ITERATIVE METHODS FOR SEMIGROUPS OF NONEXPANSIVE MAPPINGS AND VARIATIONAL INEQUALITIES

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We introduce an iterative method for finding a common fixed point of a semigroup of nonexpansive mappings in a Hilbert space, with respect to a sequence of left regular means defined on an appropriate space of bounded real valued functions of the semigroup. We prove the strong convergence of the proposed iterative algorithm to the unique solution of a variational inequality, which is the optimality condition for a minimization problem. Compared to the similar works, our results have the merit of imposing weaker hypotheses on coefficients.

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1. INTRODUCTION

Let E be a real Banach space and let C be a closed convex subset of E. Then, a mapping T of C into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. Mann [11] introduced an iteration procedure for approximation of fixed points of a nonexpansive mapping T in a Hilbert space as follows: Let $x_0 \in C$ and

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n x_n, \quad n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. See also Halpern [7].

Moudafi [13] introduced the viscosity approximation method for nonexpansive mappings. Let f be a contraction on a Hilbert space H (i.e., $\|f(x) - f(y)\| \leq \alpha\|x - y\|$, $x, y \in H$ and $0 \leq \alpha < 1$). Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n f(x_n), \quad n \geq 0, \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. It is proved [13, 21] that under certain appropriate conditions imposed on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.1) strongly converges to the unique solution x^* in $F(T)$ of the variational inequality

$$\langle (I-f)x^*,\, x-x^*\rangle \geq 0,\;\; x \in F(T).$$

On the other hand, iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [6, 19, 20, 22, 23]. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H

$$\min_{x\in F} \frac{1}{2}\langle Ax, x\rangle - \langle x, u\rangle, \tag{1.2}$$

where F is the fixed point set of a nonexpansive mapping T on H and u is a given point in H. Assume A is strongly positive; that is, there is a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \text{for all } x \in H.$$

In [19] (see also [22]), it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n u, \quad n \geq 0, \tag{1.3}$$

converges strongly to the unique solution of the minimization problem (1.2) provided the sequence $\{\alpha_n\}$ satisfies certain conditions.

Marino and Xu [12] combined the iterative method (1.3) with the viscosity approximation method (1.1) and consider the following general iterative method

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{1.4}$$

where $0 < \gamma < \overline{\gamma}/\alpha$. They proved that if the sequence $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the following conditions:

(C1) $\alpha_n \to 0$;

(C2) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$;

(C3) either $\sum\limits_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim\limits_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$;

then the sequence $\{x_n\}$ generated by (1.4) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in F, \tag{1.5}$$

which is the optimality condition for the minimization problem

$$\min_{x\in F} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$, for $x \in H$).

Finding an optimal point in the intersection F of the fixed point sets of a family of nonexpansive mappings is a task that occurs frequently in various areas of mathematical sciences and engineering. For example, the well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings; see, e.g., [2, 5]. A simple algorithmic solution to the problem of minimizing a quadratic function over F is of extreme value in many applications including set theoretic signal estimation; see, e.g., [9, 24].

See, e.g., [4, 18] for solving variational problems defined on the set of common fixed points of finitely many nonexpansive mappings.

In this paper, motivated by, Xu [21, 19], Marino and Xu [12], Colao, et al. [4], Atsushiba and Takahashi [1], Shimizu and Takahashi [15] and Takahashi [18], we introduce the following iterative algorithm: Let $x_0 \in H$, and

$$x_{n+1} = (I - \alpha_n A)T(\mu_n)x_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{1.6}$$

where $\mathcal{S} = \{T(t) : t \in S\}$ is a nonexpansive semigroup on H such that $F(\mathcal{S}) \neq \varnothing$, X is a subspace of $l^\infty(S)$ such that $1 \in X$ and the function $t \mapsto \langle T(t)x, y\rangle$ is an element of X for each $x, y \in H$, and $\{\mu_n\}$ is a sequence of means on X.

We will prove that if $\{\mu_n\}$ is left regular and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying the conditions (C1) and (C2), then $\{x_n\}$ converges in norm to a x^* in $F(\mathcal{S})$, the set of common fixed points of $\mathcal{S}$, which solves the variational inequality

$$\langle (A - \gamma f)x^*, \, x - x^*\rangle \geq 0, \quad x \in F(\mathcal{S}) \tag{1.7}$$

and is the optimality condition for the minimization problem

$$\min_{x \in F(\mathcal{S})} \frac{1}{2} \langle Ax, x\rangle - h(x),$$

where h is a potential function for γf. Various applications to the additive semigroup of nonnegative real numbers and commuting pairs of nonexpansive mappings are also presented. It is worth mentioning that we obtain our results without assuming the condition (C3).

2. PRELIMINARIES

Let S be a semigroup. We denote by $l^\infty(S)$ the Banach space of all bounded real valued functions on S with supremum norm. For each $s \in S$, we define l_s and r_s on $l^\infty(S)$ by $(l_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for each $t \in S$ and $f \in l^\infty(S)$. Let X be a subspace of $l^\infty(S)$ containing 1 and let X^* be its topological dual. An element μ of X^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be left invariant (resp. right invariant), i.e., $l_s(X) \subset X$ (resp. $r_s(X) \subset X$) for each $s \in S$. A mean μ on X is said to be left invariant (resp. right invariant) if $\mu(l_s f) = \mu(f)$ (resp. $\mu(r_s f) = \mu(f)$) for each $s \in S$

and $f \in X$. X is said to be left (resp. right) amenable if X has a left (resp. right) invariant mean. X is amenable if X is both left and right amenable. As is well known, $l^\infty(S)$ is amenable when S is a commutative semigroup or a solvable group.

A net $\{\mu_\alpha\}$ of means on X is said to be left regular if $\lim_\alpha \|l_s^*\mu_\alpha - \mu_\alpha\| = 0$ for each $s \in S$, where l_s^* is the adjoint operator of l_s.

Let C be a nonempty closed and convex subset of E. A family $\mathcal{S} = \{T(s) : s \in S\}$ is called a nonexpansive semigroup on a C if for each $s \in S$ the mapping $T(s) : C \to C$ is nonexpansive and $T(st) = T(s)T(t)$ for each $s, t \in S$. We denote by $F(\mathcal{S})$ the set of common fixed points of $\mathcal{S}$.

The open ball of radius r centered at 0 is denoted by B_r. For a subset A of E, we denote by $\overline{co}A$ and the closed convex hull of A. Weak convergence is denoted by $\rightharpoonup$.

Below, Lemmas 2.1 and 2.2 can be found in [16, 10, 14], Lemmas 2.3 and 2.5 in [17], Lemma 2.4 in [12], and Lemma 2.7 in [20].

Lemma 2.1. *Let f be a function of semigroup S into a Banach space E such that the weak closure of $\{f(t) : t \in S\}$ is weakly compact and let X be a subspace of $l^\infty(S)$ containing all the functions $t \to \langle f(t), x^*\rangle$ with $x^* \in E^*$. Then, for any $\mu \in X^*$, there exists a unique element f_μ in E such that*

$$\langle f_\mu, x^*\rangle = \mu_t\langle f(t),\ x^*\rangle$$

for all $x^* \in E^*$. Moreover, if μ is a mean on X then

$$\int f(t)\mathrm{d}\mu(t) \in \overline{co}\{f(t) : t \in S\}.$$

We can write f_μ by $\int f(t)\mathrm{d}\mu(t)$.

Lemma 2.2. *Let C be a closed convex subset of a Hilbert space H, $\mathcal{S} = \{T(s) : s \in S\}$ be a nonexpansive semigroup from C into C such that $F(\mathcal{S}) \neq \varnothing$ and X be a subspace of $l^\infty(S)$ such that $1 \in X$ and the mapping $t \mapsto \langle T(t)x, y\rangle$ be an element of X for each $x \in C$ and $y \in H$, and μ be a mean on X.*

If we write $T(\mu)x$ instead of $\int T_t x\mathrm{d}\mu(t)$, then the following hold.

(i) *$T(\mu)$ is a nonexpansive mapping from C into C.*

(ii) *$T(\mu)x = x$ for each $x \in F(\mathcal{S})$.*

(iii) *$T(\mu)x \in \overline{co}\{T_t x : t \in S\}$ for each $x \in C$.*

(iv) *If μ is left invariant, then $T(\mu)$ is a nonexpansive retraction from C onto $F(\mathcal{S})$.*

Recall the metric (nearest point) projection P_K from a Hilbert space H to a closed convex subset K of H is defined as follows: given $x \in H$, $P_K x$ is the only point in K with the property

$$\|x - P_K x\| = \inf\{\|x - y\| : y \in K\}.$$

P_K is characterized as follows.

LEMMA 2.3. *Let K be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in K$. Then $y = P_K x$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0, \quad \forall z \in K.$$

LEMMA 2.4. *Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

LEMMA 2.5. *Let C be a closed convex subset of H and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \varnothing$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

The following lemma is an immediate consequence of the inner product on H.

LEMMA 2.6. *For all $x, y \in H$, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle.$$

LEMMA 2.7. *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum\limits_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup\limits_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum\limits_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim\limits_{n\to\infty} a_n = 0$.

Throughout this paper, A will denote a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$ and f a contraction with coefficient $0 < \alpha < 1$ on Hilbert space H.

3. STRONG CONVERGENCE OF A GENERAL INTERATIVE METHOD

The following is our main result.

THEOREM 3.1. *Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on H such that $F(\mathcal{S}) \neq \varnothing$. Let X be a left invariant subspace of $l^\infty(S)$ such that $1 \in X$, and the function $t \mapsto \langle T(t)x, y \rangle$ is an element of X for each $x, y \in H$. Let $\{\mu_n\}$ be a left regular sequence of means on X and let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\alpha_n \to 0$ and $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$. Let $x_0 \in H$, $0 < \gamma < \overline{\gamma}/\alpha$ and let $\{x_n\}$ be generated by the iterative algorithm (1.6). Then $\{x_n\}$ converges in norm to $x^* \in F(\mathcal{S})$ which is a unique solution of the variational inequality (1.7). Equivalently, we have $P_{F(\mathcal{S})}(I - A + \gamma f)x^* = x^*$.*

Proof. Since $\alpha_n \to 0$ and $0 < \overline{\gamma} - \gamma\alpha$, we my assume, with no loss of generality, that $\alpha_n < \|A\|^{-1}$ and

$$0 < [2(\overline{\gamma} - \gamma\alpha) + \alpha_n\overline{\gamma}(2\gamma\alpha - \overline{\gamma})]\alpha_n < 1. \tag{3.1}$$

Let p be an arbitrary element of $F(\mathcal{S})$. Then

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(I - \alpha_n A)(T(\mu_n)x_n - p) + \alpha_n(\gamma f(x_n) - Ap)\| \\
&\le (1 - \overline{\gamma}\alpha_n)\|x_n - p\| + \alpha_n\|\gamma f(x_n) - Ap\| \ \text{(by Lemma 2.2)} \\
&\le (1 - \overline{\gamma}\alpha_n)\|x_n - p\| + \alpha_n(\gamma\|f(x_n) - f(p)\| + \|\gamma f(p) - Ap\|) \\
&\le (1 - (\overline{\gamma} - \gamma\alpha)\alpha_n)\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| \\
&= (1 - (\overline{\gamma} - \gamma\alpha)\alpha_n)\|x_n - p\| + (\overline{\gamma} - \gamma\alpha)\alpha_n\frac{\|\gamma f(p) - Ap\|}{\overline{\gamma} - \gamma\alpha}.
\end{aligned}$$

It follows from induction that

$$\|x_n - p\| \le \max\left\{\|x_n - p\|, \frac{\|\gamma f(p) - Ap\|}{\overline{\gamma} - \gamma\alpha}\right\} = M_0, \quad n \ge 0.$$

Set $D = \{y \in H : \|y - p\| \le M_0\}$. We remark D is a $\mathcal{S}$-invariant bounded closed convex set and $\{x_n\} \subseteq D$. We will show that

$$\limsup_{n\to\infty} \sup_{y\in D} \|T(\mu_n)y - T(t)T(\mu_n)y\| = 0, \quad \forall t \in S. \tag{3.2}$$

Our proof of (3.2) follows the lines of a proof in [1]. Let $\varepsilon > 0$. By [3, Theorem 1.2], there exists $\delta > 0$ such that

$$\overline{co}\, F_\delta(T(t); D) + B_\delta \subseteq F_\varepsilon(T(t); D), \quad \forall t \in S. \tag{3.3}$$

By [3, Corollary 1.1], there also exists a natural number N such that

$$\left\|\frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)y - T_t\left(\frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)y\right)\right\| \le \delta, \tag{3.4}$$

for all $t, s \in S$ and $y \in D$. Let $t \in S$. Since $\{\mu_n\}$ is strongly left regular, there exists $n_0 \in \mathbb{N}$ such that $\|\mu_n - l_{t^i}^*\mu_n\| \le \delta/(M_0 + \|p\|)$ for $n \ge n_0$ and $i = 1, \dots, N$. Then we have

$$\begin{aligned}
&\sup_{y\in D}\left\|T(\mu_n)y - \int \frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)y\,\mathrm{d}\mu_n(s)\right\| \qquad (3.5) \\
&= \sup_{y\in D}\sup_{\|z\|=1}\left|(\mu_n)_s\langle T(s)y, z\rangle - (\mu_n)_s\left\langle \frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)y, z\right\rangle\right| \\
&\le \frac{1}{N+1}\sum_{i=0}^{N}\sup_{y\in D}\sup_{\|z\|=1}|(\mu_n)_s\langle T(s)y, z\rangle - (l_{t^i}^*\mu_n)_s\langle T(s)y, z\rangle| \\
&\le \max_{i=1,\dots,N}\|\mu_n - l_{t^i}^*\mu_n\|(M_0 + \|p\|) \le \delta, \quad \forall n \ge n_0.
\end{aligned}$$

On the other hand, we note that

$$\int \frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)y\mathrm{d}\mu_n(s) \in \overline{co}\Big\{\frac{1}{N+1}\sum_{i=0}^{N} T(t)^i(T(s)y) : s\in S\Big\}.$$

From (3.3), (3.4), (3.5) and the above, we have

$$T(\mu_n)y = \int \frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)y\mathrm{d}\mu_n(s) + \Big(T(\mu_n)y - \int \frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)y\mathrm{d}\mu_n(s)\Big)$$

$$\in \overline{co}\Big\{\frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)y : s\in S\Big\} + B_\delta \subseteq \overline{co}\,F_\delta(T(t);D) + B_\delta \subseteq F_\varepsilon(T(t);D),$$

for all $y\in D$ and $n\geq n_0$. Therefore,

$$\limsup_n \sup_{y\in D} \|T(t)T(\mu_n)y - T(\mu_n)y\| \leq \varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we get (3.2). In this stage, we will show

$$\lim_n \|x_n - T(t)x_n\| = 0, \quad \forall t\in S. \tag{3.6}$$

Let $t\in S$ and $\varepsilon>0$. Then there exists $\delta>0$ which satisfies (3.3). Take $L_0 = (\gamma\alpha + \|A\|)M_0 + \|\gamma f(p) - Ap\|$. From $\lim \alpha_n = 0$ and (3.2), there exists $k_0\in\mathbb{N}$ such that $\alpha_n < \delta/L_0$ and $T(\mu_n)x_n \in F_\delta(T(t))$, for all $n>k_0$. We note that

$$\begin{aligned}
&\|\alpha_n(\gamma f(x_n) - AT(\mu_n)x_n)\|\\
&\leq \alpha_n(\gamma\|f(x_n)-f(p)\| + \|\gamma f(p) - Ap\| + \|AT(\mu_n)x_n - Ap\|)\\
&\leq \alpha_n(\gamma\alpha\|x_n - p\| + \|A\|\|x_n-p\| + \|\gamma f(p) - Ap\|)\\
&\leq \alpha_n((\gamma\alpha + \|A\|)M_0 + \|\gamma f(p) - Ap\|) \leq \frac{\delta}{L_0}L_0 = \delta,
\end{aligned}$$

for all $n>k_0$. Therefore, we have

$$x_{n+1} = T(\mu_n)x_n + \alpha_n(\gamma f(x_n) - AT(\mu_n)x_n) \in F_\delta(T(t)) + B_\delta \subseteq F_\varepsilon(T(t)),$$

for all $n>k_0$. This shows that

$$\limsup_n \|x_n - T(t)x_n\| \leq \varepsilon,$$

and since $\varepsilon>0$ is arbitrary, we get (3.6).

Banach' s Contraction Mapping Principal guarantees that $P_{F(\mathcal{S})}(\gamma f + (I-A))$ has a unique fixed point x^* which is the unique solution of the variational inequality

$$\langle (A-\gamma f)x^*,\, x - x^*\rangle \geq 0, \quad x\in F(\mathcal{S}).$$

We show that

$$\limsup_{n\to\infty} \langle x_n - x^*,\, \gamma f(x^*) - Ax^*\rangle \leq 0. \tag{3.7}$$

To see this, we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle x_n - x^*,\, \gamma f(x^*) - Ax^*\rangle = \lim_{k\to\infty} \langle x_{n_k} - x^*,\, \gamma f(x^*) - Ax^*\rangle.$$

We may also assume that $x_{n_k} \rightharpoonup z$. Note that $z \in F(\mathcal{S})$ in virtue of Lemma 2.5 and (3.6). Therefore,

$$\limsup_{n\to\infty} \langle x_n - x^*,\, \gamma f(x^*) - Ax^*\rangle = \langle z - x^*,\, \gamma f(x^*) - Ax^*\rangle \le 0.$$

Finally, we prove $\|x_n - x^*\| \to 0$. To this end, we calculate

$$\|x_{n+1} - x^*\|^2 = \|(I - \alpha_n A)(T(\mu_n)x_n - x^*) + \alpha_n(\gamma f(x_n) - Ax^*)\|^2 \tag{3.8}$$
$$\le \|(I-\alpha_n A)(T(\mu_n)x_n - x^*)\|^2 + 2\alpha_n\langle \gamma f(x_n) - Ax^*,\, x_{n+1} - x^*\rangle \quad \text{(by Lemma 2.6)}$$
$$\le (1 - \alpha_n\overline{\gamma})^2\|x_n - x^*\|^2 + 2\alpha_n\langle \gamma f(x_n) - Ax^*,\, x_{n+1} - x^*\rangle.$$

On the other hands,

$$\langle \gamma f(x_n) - \gamma f(x^*), x_{n+1} - x^*\rangle \le \gamma\alpha\|x_n - x^*\|\,\|x_{n+1} - x^*\|$$
$$\le \gamma\alpha\|x_n - x^*\|\sqrt{(1-\alpha_n\overline{\gamma})^2\|x_n - x^*\|^2 + 2\alpha_n\langle \gamma f(x_n) - Ax^*,\, x_{n+1} - x^*\rangle} \quad \text{(by (3.8))}$$
$$\le \gamma\alpha(1 - \alpha_n\overline{\gamma})\|x_n - x^*\|^2 + \gamma\alpha\|x_n - x^*\|\sqrt{2|\langle \gamma f(x_n) - Ax^*,\, x_{n+1} - x^*\rangle|}\sqrt{\alpha_n}.$$

Since $\{x_n\}$ is bounded, we can take a constant $G_0 > 0$ such that

$$\gamma\alpha\|x_n - x^*\|\sqrt{2|\langle \gamma f(x_n) - Ax^*, x_{n+1} - x^*\rangle|} < G_0, \quad (\forall n).$$

So, from the above, we reach the following

$$\langle \gamma f(x_n) - \gamma f(x^*), x_{n+1} - x^*\rangle \le \gamma\alpha(1 - \alpha_n\overline{\gamma})\|x_n - x^*\|^2 + G_0\sqrt{\alpha_n}. \tag{3.9}$$

Now, combining (3.8) and (3.9), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le (1 - \alpha_n\overline{\gamma})^2\|x_n - x^*\|^2 + 2\alpha_n\langle \gamma f(x_n) - \gamma f(x^*), x_{n+1} - x^*\rangle \\
&\quad + 2\alpha_n\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle \\
&\le (1 - \alpha_n\overline{\gamma})^2\|x_n - x^*\|^2 + 2\alpha_n(\gamma\alpha(1 - \alpha_n\overline{\gamma})\|x_n - x^*\|^2 + G_0\sqrt{\alpha_n}) \\
&\quad + 2\alpha_n\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle \\
&= (1 - 2\alpha_n\overline{\gamma} + \alpha_n^2\overline{\gamma}^2 + 2\alpha_n\gamma\alpha(1 - \alpha_n\overline{\gamma}))\|x_n - x^*\|^2 + 2\alpha_n G_0\sqrt{\alpha_n} \\
&\quad + 2\alpha_n\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle \\
&= (1 - [2(\overline{\gamma} - \gamma\alpha) + \alpha_n\overline{\gamma}(2\gamma\alpha - \overline{\gamma})]\alpha_n)\|x_n - x^*\|^2 \\
&\quad + 2\alpha_n(G_0\sqrt{\alpha_n} + \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle).
\end{aligned}$$

It then follows that

$$\|x_{n+1} - x^*\|^2 \le (1 - [2(\overline{\gamma} - \gamma\alpha) + \alpha_n\overline{\gamma}(2\gamma\alpha - \overline{\gamma})]\alpha_n)\|x_n - x^*\|^2 + \alpha_n\beta_n, \tag{3.10}$$

where

$$\beta_n = 2(G_0\sqrt{\alpha_n} + \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle).$$

By (3.7), we get $\limsup_{n\to\infty} \beta_n \le 0$. Now, considering (3.1), applying Lemma 2.7 to (3.10) concludes that $\|x_n - x^*\| \to 0$. $\square$

Corollary 3.2. *Let $\mathcal{S}$, X, $\{\mu_n\}$ and $\{\alpha_n\}$ be as in Theorem* 3.1. *Let $u, x_0 \in H$ and define a sequence $\{x_n\}$ by the iterative algorithm*

$$x_{n+1} = (I - \alpha_n A)T(\mu_n)x_n + \alpha_n u, \quad n \geq 0.$$

Then $\{x_n\}$ converges in norm to a $x^* \in F(\mathcal{S})$ which is the unique solution of the minimization problem

$$\min_{x \in F(\mathcal{S})} \frac{1}{2} \langle Ax, x \rangle - \langle x, u \rangle. \tag{3.11}$$

Proof. It suffices to take $f \equiv u$ and $\gamma = 1$ in Theorem 3.1. □

4. SOME APPLICATIONS

Corollary 4.1. *Let S and T be nonexpansive mappings on H with $ST = TS$ such that $F(S) \cap F(T) \neq \varnothing$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ satisfying conditions $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Let $x_0 \in H$, $0 < \gamma < \overline{\gamma}/\alpha$ and define a sequence $\{x_n\}$ by the iterative algorithm*

$$x_{n+1} = (I - \alpha_n A)\frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^i T^j x_n + \alpha_n \gamma f(x_n), \quad n \geq 0.$$

Then $\{x_n\}$ converges in norm to a unique $x^* \in F(S) \cap F(T)$ which solves the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in F(S) \cap F(T).$$

In particular, taking $f \equiv u \in H$ and $\gamma = 1$, x^* is the unique solution of the minimization problem

$$\min_{x \in F(S) \cap F(T)} \frac{1}{2} \langle Ax, x \rangle - \langle x, u \rangle.$$

Proof. Let $T(i,j) = S^i T^j$ for each $i, j \in \mathbb{N} \cup \{0\}$. Then $\{T(i,j) : i, j \in \mathbb{N} \cup \{0\}\}$ is a semigroup of nonexpansive mappings on C. Now, for each $n \in \mathbb{N}$, define $\mu_n(f) = \frac{1}{n^2} \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} f(i,j)$ for each $f \in l^\infty((\mathbb{N} \cup \{0\})^2)$. Then, $\{\mu_n\}$ is a regular sequence of means [17]. Next, for each $x \in C$ and $n \in \mathbb{N}$, we have

$$T(\mu_n)x = \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^i T^j x,$$

for each $n \in \mathbb{N}$. Therefore, applying Theorem 3.1, the result follows. □

Corollary 4.2. *Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on H such that $F(\mathcal{S}) \neq \varnothing$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ satisfying conditions $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Let $x_0 \in H$, $0 < \gamma < \overline{\gamma}/\alpha$ and define a sequence $\{x_n\}$ by the iterative algorithm*

$$x_{n+1} = (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n \mathrm{d}s + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{4.1}$$

where $\{t_n\}$ is an increasing sequence in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} t_n/t_{n+1} = 1$. Then $\{x_n\}$ converges in norm to a unique $x^* \in F(\mathcal{S})$ which solves the variational inequality (1.7).

Proof. For $n \in \mathbb{N}$, define $\mu_n(f) = \frac{1}{t_n}\int_0^{t_n} f(t)\mathrm{d}t$ for each $f \in C(\mathbb{R}_+)$, where $f \in C(\mathbb{R}_+)$ denotes the space of all real valued bounded continuous functions on $\mathbb{R}_+$ with supremum norm. Then, $\{\mu_n\}$ is a regular sequence of means [17]. Further, for each $x \in C$, we have $T(\mu_n)x = \frac{1}{t_n}\int_0^{t_n} T(s)x\mathrm{d}s$. Now, apply Theorem 3.1 to conclude the result. □

Corollary 4.3. *Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on H such that $F(\mathcal{S}) \neq \varnothing$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ satisfying conditions $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Let $x_0 \in H$, $0 < \gamma < \overline{\gamma}/\alpha$ and define a sequence $\{x_n\}$ by the iterative algorithm*

$$x_{n+1} = (I - \alpha_n A)r_n\int_0^{\infty} \exp(-r_n s)T(s)x_n \mathrm{d}s + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{4.2}$$

where $\{r_n\}$ is a decreasing sequence in $(0,\infty)$ such that $\lim_{n\to\infty} r_n = 0$. Then $\{x_n\}$ converges in norm to a unique $x^* \in F(\mathcal{S})$ which solves the variational inequality (1.7).

Proof. For each $n \in \mathbb{N}$, define $\mu_n(f) = r_n\int_0^{\infty} \exp(-r_n t)f(t)\mathrm{d}t$ for each $f \in C(\mathbb{R}_+)$. Then, $\{\mu_n\}$ is a regular sequence of means [17]. Further, for each $x \in C$, we have $T(\mu_n)x = r_n\int_0^{\infty} \exp(-r_n t)T(t)x\mathrm{d}t$. Now, apply Theorem 3.1 to conclude the result. □

Corollary 4.4. *In Corollaries* 4.3 *and* 4.4, *if we take $f \equiv u \in H$ and $\gamma = 1$, then $\{x_n\}$, defined by* (4.1) *and* (4.2) *converge in norm to the unique solution of the minimization problem* (3.11).

Corollary 4.5. *Let T be a nonexpansive mapping on H such that $F(T) \neq \varnothing$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ satisfying conditions $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$ and let $Q = \{q_{n,m}\}$ be a strongly regular matrix. Let $x_0 \in H$,*

$0 < \gamma < \overline{\gamma}/\alpha$ *and define a sequence* $\{x_n\}$ *by the iterative algorithm*

$$x_{n+1} = (I - \alpha_n A)\sum_{m=0}^{\infty} q_{n,m}T^m x_n + \alpha_n \gamma f(x_n), \quad n \geq 0.$$

Then $\{x_n\}$ *converges in norm to a unique* $x^* \in F(T)$ *which solves the variational inequality* (1.5). *In particular, taking* $f \equiv u \in H$ *and* $\gamma = 1$, x^* *is the unique solution of the minimization problem* (1.2).

Proof. For each $n \in \mathbb{N}$, define

$$\mu_n(f) = \sum_{m=0}^{\infty} q_{n,m} f(m)$$

for each $f \in l^\infty(\mathbb{N} \cup \{0\})$. Since Q is a strongly regular matrix, for each m, we have $q_{n,m} \to 0$, as $n \to \infty$; see [8]. Then, it is easy to see $\{\mu_n\}$ is a regular sequence of means. Further, for each $x \in C$, we have $T(\mu_n)x = \sum_{m=0}^{\infty} q_{n,m}T^m x$. Now, apply Theorem 3.1 to conclude the result. $\square$

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