RELATIVE TOPOLOGICAL PRESSURE

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In this paper a mathematical model for the observer is considered. The notion of topological pressure from the viewpoint of an observer is introduced. As a result an extension of the notion of topological pressure for the non-compact metric spaces with a special kind of semi-dynamics is deduced. The properties of the pressure for the notion of relative topological pressure are studied.

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1. INTRODUCTION

In the direction of the study of chaotic systems, homoclinic points play an essential role. In many cases homoclinic points are members of the non-compact unstable manifolds.

When we restrict the inverse dynamics on the unstable manifold then the notion of topological pressure can be a good candidate in the direction of the study of dynamics.

The notion of topological pressure was introduced by Ruelle [7] and extended by Walters [8] for compact metric spaces, but in the study of homoclinic points we accept with the non-compact metric spaces. So the first problem is:

What does the notion of topological pressure for non-compact metric spaces mean?

The other problem arises from the engineering of the problem, because in the study of a nature's phenomenon we have to use many approximations. One of them is the observer viewpoint. Let us explain the "observer" mathematically. In fact if X is a nonempty set then a one dimensional observer of X, [1, 3, 6] is a fuzzy set λ, as a function from X to the closed interval $[0, 1]$. So, the second problem is:

How can we add the observer viewpoint in the notion of topological pressure?

To find a solution for these questions we restrict ourself to a metric space (X, d) with a semi-dynamics created by a continuous map $f : X \to X$ with a

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base K. In fact a compact subset K of X is called a base for f, [4] if for given compact set $C \subseteq X$ there exists a natural number $m(C)$ such that $f^n(K) \supseteq C$ for all $n \geq m(C)$.

Moreover we assume that λ is a one dimensional observer of X.

With these assumptions, we will introduce a mathematical notion containing the answers for the above questions.

2. TOPOLOGICAL PRESSURE FROM AN OBSERVER'S VIEWPOINT

We assume that $g : X \to X$ is a continuous map. If m and n are natural numbers, and ε is a positive real number, then a subset $R^{m,n}_{f,\varepsilon}$ of $f^m(K)$ is called an (m, n, f, ε) spanning set if for given $x \in f^m(K)$ there exists $y \in R^{m,n}_{f,\varepsilon}$ such that $d(g^i(x), g^i(y)) < \varepsilon$ for all $0 \leq i \leq n-1$.

An (m, n, f, ε) separating set $S^{m,n}_{f,\varepsilon} \subseteq f^m(K)$ is a set such that for all $x, y \in S^{m,n}_{f,\varepsilon}$ there is $0 \leq i \leq n-1$ so that $d(g^i(x), g^i(y)) \geq \varepsilon$.

For each $m, n \in \mathbf{N}$ and $\varepsilon > 0$ there is $k \in \mathbf{N}$ such that the cardinality of $S^{m,n}_{f,\varepsilon}$ is smaller than k. So, there is a real number which is the maximal cardinal number of such $S^{m,n}_{f,\varepsilon}$, and we denote it by $s(m, n, f, \varepsilon)$.

Let $S^{m,n}_{f,\varepsilon}$ be a set such that $|S^{m,n}_{f,\varepsilon}| = s(m, n, f, \varepsilon)$. Then it is an (m, n, f, ε) spanning set. So there exists an (m, n, f, ε) spanning set with a minimal cardinality which is a natural number and we denote its cardinality by $r(m, n, f, \varepsilon)$.

We also have

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \big(s(m, n, f, \varepsilon)\big) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \big(r(m, n, f, \varepsilon)\big).$$

This common limit is denoted by $h(m, f, g)$ which is also a kind of topological entropy beside other extensions of it [2].

Let $C(f^m(K), R)$ be the Banach algebra of real valued continuous functions on $f^m(K)$ with the supremum norm. For $u \in C(f^m(K), R)$, $n \in \mathbf{N}$, and $x \in f^m(K)$ the summation $\sum_{i=0}^{n-1} \delta^{g^i(x)}_{f^m(K)} \lambda(g^i(x)) u(g^i(x))$ is denoted by $(S^\lambda_{n,m} u)(x)$, where

$$\delta^{g^i(x)}_{f^m(K)} = \begin{cases} 1 & \text{if } g^i(x) \in f^m(K), \\ 0 & \text{if } g^i(x) \notin f^m(K). \end{cases}$$

We denote the infimum of the set

$$\left\{ \sum_{x \in D} \mathrm{e}^{(S^\lambda_{n,m} u)(x)} : D \text{ is an } (m, n, f, \varepsilon) \text{ spanning set} \right\}$$

by $Q^\lambda_{m,n,f}(g, u, \varepsilon)$.

The $\limsup_{n\to\infty} \frac{1}{n} \log Q^{\lambda}_{m,n,f}(g,u,\varepsilon)$ is denoted by $Q^{\lambda}(m,f,g,u,\varepsilon)$.
We denote the $\lim_{\varepsilon\to 0} Q^{\lambda}(m,f,g,u,\varepsilon)$ by $P^{\lambda}(m,f,g,K,u)$.

Definition 2.1. If $C(X,R)$ is the set of real valued continuous functions on X, then the mapping $P^{\lambda}(f,g,K,\cdot) : C(X,R) \to R \cup \{\infty\}$ defined by $P^{\lambda}(f,g,K,u) = \limsup_{m\to\infty} P^{\lambda}(m,f,g,K,u)$ is the relative topological pressure of u up to the semi-dynamics f with the base K and observer λ.

This definition is a natural generalization of the notion of topological pressure [9]. Because when X is a compact metric space then $P^{\chi_X}(I,g,X,u)$ will be the topological pressure of u, where I is the identity map of X, and χ_X is the characteristic function of X.

THEOREM 2.1. *Let* $u \in C(f^m(K),R)$ *and* $0 < \varepsilon < \varepsilon'$. *Then:*
(a) $Q^{\lambda}_{m,n,f}(g,u,\varepsilon) \geq Q^{\lambda}_{m,n,f}(g,u,\varepsilon')$;
(b) *if* $S^{\lambda}_{n,m}u$ *is a continuous function on* $f^m(K)$ *then* $Q^{\lambda}_{m,n,f}(g,u,\varepsilon) \leq \|e^{S^{\lambda}_{n,m}u}\| r(m,n,f,\varepsilon)$;
(c) $Q^{\lambda}_{m,n,f}(g,0,\varepsilon) = r(m,n,f,\varepsilon)$.

Proof. (a) Since

$$\left\{ \sum_{x\in D} e^{(S^{\lambda}_{n,m}u)(x)} : D \text{ is an } (m,n,f,\varepsilon) \text{ spanning set} \right\}$$

is a subset of

$$\left\{ \sum_{x\in D} e^{(S^{\lambda}_{n,m}u)(x)} : D \text{ is an } (m,n,f,\varepsilon') \text{ spanning set} \right\}$$

then

$$Q^{\lambda}_{m,n,f}(g,u,\varepsilon) \geq Q^{\lambda}_{m,n,f}(g,u,\varepsilon').$$

(b) If D_0 is an (m,n,f,ε) spanning set such that the cardinality of D_0 is equal to $r(m,n,f,\varepsilon)$, then

$$Q^{\lambda}_{m,n,f}(g,u,\varepsilon) \leq \sum_{x\in D_0} e^{(S^{\lambda}_{n,m}u)(x)} \leq \|e^{S^{\lambda}_{n,m}u}\|\, |D_0|.$$

(c) $Q^{\lambda}_{m,n,f}(g,0,\varepsilon) = \inf\{|D| : D \text{ is an } (m,n,f,\varepsilon) \text{ spanning set}\} = r(m,n,f,\varepsilon)$. □

If $f_1 : X \to X$ is a continuous map with a base K, and $f_2 : X \to X$ is a continuous map conjugate to f_1, i.e., there is a homeomorphism $\psi : X \to X$ such that $f_2 \circ \psi = \psi \circ f_1$, then the topological conjugacy implies that f_2 is a map with the base $\psi(K)$.

Theorem 2.2. *If* $\psi(K) = K$ *then*

$$P^{\lambda}(f_1, g, K, u) = P^{\lambda}(f_2, g, K, u).$$

Proof. Let $D \subseteq f_1^m(K)$ be a given (m, n, f, ε) spanning set. Since K is a base for f_2 then there exists m_2 such that

$$f_1^m(K) \subseteq f_2^{m_3}(\psi(K)) = f_2^{m_3}(K),$$

for all $m_3 > m_2$. Hence

$$P^{\lambda}(m, f_1, g, K, u) \leq P^{\lambda}(m_3, f_2, g, K, u).$$

Therefore,

$$P^{\lambda}(f_1, g, K, u) \leq P^{\lambda}(f_2, g, K, u).$$

Now, by replacing f_1 with f_2 we have

$$P^{\lambda}(f_2, g, K, u) \leq P^{\lambda}(f_1, g, K, u).$$

Thus

$$P^{\lambda}(f_1, g, K, u) = P^{\lambda}(f_2, g, K, u). \quad \square$$

3. PROPERTIES OF $P^{\lambda}(f, g, K, \cdot)$ AS A PRESSURE MAP

Now let us mention the properties of the relative topological pressure.

Theorem 3.1. *Let* $u, v \in C(X, R)$ *and let* $c \in R$. *Then:*

(i) *If* $u \leq v$ *then* $P^{\lambda}(f, g, K, u) \leq P^{\lambda}(f, g, K, v)$.

(ii) *If* λ *is a non-vanishing function on* $f^m(K)$ *then* $P^{\lambda}(m, f, g, K, \cdot)$ *is a real number or it is identically* ∞.

(iii) *If* $c \geq 0$ *then* $P^{\lambda}(f, g, K, u + c) \leq P^{\lambda}(f, g, K, u) + c$.

(iv) $P^{\lambda}(f, g, K, u + v) \leq P^{\lambda}(f, g, K, u) + P^{\lambda}(f, g, K, v)$.

(v) $P^{\lambda}(f, g, K, cu) \leq cP^{\lambda}(f, g, K, u)$ *if* $c \geq 1$, *and* $P^{\lambda}(f, g, K, cu) \geq cP^{\lambda}(f, g, K, u)$ *if* $c \leq 1$.

(vi) $|P^{\lambda}(f, g, K, u)| \leq P^{\lambda}(f, g, K, |u|)$.

Proof. (i) This follows from the fact that

$$S^{\lambda}_{n,m} u \leq S^{\lambda}_{n,m} v.$$

(ii) If $u_0 = u|_{f^m(K)}$, $a = \inf\{u_0(x)\lambda(x),\ 0 \mid x \in f^m(K)\}$ and $b = \sup\{u_0(x),\ 0 \mid x \in f^m(K)\}$ then the inequalities

$$na \leq (S^{\lambda}_{n,m} u)(x) \leq nb,$$

imply

$$r(m, n, f, \varepsilon)\mathrm{e}^{na} \leq Q^{\lambda}_{m,n,f}(g, u, \varepsilon) \leq s(m, n, f, \varepsilon)\mathrm{e}^{nb}.$$

So,

$$h(m, f, g) + a \leq P^{\lambda}(m, f, g, K, u) \leq h(m, f, g) + b.$$

Thus $P^\lambda(m,f,g,K,u)$ is a real number if and only if $h(m,f,g)$ is a real number.

(iii) Since

$$(S^\lambda_{n,m}(u+c))(x) \le (S^\lambda_{n,m}u)(x) + nc$$

then

$$Q^\lambda_{m,n,f}(g,u+c,\varepsilon) \le Q^\lambda_{m,n,f}(g,u,\varepsilon)\mathrm{e}^{nc}.$$

Hence

$$P^\lambda(m,f,g,K,u+c) \le P^\lambda(m,f,g,K,u) + c.$$

Thus

$$P^\lambda(f,g,K,u+c) \le P^\lambda(f,g,K,u) + c.$$

(iv) Since

$$S^\lambda_{n,m}(u+v) = S^\lambda_{n,m}(u) + S^\lambda_{n,m}(v)$$

then

$$Q^\lambda_{m,n,f}(g,u+v,\varepsilon) \le Q^\lambda_{m,n,f}(g,u,\varepsilon)Q^\lambda_{m,nf,}(g,v,\varepsilon).$$

Thus

$$P^\lambda(f,g,K,u+v) \le P^\lambda(f,g,K,u) + P(f,g,K,v).$$

(v) This follows from the following inequalities.

If D is a finite subset of X then

$$\sum_{x\in D} \mathrm{e}^{c(S^\lambda_{n,m}u)(x)} \le \left(\sum_{x\in D} \mathrm{e}^{(S^\lambda_{n,m}u)(x)}\right)^c \quad \text{if } c \ge 1$$

and

$$\sum_{x\in D} \mathrm{e}^{c(S^\lambda_{n,m}u)(x)} \ge \left(\sum_{x\in D} \mathrm{e}^{(S^\lambda_{n,m}u)(x)}\right)^c \quad \text{if } c \le 1.$$

(vi) This is a consequence of parts (i) and (v) and the inequalities $-|u| \le u \le |u|$. □

4. CONCLUSION

Another approach to the notion of relative topological pressure can be deduced by taking $m = n$. The consideration of this case can be a topic for further research, and the paper [5] is a suitable reference for this work.

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