \hat{v}(x)$,

$$\hat{u}(t,x) = \int_{R^n} u_0(y) P(t,x,y) \mathrm{d}y + \tag{11}$$

$$+ \int_0^t \mathrm{d}s \int_{R^n} f(y, \hat{u}(s,y), \hat{v}(y)) P(t-s, x, y) \mathrm{d}y.$$

Then $u = \hat{u}(t,x)$, $t \in [0,a]$, $x \in \mathbf{R}^n$ fulfills the first parabolic equation from (1) for $v = \hat{v}(x)$, i.e.,

$$\begin{cases} \partial_t \hat{u}(t,x) = \Delta_x \hat{u}(t,x) + f(x, \hat{u}(t,x), \hat{v}(x)), & t \in (0,a], \\ \hat{u}(0,x) = u_0(x), & x \in \mathbf{R}^n. \end{cases} \tag{12}$$

3. THE REDUCED SYSTEM AND THE DYNAMICAL LIMITS

It can be noticed that the reduced system associated to (1) follows to be the parabolic-elliptic one given by (12) and (10). Indeed, putting together the remarks and results in the previous section we obtain the next theorem.

THEOREM 3.1. *Let $f(x,u,v) \in \mathbf{R}^m$ and $h(x,u,v) \in \mathbf{R}^k$ such that the hypotheses* (i_1), (i_2) *and* (i_3) *are fulfilled. For each $\varepsilon \in (0,1]$ we consider $(u_\varepsilon(t,x), v_\varepsilon(t,x))$ the solution of* (1) *written by the means of the integral equations* (6). *Let $(\hat{u}(t,x), \hat{v}(x))$, $t \in [0,a]$, $x \in \mathbf{R}^n$ given by* (12), *respectively* (8), *i.e.,*

$$\begin{cases} \partial_t \hat{u}(t,x) = \Delta_x \hat{u}(t,x) + f(x, \hat{u}(t,x), \hat{v}(x)), & t \in (0,a], \\ \hat{u}(0,x) = u_0(x), & x \in \mathbf{R}^n, \\ \hat{v}(x) = \int_0^\infty \Phi(\sigma) \left[\int_{R^n} h_0(y) P(\sigma,x,y) \mathrm{d}y \right] \mathrm{d}\sigma. \end{cases}$$

Then

$$\lim_{\varepsilon\downarrow 0}(u_\varepsilon(t,x),v_\varepsilon(t,x)) = (\hat{u}(t,x),\hat{v}(x))$$

for each $t \in [0,a]$ *and uniformly with respect to* $x \in \mathbf{R}^n$.

Proof. Indeed, by using the Remark 2.1, the unique solution of (6), $(u_\varepsilon(t,x),v_\varepsilon(t,x))$, $(t,x)\in[0,a]\times\mathbf{R}^n$ is also the unique solution of the integral system (1).

In the same time, $\hat{v}(x)$, $x\in\mathbf{R}^k$ defined by (8) fulfills (9) and (10).

We need only to prove that $u_\varepsilon(t,x)$ defined in (6) has the property

$$\lim_{\varepsilon\downarrow 0} u_\varepsilon(t,x) = \hat{u}(t,x) \tag{13}$$

for each $t\in[0,a]$, $x\in\mathbf{R}^n$, where $\hat{u}(\cdot)$ fulfills (11) and (12).

We have, successively

$$\begin{aligned} &|u_\varepsilon(t,x)-\hat{u}(t,x)| \le \qquad (14)\\ &\le \int_0^t \mathrm{d}s \int_{R^n} |f(y,u_\varepsilon(s,y),v_\varepsilon(s,y)) - f(y,\hat{u}(s,y),\hat{v}(y))|\, P(t-s,x,y)\mathrm{d}y \le\\ &\le L\int_0^t \mathrm{d}s \int_{R^n} \left[|u_\varepsilon(s,y)-\hat{u}(s,y)| + |v_\varepsilon(s,y)-\hat{v}(y)|\right] P(t-s,x,y)\mathrm{d}y \le\\ &\le L\int_0^t \|u_\varepsilon(s,\cdot)-\hat{u}(s,\cdot)\|\,\mathrm{d}s + C\int_0^t \exp\left(-\alpha\frac{t}{\varepsilon}\right)\mathrm{d}s, \end{aligned}$$

where $\|u_\varepsilon(s,\cdot)-\hat{u}(s,\cdot)\| \overset{\Delta}{=} \sup_{x\in\mathbf{R}^n}\|u_\varepsilon(s,x)-\hat{u}(s,x)\|$ and $|v_\varepsilon(t,x)-\hat{v}_\xi(x)| \le C\exp\left(-\alpha\frac{t}{\varepsilon}\right)$ for any $x\in\mathbf{R}^n$.

By using now Gronwall's lemma, from(14) we obtain

$$\|u_\varepsilon(t,\cdot)-\hat{u}(t,\cdot)\| \le \varepsilon C_1, \quad t\in[0,a], \tag{15}$$

where $C_1 > 0$ is constant.

We may conclude then that (13) holds for any $t\in[0,a]$ and uniformly with respect to $x\in\mathbf{R}^n$, which completes the proof. □

Remark 3.2. The result from Theorem 3.1 does not change if the hypothesis (i_3) is replaced by the more relaxed (i_3').

To begin proving this, let us choose $\rho_2 > 0$ sufficiently large such that $\rho_2 \ge 2\frac{C}{\alpha}$, where $|h_0(y,u)| \le C$ for any $y\in\mathbf{R}^n$ and $u\in B(0,\rho_1)\le\mathbf{R}^m$, $\alpha>0$ being the constant from (i_3').

We have to replace the integral system (6) with

$$(6') \quad \begin{cases} u_\varepsilon(t,x) = \int_{R^n} u_0(y)P(t,x,y)\mathrm{d}y + \\ \qquad + \int_0^t \mathrm{d}s \int_{R^n} f(y,u_\varepsilon(s,y),v_\varepsilon(s,y))P(t-s,x,y)\mathrm{d}y, \\ v_\varepsilon(t,x) = \Phi(\tau)\int_{R^n} v_0(y)P(\tau,x,y)\mathrm{d}y + \\ \qquad + \int_0^\tau \Phi(\tau-s)\mathrm{d}s \int_{R^n} h_0(y,u_\varepsilon(s,y))P(\tau-s,x,y)\mathrm{d}y \end{cases}$$

for $\tau \overset{\Delta}{=} \frac{t}{\varepsilon}$ and by using again the successive approximations method we conclude this new system has a unique solution

$$(u_\varepsilon(t,x), v_\varepsilon(t,x)) \in B(0,\rho_1)\times B(0,\rho_2), \quad t\in[0,a]\,,\ x\in \mathbf{R}^n$$

for any $\varepsilon \in (0,1]$.

We define for each $u(t,x) \in B(0,\rho_1)$, $x \in \mathbf{R}^n$ continuous and each $t \in [0,a]$ the map $V(t,x;u)$ by

$$(16) \quad V(t,x;u) = \int_0^\infty \Phi(\sigma)\left[\int_{R^n} h_0(y,u(t,y))P(\sigma,x,y)\mathrm{d}y\right]\mathrm{d}\sigma, \quad x\in\mathbf{R}^n.$$

A straightforward calculation shows that $V(t,x;u)$, $x \in \mathbf{R}^n$ verifies $V(t,x;u) \in B(0,\rho_2)$ and also the equation

$$(17) \quad \Delta_x V(t,x) + AV(t,x) + h_0(x,u(t,x)) = 0, \quad x\in\mathbf{R}^n$$

for each $t \in [0,a]$.

By using the Lipschitz continuity of $h_0(y,u)$ in $u \in B(0,\rho_1)$ (according to (i_1) for h_0), we obtain from (16) that $V(t,x;u)$ fulfills also

$$(18) \quad \left|V(t,x;u'') - V(t,x;u')\right| \le L\left\|u''(t,\cdot)-u'(t,\cdot)\right\|, \quad t\in[0,a]\,,\ x\in\mathbf{R}^n$$

for any continuous applications $u''(t,x)$, $u'(t,x) \in B(0,\rho_1)$, where $\|u''(t,\cdot) - u'(t,\cdot)\| \overset{\Delta}{=} \sup_{x\in\mathbf{R}^n} |u''(t,x)-u'(t,x)|$. Therefore, $V(t,x;u)$ is Lipschitz continuous regarded as a map from the space of continuous bounded functions $u(t,\cdot) \in \mathbf{C}(\mathbf{R}^n; B(0,\rho_1))$ to the space of continuous bounded functions $v(t,\cdot) \in \mathbf{C}(\mathbf{R}^n; B(0,\rho_2))$ and (18) could be rewritten as

$$(19) \quad \left\|V(t,\cdot\,;u'') - V(t,\cdot\,;u')\right\| \le L\left\|u''(t,\cdot)-u'(t,\cdot)\right\|, \quad \forall t\in[0,a]\,,$$

where $u'(\cdot)$, $u''(\cdot) \in C\left([0,a]\times\mathbf{R}^n; B(0,\rho_1)\right)$ and $L > 0$ is constant. The standard process of successive approximations defines then $u = \hat{u}(t,x)$ with

$t \in [0,a]$, $x \in \mathbf{R}^n$ as a unique solution of the integral system

$$(20)\qquad \begin{cases} \hat{u}(t,x) = \displaystyle\int_{R^n} u_0(y)P(t,x,y)\mathrm{d}y + \\ \qquad\qquad + \displaystyle\int_0^t \mathrm{d}s \int_{R^n} f(y, \hat{u}(s,y), V(s,y;\hat{u})P(t-s,x,y)\mathrm{d}y \\ \hat{u}(t,x) \in B(0,\rho_1), \quad (t,x) \in [0,a] \times \mathbf{R}^n, \end{cases}$$

where the map $V(t,x;u)$ defined by (16) fulfills (17) and (19).

We denote now

$$\hat{v}(t,x) = V(t,x;\hat{u}), \quad (t,x) \in [0,a] \times \mathbf{R}^n$$

and by using (17) and (20) see easily that the pair of continuous functions $(\hat{u}(t,x),\ \hat{v}(t,x)) \in B(0,\rho_1) \times B(0,\rho_2)$ is a solution of the reduced system

$$(21)\qquad \begin{cases} \partial_t \hat{u} = \Delta_x \hat{u} + f(x,\hat{u},\hat{v}(t,x)),\ \hat{u}(0,x) = u_0(x), & t \in (0,a]\,,\ x \in \mathbf{R}^n, \\ \Delta_x \hat{v} + A\hat{v} + h_0(x,\hat{u}(t,x)) = 0, & (t,x) \in [0,a] \times \mathbf{R}^n. \end{cases}$$

We will show that taking the limit of the solution $(u_\varepsilon(t,x),\ v_\varepsilon(t,x))$, $(t,x) \in [0,a] \times \mathbf{R}^n$ of the integral system $(6')$ for $\varepsilon \downarrow 0$ we obtain the solution $(\hat{u}(t,x),\ \hat{v}(t,x))$, $(t,x) \in (0,a] \times \mathbf{R}^n$ of the reduced system (21).

For this purpose we first define for any $\varepsilon \in (0,1]$ the mapping

$$V_\varepsilon(t,x;u), \quad (t,x) \in [0,a] \times \mathbf{R}^n, \quad u(\cdot) \in \mathbf{C}\left([0,a] \times \mathbf{R}^n; B(0,\rho_1)\right)$$

as

$$(22)\qquad \begin{cases} V_\varepsilon(t,x;u) \overset{\Delta}{=} \Phi(\tau) \displaystyle\int_{R^n} v_0(y)P(\tau,x,y)\mathrm{d}y + \\ \qquad\qquad + \displaystyle\int_0^\tau \Phi(\tau - s)\mathrm{d}s \int_{R^n} h_0(y, u_\varepsilon(s,y))P(\tau - s,x,y)\mathrm{d}y, \\ V_\varepsilon(t,x;u) \in B(0,\rho_2), \quad \tau \overset{\Delta}{=} \frac{t}{\varepsilon}. \end{cases}$$

The same straightforward computations that prove $V(t,x;u)$ defined in (16) has the properties (17), (19) could be used to check that $V_\varepsilon(t,x;u)$ fulfills

$$(23)\qquad \begin{cases} \partial_t V_\varepsilon = \frac{1}{\varepsilon}\left[\Delta_x V_\varepsilon + AV_\varepsilon + h_0(x,u(t,x))\right], & t \in (0,a]\,, \\ V_\varepsilon(0,x) = v_0(x), & x \in \mathbf{R}^n, \end{cases}$$

and

$$(24)\qquad \left\| V_\varepsilon(t,\cdot\,;u'') - V_\varepsilon(t,\cdot\,;u') \right\| \le L_1' \left\| u''(t,\cdot) - u'(t,\cdot) \right\|,$$

where $u'(\cdot)$, $u''(\cdot) \in C\left([0,a] \times \mathbf{R}^n; B(0,\rho_1)\right)$ and $L_1 > 0$ is constant.

By using now $V_\varepsilon(t,x;u)$ we define (just as we previously defined the solution of (20)) $u=\tilde{u}_\varepsilon(t,x)$ as the unique solution of the system

$$\tilde{u}_\varepsilon(t,x)=\int_{R^n}u_0(y)P(t,x,y)\mathrm{d}y+ \tag{25}$$

$$+\int_0^t \mathrm{d}s\int_{R^n}f(y,\tilde{u}_\varepsilon(s,y),V_\varepsilon(s,y;\tilde{u}_\varepsilon))P(t-s,x,y)\mathrm{d}y$$

with $\tilde{u}_\varepsilon(0,x)=u_0(x)$ and $\tilde{u}_\varepsilon(t,x)\in B(0,\rho_1)$ for any $t\in[0,a]$, $x\in\mathbf{R}^n$ and $\varepsilon\in(0,1]$.

Taking now $\tilde{v}_\varepsilon(t,x)\overset{\Delta}{=}V_\varepsilon(t,x;\tilde{u}_\varepsilon)$, $(t,x)\in[0,a]\times\mathbf{R}^n$ and using (22) and (23) it follows with no difficulty that $(\tilde{u}_\varepsilon(t,x),\,\tilde{v}_\varepsilon(t,x))\,,\,(t,x)\in[0,a]\times\mathbf{R}^n$ is a solution of the integral system $(6')$.

The uniqueness of the solution of $(6')$ leads to the identities

$$\tilde{u}_\varepsilon(t,x)=u_\varepsilon(t,x),\quad \tilde{v}_\varepsilon(t,x)=v_\varepsilon(t,x)$$

for any $(t,x)\in[0,a]\times\mathbf{R}^n$, $\varepsilon\in(0,1]$ and it also follows that $(\tilde{u}_\varepsilon(t,x),\tilde{v}_\varepsilon(t,x))$ fulfills the perturbed parabolic system (1).

We will prove the next auxiliary result:

Lemma 3.3. *Let $V(t,x;u)$ and $V_\varepsilon(t,x;u)$ the maps defined by* (16) *and* (22) *respectively, for $(t,x)\in[0,a]\times\mathbf{R}^n$ and $u(\cdot)\in\mathbf{C}\left([0,a]\times\mathbf{R}^n;B(0,\rho_1)\right)$ and taking values in $\mathbf{C}\left([0,a]\times\mathbf{R}^n;B(0,\rho_2)\right)$. Assume the hypotheses* (i_1), (i_2) *and* (i'_3) *are fulfilled. Then $\lim\limits_{\varepsilon\downarrow 0}V_\varepsilon(t,x;u)=V(t,x;u)$ uniformly with respect to $x\in\mathbf{R}^n$ and $u(\cdot)\in\mathbf{C}_\gamma\left([0,a]\times\mathbf{R}^n;B(0,\rho_1)\right)$ for each $t\in(0,a]$, where the space $\mathbf{C}_\gamma\left([0,a]\times\mathbf{R}^n;B(0,\rho_1)\right)\subseteq\mathbf{C}\left([0,a]\times\mathbf{R}^n;B(0,\rho_1)\right)$ is defined below by* (30).

Proof. By using the change $\tau-s=\sigma\in[0,\tau]$, $s\in[0,\tau]$ and by decomposing $[0,\tau]\overset{\Delta}{=}\left[0,\frac{t}{\varepsilon}\right]=\left[0,\frac{t}{\sqrt{\varepsilon}}\right]\cup\left(\frac{t}{\sqrt{\varepsilon}},\frac{t}{\varepsilon}\right]$ we may rewrite (22) as

$$V_\varepsilon(t,x;u)=\Phi(\tau)\int_{R^n}v_0(y)P(\tau,x,y)\mathrm{d}y+ \tag{26}$$

$$+\int_0^{t/\sqrt{\varepsilon}}\Phi(\tau)\left[\int_{R^n}h_0(y,u(t-\varepsilon\sigma,y))P(\sigma,x,y)\mathrm{d}y\right]\mathrm{d}\sigma+$$

$$+\eta_\varepsilon(t,z)\overset{\Delta}{=}\hat{v}_\varepsilon(t,x)+\eta_\varepsilon(t,x),$$

where

$$\eta_\varepsilon(t,x)\overset{\Delta}{=}\int_{t/\sqrt{\varepsilon}}^{t/\varepsilon}\Phi(\sigma)\left[\int_{R^n}h_0(y,u(t-\varepsilon\sigma,y))P(\sigma,x,y)\mathrm{d}y\right]\mathrm{d}\sigma \tag{27}$$

is bounded for $\varepsilon \in (0,1]$, $(t,x) \in [0,a] \times \mathbf{R}^n$ and has the property

$$\lim_{\varepsilon \downarrow 0} \eta_\varepsilon(t,x) = 0 \tag{28}$$

uniformly for $(t,x) \in [0,a] \times \mathbf{R}^n$ and $u(\cdot) \in C\left([0,a] \times \mathbf{R}^n; B(0,\rho_1)\right)$.

On the other hand, for $\varepsilon \to 0$ and $\sigma \in [0, t/\sqrt{\varepsilon}]$ we have

$$\lim_{\varepsilon \downarrow 0} \varepsilon\sigma = 0 \tag{29}$$

uniformly for $t \in [0,a]$ and by choosing the space of functions

$$u(\cdot) \in \mathbf{C}_\gamma\left([0,a] \times \mathbf{R}^n; B(0,\rho_1)\right)$$

equally uniformly continuous with respect to $t \in [0,a]$, i.e.,

$$\left\| u(t'',\cdot) - u(t',\cdot) \right\| \le \gamma\left(\left|t'' - t'\right|\right), \tag{30}$$

where $\lim_{\delta \downarrow 0} \gamma(\sigma) = 0$ we obtain

$$\lim_{\varepsilon \downarrow 0} u(t - \varepsilon\sigma, x) = u(t,x) \tag{31}$$

uniformly for $x \in \mathbf{R}^n$ and $u(\cdot) \in \mathbf{C}_\gamma\left([0,a] \times \mathbf{R}^n; B(0,\rho_1)\right)$ for each $t \in (0,a]$. By using (31) in (26) we obtain with no difficulty

$$\lim_{\varepsilon \downarrow 0} \hat{v}_\varepsilon(t,x) = V(t,x;u) \tag{32}$$

uniformly for $x \in \mathbf{R}^n$ and $u(\cdot) \in \mathbf{C}_\gamma\left([0,a] \times \mathbf{R}^n; B(0,\rho_1)\right)$ for each $t \in (0,a]$. The equations (28) and (32) complete the proof of the lemma. □

Remark 3.4. It appears as necessary for the family of solutions

$$u_\varepsilon(t,x), \quad \varepsilon \in (0,1], \quad (t,x) \in [0,a] \times \mathbf{R}^n$$

defined by (6′) and corresponding to $v = V_\varepsilon(t,x;u)$ to have an "equal uniform continuity property" for $t \in [0,a]$, property of the type

$$\left\| u_\varepsilon(t'',\cdot) - u_\varepsilon(t',\cdot) \right\| \le \gamma\left(\left|t'' - t'\right|\right), \quad \varepsilon \in (0,1], \tag{33}$$

where $\gamma(\delta)$ verifies $\lim_{\delta \downarrow 0} \gamma(\delta) = 0$.

By definition $u = u_\varepsilon(t,x) \in B(0,\rho_1)$, $t \in [0,a]$, $x \in \mathbf{R}^n$ is the unique solution of the integral system

$$u_\varepsilon(t,x) = \int_{R^n} u_0(y) P(t,x,y) \mathrm{d}y + \tag{34}$$
$$+ \int_0^t \mathrm{d}s \int_{R^n} f(y, u_\varepsilon(s,y), v_\varepsilon(s,y)) P(t-s,x,y) \mathrm{d}y,$$

where $v_\varepsilon(t,x) \triangleq V_\varepsilon(t,x;u_\varepsilon)$ and the map $V_\varepsilon(t,x;u)$ defined in (22) has the properties (23) and (24).

Moreover, if $u_0(\cdot) \in C^1(\mathbf{R}^n;\mathbf{R}^m)$ and fulfills

$$p_0^i(x) \triangleq \partial_i u_0(x) \triangleq \frac{\partial u_0}{\partial x_i}(x) \in B(0,\rho_1/2) \tag{i$_4$}$$

for $x \in \mathbf{R}^n$, $i \in \{1,\ldots,n\}$, then the solution of (34) is $\mathbf{C}^1$ for $x \in \mathbf{R}^n$ and $p_\varepsilon^i(t,x) \triangleq \partial_i u_\varepsilon(t,x)$ is continuous and verifies the following system

$$p_\varepsilon^i(t,x) = \int_{R^n} p_0^i(y)P(t,x,y)\mathrm{d}y + \tag{35}$$

$$+\int_0^t \mathrm{d}s \int_{R^n} f\left(y,u_\varepsilon(s,y),v_\varepsilon(s,y)\right)\partial_{x_i}P(t-s,x,y)\mathrm{d}y,$$

$p_\varepsilon^i(t,x) \in B(0,\rho_1)$, $\forall(t,x) \in [0,a]\times\mathbf{R}^n$, $i \in \{1,\ldots,n\}$, for $0 < a$ sufficiently small.

The statements related to (35) are based upon the standard successive approximations procedure applied to the combined system of integral equations (34) and (35), which leads to the unique continuous and uniformly bounded solution $\left(u_\varepsilon(t,x),p_\varepsilon^1(t,x),\ldots,p_\varepsilon^n(t,x)\right)$, $(t,x)\in[0,a]\times\mathbf{R}^n$.

Therefore, if the initial value $u_0(\cdot)$ fulfils (i$_4$), then the solution $u_\varepsilon(t,x)$, $x \in \mathbf{R}^n$ defined by (34) is Lipschitz continuous and

$$\left|u_\varepsilon(t,x'') - u_\varepsilon(t,x')\right| \le C_0\left|x''-x'\right| \tag{36}$$

for any $x',x'' \in \mathbf{R}^n$, $t \in [0,a]$ and $\varepsilon \in (0,1]$, where $C_0 > 0$ is constant.

On the other hand, for $t \in [t',t''] \le [0,a]$ we define

$$\hat{u}_\varepsilon(s,x) \triangleq u_\varepsilon(t'+s,x), \quad s \in \left[0,t''-t'\right], x \in \mathbf{R}^n, \tag{37}$$

and by using the related parabolic system

$$\begin{cases} \partial_s\hat{u}_\varepsilon(s,x) = \Delta_x\hat{u}_\varepsilon(s,x) + f(x,\hat{u}_\varepsilon(s,x),\hat{v}_\varepsilon(s,x)), & s\in(0,t''-t'],\\ \hat{u}_\varepsilon(0,x) = u_\varepsilon(t',x), & x\in\mathbf{R}^n,\end{cases}$$

we represent the solution $\hat{v}_\varepsilon(s,x) \triangleq v_\varepsilon(t'+s,x)$ by means of the integral system

$$\hat{u}_\varepsilon(s,x) = \int_{R^n} u_\varepsilon(t',y)P(s,x,y)\mathrm{d}y + \tag{38}$$

$$+\int_0^s \mathrm{d}\sigma \int_{R^n} f\left(y,\hat{u}_\varepsilon(\sigma,y),\hat{v}_\varepsilon(\sigma,y)\right)P(s-\sigma,x,y)\mathrm{d}y$$

for any $s \in [0,t''-t']$, $x \in \mathbf{R}^n$, $\varepsilon \in (0,1]$, where $\hat{u}_\varepsilon(s,x) \in B(0,\rho_1)$ and $\hat{v}_\varepsilon(s,x) \in B(0,\rho_2)$.

By using now (37) and (38) for $s = t''-t'$ we obtain

$$\left|u_\varepsilon(t'',x)-u_\varepsilon(t',x)\right| \le \int_{R^n}\left|u_\varepsilon(t',y)-u_\varepsilon(t',x)\right|P(t''-t',x,y)\mathrm{d}y + C\left|t''-t'\right| \tag{39}$$

for any $x \in \mathbf{R}^n$, $\varepsilon \in (0,1]$, where $C > 0$ is the boundedness constant given by (i_1) for the function f.

We also interpret the first term in the right side of (38) as the expectation on the probability space $\{\Omega, F, P\}$ related to a Wiener standard process $w(t) \in \mathbf{R}^n$, $t \in [0,\infty)$ and we define $y(t;x) \overset{\Delta}{=} x + \sqrt{2}w(t)$, to obtain

$$\int_{R^n} \left|u_\varepsilon(t',y) - u_\varepsilon(t',x)\right| P(t''-t',x,y)\mathrm{d}y = \\ = E\left|u_\varepsilon\left(t', y(t''-t';x)\right) - u_\varepsilon(t',x)\right|. \tag{40}$$

Taking advantage of (36) in (40) we are led to

$$\int_{R^n} \left|u_\varepsilon(t',y) - u_\varepsilon(t',x)\right| P(t''-t',x,y)\mathrm{d}y \leq \\ \leq C_1 E\left|w(t''-t')\right| \leq C_1\sqrt{|t''-t'|} \tag{41}$$

for any $[t',t''] \subseteq [0,a]$, $x \in \mathbf{R}^n$, $\varepsilon \in (0,1]$, where $C_1 > 0$ is constant.

We use now (41) in (39) to deduce

$$\left\|u_\varepsilon(t'',\cdot) - u_\varepsilon(t',\cdot)\right\| \leq C_2\sqrt{|t''-t'|}, \quad \forall\left[t',t''\right] \subseteq [0,a], \ \varepsilon \in (0,1], \tag{42}$$

where $C_2 > 0$ is constant.

The calculations above may be resumed in

LEMMA 3.5. *Let $f(x,u,v) \in \mathbf{R}^m$, $h(x,u,v) \in \mathbf{R}^k$ be such that* (i_1), (i_2) *and* (i_3') *are fulfilled. Let $u_0(\cdot)$ also fulfill hypothesis* (i_4). *Consider $u_\varepsilon(t,x)$ and $\hat{u}(t,x)$, $t \in [0,a]$, $x \in \mathbf{R}^n$, $\varepsilon \in (0,1]$ the solutions of the integral system* $(6')$ *corresponding to $v = v_\varepsilon(t,x) \overset{\Delta}{=} V_\varepsilon(t,x;u_\varepsilon)$, respectively to $v = \hat{v}(t,x) = V(t,x;\hat{u})$. Then $u_\varepsilon(\cdot), \hat{u}(\cdot) \in C_\gamma\left([0,a] \times \mathbf{R}^n; B(0,\rho_1)\right)$, where $\gamma(\delta) = C_2\sqrt{\delta}$ is given by* (42) *and the space C_γ is defined by* (30).

The synthesis of the considerations presented in all the above remarks and lemmas is given in the next

THEOREM 3.6. *Assume $f(x,u,v) \in \mathbf{R}^m$, $h(x,u,v) \in \mathbf{R}^k$ and $u_0(\cdot) \in C^1(\mathbf{R}^n;\mathbf{R}^m)$ are given such that the hypotheses* (i_1), (i_2) *and* (i_3'), *respectively $\partial_i u_0(x) \in B(0,\rho_1/2)$, $i \in \{1,\ldots,n\}$ are fulfilled.*

Let $(u_\varepsilon(t,x), v_\varepsilon(t,x))$, $\varepsilon \in (0,1]$, $(t,x) \in [0,a] \times \mathbf{R}^n$ be the solution of the system (1) *represented by the integral equations* $(6')$.

Let $(\hat{u}(t,x), \hat{v}(t,x))$, $(t,x) \in [0,a] \times \mathbf{R}^n$ be the solution of the reduced system (21), *where $\hat{v}(t,x) \overset{\Delta}{=} V(t,x;\hat{u})$, $v_\varepsilon(t,x) \overset{\Delta}{=} V_\varepsilon(t,x;u_\varepsilon)$, the applications $V(t,x;u)$, $V_\varepsilon(t,x;u)$ defined by* (16) *and* (22) *and related by Lemma* 3.3.

Then $\lim\limits_{\varepsilon\downarrow 0}(u_\varepsilon(t,x), v_\varepsilon(t,x)) = (\hat{u}(t,x), \hat{v}(t,x))$ uniformly with respect to $x \in \mathbf{R}^n$, for each fixed $t \in [0,a]$.

Proof. The assumption is that the conditions in Lemmas 3.3 and 3.5 are fulfilled, hence $u_\varepsilon(\cdot), \hat{u}(\cdot) \in C_\gamma([0,a]\times\mathbf{R}^n; B(0,\rho_1))$, $\varepsilon\in(0,1]$, where $\gamma(\delta)=C_2\sqrt{\delta}$. On the other hand, by using (i_2) we deduce

$$|u_\varepsilon(t,x)-\hat{u}(t,x)|\leq \tag{43}$$

$$\leq\int_0^t \mathrm{d}s\left[\int_{R^n}|f(y,u_\varepsilon(s,y),v_\varepsilon(s,y))-f(y,\hat{u}(s,y),\hat{v}(s,y))|\,P(t-s,x,y)\mathrm{d}y\right]\leq$$

$$\leq L\int_0^t\|u_\varepsilon(s,\cdot)-\hat{u}(s,\cdot)\|\,\mathrm{d}s+L\int_0^t\|v_\varepsilon(s,\cdot)-\hat{v}(s,\cdot)\|\,\mathrm{d}s,$$

for any $x\in\mathbf{R}^n$ and any $t\in[0,a]$, where $v_\varepsilon(t,x)\overset{\Delta}{=}V_\varepsilon(t,x;u_\varepsilon)=V_\varepsilon(t,x;\hat{u})+[V_\varepsilon(t,x;u_\varepsilon)-V_\varepsilon(t,x;\hat{u})]$, $\hat{v}(t,x)\overset{\Delta}{=}V(t,x;\hat{u})$.

From (24) we have

$$\|V_\varepsilon(t,\cdot\,;u_\varepsilon)-V_\varepsilon(t,\cdot\,;\hat{u})\|\leq L_1\|u_\varepsilon(t,\cdot)-\hat{u}(t,\cdot)\|\,,\quad\forall t\in[0,a] \tag{44}$$

and we rewrite (43) as

$$\|u_\varepsilon(t,\cdot)-\hat{u}(t,\cdot)\|\leq \tag{45}$$

$$\leq L_2\int_0^t\|u_\varepsilon(s,\cdot)-\hat{u}(s,\cdot)\|\,\mathrm{d}s+L\int_0^t\|V_\varepsilon(s,\cdot\,;\hat{u})-V(s,\cdot\,;u)\|\,\mathrm{d}s,$$

where $u_\varepsilon(\cdot),\hat{u}(\cdot)\in C_\gamma([0,a]\times\mathbf{R}^n;B(0,\rho_1))$, $\varepsilon\in(0,1]$ and $V_\varepsilon(t,x;u)\in B(0,\rho_2)$, $V(t,x;u)\in B(0,\rho_2)$ fulfill the conclusion of Lemma 3.3.

We apply Gronwall's lemma and from (45) we obtain

$$\|u_\varepsilon(t,\cdot)-\hat{u}(t,\cdot)\|\leq\beta(\varepsilon) \tag{46}$$

for any $t\in[0,a]$, where $\beta(\varepsilon)\overset{\Delta}{=}L\left(\int_0^a\|V_\varepsilon(t,\cdot\,;\hat{u})-V(t,\cdot\,;\hat{u})\|\,\mathrm{d}t\right)\exp(L_2\cdot a)$ has the property

$$\lim_{\varepsilon\downarrow 0}\beta(\varepsilon)=0. \tag{47}$$

Therefore, we also obtain that $v_\varepsilon(t,x)-\hat{v}(t,x)\overset{\Delta}{=}V_\varepsilon(t,x;u_\varepsilon)-V(t,x;\hat{u})=[V_\varepsilon(t,x;\hat{u})-V(t,x;\hat{u})]+[V_\varepsilon(t,x;u_\varepsilon)-V_\varepsilon(t,x;\hat{u})]$ fulfills

$$\lim_{\varepsilon\downarrow 0}\|v_\varepsilon(t,\cdot)-\hat{v}(t,\cdot)\|=0 \tag{48}$$

for each $t\in(0,a]$ and this completes the proof. $\square$

FINAL REMARKS

The solution $v = \hat{v}(t,x) \in B(0,\rho_2) \subset \mathbf{R}^k$ in Theorem 3.6 is related to the elliptic system (21) and has the explicit form (16).

It is easily seen that for $h_0(x,u) = 0$ the solution $\hat{v}(\cdot)$ is also identically 0, even if the related elliptic equations $\Delta v + Av = 0$ may have non-trivial bounded solutions. For example, the equation $v''(x) - v(x) = 0$, $x \in \mathbf{R}$ admits a one-dimensional space $N \overset{\Delta}{=} \{v(x) = \lambda\tilde{v}(x),\ \lambda, x \in \mathbf{R}\}$ as set of solutions (where $\tilde{v}(x) = \begin{cases} \mathrm{e}^{-x}, & x \in [0,\infty), \\ \mathrm{e}^{x}, & x \in (-\infty, 0] \end{cases}$ fulfills the equation $v'' - v = 0$ for any $x \neq 0$).

The presence of the stable matrix A in the elliptic equations $\Delta v + Av = 0$ is very helpful for defining solutions of the equations $\Delta v + Av + \mu h_0(x,t) = 0$, $v = \hat{v}_\mu(t,x)$, $\mu \in [-a,a]$ continuous with respect to the parameter μ and avoiding the bifurcation phenomenon related to those elliptic equations.

On the other hand, the solution of the perturbed part in (1)

$$(*)\qquad \begin{cases} \varepsilon\partial_t v = \Delta_x v + Av + h_0(x, u_\varepsilon(t,x)), & t \in [0,a]\,,\ x \in \mathbf{R}^n, \\ v(0,x) = v_0(x) \end{cases}$$

could be represented by using a singularly perturbed process $y_\varepsilon(t,x) = x + \sqrt{\frac{2}{\varepsilon}}w(t)$ determined by the Wiener standard process.

The solution of $(*)$ could be written

$$v_\varepsilon(t,x) = \Phi\left(\frac{t}{\varepsilon}\right) Ev_0(y_\varepsilon(t,x)) +$$

$$+\frac{1}{\varepsilon}\int_0^t \Phi\left(\frac{t-s}{\varepsilon}\right) Eh_0\left(y_\varepsilon(t-s), u_\varepsilon(s, y_\varepsilon(t-s,x))\right) \mathrm{d}s \overset{\Delta}{=} E\tilde{v}_\varepsilon(t,x,\omega),$$

where the process $\tilde{v}_\varepsilon(t,x,\omega)$ fulfills a stochastic parabolic system

$$(**)\quad \varepsilon d_t\tilde{v} = [\Delta_x\tilde{v} + A\tilde{v} + h_0(x,u_\varepsilon(t,x))]\,\mathrm{d}t + \sqrt{\varepsilon}\,(\partial_x\tilde{v})_*\,\mathrm{d}W(t), \quad t \in (0,a]\,,$$

if $v_0(\cdot)$ and $h_0(\cdot)$ are $\mathbf{C}^1$ and where $_*$ is the Ito integral.

The analysis from Theorem 3.6 could be extended to stochastic systems of the form $(**)$.

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Received 27 April 2009

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