ON LINEAR RECURRENCES WITH POSITIVE VARIABLE COEFFICIENTS IN BANACH SPACES

ALEXANDRU MIHAIL

The purpose of the paper is to study the convergence of a linear recurrence with positive variable coefficients with elements from a Banach space and to estimate the speed of the convergence. The conditions depend of the sum of the coefficients and the minimum of the coefficients. In this paper we study the case when the sequence of the sum of the coefficients is convergent to 1. The results are first proved for recurrences of real numbers. These results are extended later for Banach spaces.

AMS 2000 Subject Classification: 40A05.

Key words: linear recurrence, homogeneous linear recurrence, Banach space.

1. INTRODUCTION AND PRELIMINARIES

The aim of the paper is the study of linear recurrences with positive variable coefficients in Banach spaces (see Definition 1.1 below). The paper extends the results obtained in [2], where the convergence or the divergence of the linear recurrence was studied. Here we want to estimate the speed of the convergence of the linear recurrence.

The Banach spaces will be supposed to be real Banach spaces although this is not necessary. The results are also valid for complex Banach spaces because every complex Banach space can be seen as a real Banach space. For a Banach space X, 0_X denotes the identity element of X and for a set $A \subset X$, $\langle A \rangle$ denotes the Banach subspace of X generated by A and $\mathbf{conv}\langle A \rangle$ denotes the convex closure of the set A, that is

$$\mathbf{conv}A = \overline{\left\{ \sum_{i=\overline{1,n}} a_i x_i | x_1, x_2, \dots, x_n \in A,\ a_1, a_2, \dots, a_n \in [0,1], \sum_{i=\overline{1,n}} a_i = 1 \right\}}.$$

n, m, k, l, i, j, p denotes natural numbers if we do not say otherwise, $\delta_j^i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

MATH. REPORTS **12(62)**, *1* (2010), 9–29

Definition 1.1. Let X be a real normed space and $k \geq 1$ be fixed. Let $\big(a_n = (a_n^1, a_n^2, \ldots, a_n^k)\big)_{n \geq k+1}$ be a sequence of elements from $\mathbb{R}^k$ and $(b_n)_{n \geq k+1}$ be a sequence of elements from X. The sequence $(x_n)_{n \geq 1}$ given by

$$x_{n+k} = a_{n+k}^1 x_{n+k-1} + a_{n+k}^2 x_{n+k-2} + \cdots + a_{n+k}^k x_n + b_{n+k}$$

for $n \geq 1$ is called the linear recurrence of order k with coefficients $(a_n)_{n \geq k+1}$, free terms $(b_n)_{n \geq k+1}$ and initial values $x_1, x_2, \ldots, x_k \in X$. The sequence is called homogeneous if $b_n = 0$ for every $n \geq k+1$.

For two sequences of real numbers $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$, $(a_n)_{n \geq 1} \sim (b_n)_{n \geq 1}$ means that $(a_n)_{n \geq 1}$ is convergent if and only if $(b_n)_{n \geq 1}$ is convergent.

The infinite series and products are denoted by $\sum\limits_{n \geq 1}$ and $\prod\limits_{n \geq 1}$.

Notation. $\sum\limits_{l=\overline{m,n}} a_l = 0$ and $\prod\limits_{l=\overline{m,n}} a_l = 1$ if $m > n$.

The following lemma is well-known (see [1] or [3]).

Lemma 1.1. *For a sequence of real positive numbers $(a_n)_{n \geq 1}$ we have*
a) $\prod\limits_{n \geq 1} (1 + a_n) \sim \sum\limits_{n \geq 1} a_n$;
b) $\prod\limits_{n \geq 1} (1 - a_n) > 0 \Leftrightarrow \sum\limits_{n \geq 1} a_n < \infty$ *if* $a_n < 1$ *for every* $n \in \mathbb{N}^*$.

We recall, from [2], the following important remark on homogenous real linear recurrences with variable coefficients:

Remark 1.1. Let $(a_n)_{n \geq k+1}$ be a fixed sequence with $a_n \in \mathbb{R}_+^k$ and $(x_n)_{n \geq 1}$ be the homogeneous linear recurrence of order k associated with the sequence $(a_n)_{n \geq k+1}$ with initial values $x_1, x_2, \ldots, x_k > 0$ that is

$$x_{n+k} = a_{n+k}^1 x_{n+k-1} + a_{n+k}^2 x_{n+k-2} + \cdots + a_{n+k}^k x_n$$

for $n \geq 1$. Then

a) If there exist initial values $x_1, x_2, \ldots, x_k > 0$ such that $\lim\limits_{n \to \infty} x_n = 0$, $\lim\limits_{n \to \infty} x_n = \infty$ and respectively $(x_n)_{n \geq 1}$ is bounded, then for every initial values $x_1, x_2, \ldots, x_k$ (not necessary greater than 0 in the first and the third case) $\lim\limits_{n \to \infty} x_n = 0$, $\lim\limits_{n \to \infty} x_n = \infty$ and respectively $(x_n)_{n \geq 1}$ is bounded.

b) If there exists initial values $x_1, x_2, \ldots, x_k > 0$ such that $\lim\limits_{n \to \infty} x_n \in (0, \infty)$, then $\lim\limits_{n \to \infty} \sum\limits_{j=\overline{1,k}} a_n^j = 1$.

This remark suggested us to divide the problem when x_n is real in the cases $\lim\limits_{n \to \infty} x_n = 0$, $\lim\limits_{n \to \infty} x_n = \infty$ and $(x_n)_{n \geq 1}$ is a bounded sequence, in particular when $\lim\limits_{n \to \infty} x_n \in (0, \infty)$. These cases depend on the behavior of the sequence $\left(\sum_{j=\overline{1,k}} a_n^j\right)_{n\geq k+1}$. In Banach spaces we have similar cases. When $\sum_{j=\overline{1,k}} a_n^j$ is smaller than 1, $(x_n)_{n\geq 1}$ tends to be convergent to 0, for example when $\limsup_{n\to\infty} \sum_{j=\overline{1,k}} a_n^j < 1$ or when $\sum_{j=\overline{1,k}} a_n^j < 1$ and $\prod_{n\geq k+1}\left(\sum_{j=\overline{1,k}} a_n^j\right) = 0$. When $\sum_{j=\overline{1,k}} a_n^j$ is greater than 1, $(x_n)_{n\geq 1}$ tends to be divergent, for example when $\liminf_{n\to\infty} \sum_{j=\overline{1,k}} a_n^j > 1$ or when $\sum_{j=\overline{1,k}} a_n^j > 1$ and $\prod_{n\geq k+1}\left(\sum_{j=\overline{1,k}} a_n^j\right) = \infty$. In particular if $x_1, x_2, \ldots, x_k$ are in a cone then $(\|x_n\|)_{n\geq 1}$ tends to be divergent to ∞. Also when $\sum_{j=\overline{1,k}} a_n^j$ is convergent to 1 quickly enough, the sequence $(x_n)_{n\geq 1}$ is bounded and if the coefficients are not to small then it is convergent to a limit which is different from 0 in general. These cases are also valid for inhomogeneous linear recurrences. In this paper we will study the last case when the series $\sum_{n\geq k+1}\left|\left(\sum_{j=\overline{1,k}} a_n^j\right) - 1\right|$ is convergent. This case seems to be difficult in the sense that the calculations are longer than in the other cases. The paper is divided into 6 sections. The first section is the introduction and the second one gives some particular cases in which is possible to calculate the general term. These cases give an idea of what happens in the general case. The third section contains the study of the real homogenous case when the sum of the coefficients is 1. The next section contains the study of the general homogenous case when the sum of the coefficients is 1. The fifth section contains the results obtained in the general inhomogeneous case when the sum of the coefficients is 1. The last section contains the general case.

2. PARTICULAR CASES

In this part we will give some particular cases when it is possible to find the general term of the recurrence. We study first the case $k = 1$.

Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be sequences of real numbers with $a_n \neq 0$ and let $(x_n)_{n\geq 0}$ be the sequence given by the linear recurrence

$$x_{n+1} = a_n x_n + b_n$$

for $n \geq 0$. An inductive calculation shows us that

$$x_n = \left(\prod_{l=\overline{1,n}} a_l\right) x_0 + \sum_{l=\overline{1,n}}\left(\left(\prod_{j=\overline{l+1,n}} a_j\right) b_l\right).$$

We remark that $(x_n)_{n\geq 0}$ is convergent for every x_0 if and only if the product $\prod_{l\geq 1} a_l$ is convergent and the sequence $\sum_{l=\overline{1,n}} \Big(b_l \prod_{j=\overline{l+1,n}} a_j\Big)$ is also convergent.

Let us suppose that the series $\sum_{n\geq 1} |a_n - 1|$ is convergent (this imply that the product $\prod_{l\geq 1} a_l$ is convergent and different from 0). Because the convergence of the product $\prod_{l\geq 1} a_l$ and of the sequence $\sum_{l=\overline{1,n}} \Big(b_l \prod_{j=\overline{l+1,n}} a_j\Big)$ does not change when we change a finite number of terms of the sequence $(a_n)_{n\geq 1}$ (maintaining the condition $a_n \neq 0$) we can suppose also that in this case we have $a_n \in (1-\varepsilon, 1+\varepsilon)$ for an $\varepsilon > 0$. If $b_n \geq 0$ the sequence $(x_n)_{n\geq 0}$ is convergent for every x_0 if and only if the series $\sum_{l\geq 1} b_l$ is convergent. To see this let us remark that $0 < \prod_{n\geq 1}(1 - |a_n - 1|) \leq \prod_{j=\overline{l+1,n}} a_j \leq \prod_{n\geq 1}(1 + |a_n - 1|) < +\infty$. If the series $\sum_{l\geq 1} |b_l|$ is convergent then $(x_n)_{n\geq 0}$ is also convergent for every x_0. If the series $\sum_{l\geq 1} |b_l|$ is divergent there exists a sequence $(a_n)_{n\geq 1}$ such that $(x_n)_{n\geq 0}$ is divergent.

We now study the case $k = 2$ when the sum of the coefficients is 1.

Lemma 2.1. *Let $(x_n)_{n\geq 0}$ be the sequence of real numbers given by the linear recurrence*

$$x_{n+1} = (1 - a_n)x_n + a_n x_{n-1}$$

for $n \geq 1$ where $0 < a_n < 1$. If $x_1 \neq x_0$ the sequence $(x_n)_{n\geq 1}$ is convergent if and only if the product $\prod_{j\geq 1} a_j$ is convergent to 0.

Proof. We have

$$x_{n+1} - x_n = -a_n(x_n - x_{n-1}),$$

$$x_{n+1} - x_n = (-1)^n (x_1 - x_0) \prod_{l=\overline{1,n}} a_l$$

and

$$x_{n+1} = (x_1 - x_0)\left(1 + \sum_{l=\overline{1,n}} \left((-1)^l \prod_{j=\overline{1,l}} a_j\right)\right) + x_0.$$

If the sequence $(x_n)_{n\geq 0}$ is convergent then the sequence $(x_{n+1} - x_n)_{n\geq 0}$ is convergent to 0 and then $\prod_{j\geq 1} a_j$ is also convergent to 0.

If $\prod\limits_{j\geq 1} a_j$ is convergent to 0, because $0 < a_n < 1$, the sequence $\Big(\prod\limits_{l=\overline{1,n}} a_l\Big)_{n\geq 1}$ is decreasing to 0 and the series $\sum\limits_{l\geq 1}\Big((-1)^l \prod\limits_{j=\overline{1,l}} a_j\Big)$ is convergent (by the Leibnitz corollary for alternate series).

Lemma 2.2. *Let $(x_n)_{n\geq 0}$ be the sequence given by the linear recurrence*

$$x_{n+1} = (1 - a_n)x_n + a_n x_{n-1} + b_n$$

for $n \geq 1$, where $0 < a_n < 1$ and $(b_n)_{n\geq 1}$ is a sequence of real numbers. Then

a) *If the sequence $(x_n)_{n\geq 0}$ is convergent for every x_0 and every x_1 then the product $\prod\limits_{j\geq 1} a_j$ is convergent to 0, the sequence $\sum\limits_{l=\overline{1,n}}\Big((-1)^l\Big(\prod\limits_{j=\overline{l+1,n}} a_j\Big)b_l\Big)$ is convergent and the sequence $\Big(\sum\limits_{l=\overline{1,n}}(-1)^l b_l\Big(\sum\limits_{j=\overline{l,n}}(-1)^j\Big(\prod\limits_{i=\overline{l+1,j}} a_i\Big)\Big)\Big)_{n\geq 1}$ is also convergent.*

b) *If the product $\prod\limits_{j\geq 1} a_j$ is convergent to 0 and the sequence*

$$\Bigg(\sum_{l=\overline{1,n}}(-1)^l b_l\Bigg(\sum_{j=\overline{l,n}}(-1)^j\Bigg(\prod_{i=\overline{l+1,j}} a_i\Bigg)\Bigg)\Bigg)_{n\geq 1}$$

is convergent then the sequence $(x_n)_{n\geq 0}$ is convergent for every x_0 and every x_1.

c) *If the product $\prod\limits_{j\geq 1} a_j$ is convergent to 0, the series $\sum\limits_{j\geq 1}|b_j|$ is convergent and the sequence $\Big(\sum\limits_{j\geq n}\Big(\prod\limits_{i=\overline{n+2,j+1}} a_i\Big)\Big)_{n\geq 1}$ is bounded then the sequence $(x_n)_{n\geq 0}$ is also convergent for every x_0 and every x_1.*

d) *If there is an $\varepsilon > 0$ such that $a_n \leq 1 - \varepsilon$ for $n \geq 1$ and the series $\sum\limits_{j\geq 1}|b_j|$ and $\sum\limits_{j\geq n+1}\Big(\prod\limits_{i=\overline{n,j}} a_i\Big)$ are convergent, then the sequence $(x_n)_{n\geq 1}$ is also convergent for every x_0 and every x_1.*

Proof. Let $(x_n)_{n\geq 0}$ be the sequence given by the linear recurrence

$$x_{n+1} = (1 - a_n)x_n + a_n x_{n-1} + b_n$$

for $n \geq 0$ where $0 < a_n < 1$.

Then $x_{n+1} - x_n = a_n(x_{n-1} - x_n) + b_n = -a_n(x_n - x_{n-1}) + b_n$.

If we denote $x_{n+1} - x_n$ by y_n then the sequence $(y_n)_{n\geq 0}$ is defined by $y_n = -a_n y_{n-1} + b_n$ and $y_0 = x_1 - x_0$.

We have $y_n = (-1)^n\Big[\Big(\prod_{l=\overline{1,n}} a_l\Big)(x_1 - x_0) + \sum_{l=\overline{1,n}}\Big((-1)^l\Big(\prod_{j=\overline{l+1,n}} a_j\Big)b_l\Big)\Big]$.

Then

$$x_n = x_0 + \sum_{j=\overline{1,n}}(x_j - x_{j-1}) =$$

$$= \sum_{j=\overline{1,n}}(-1)^j\Big[\Big(\prod_{l=\overline{1,j}} a_l\Big)(x_1 - x_0) + \sum_{l=\overline{1,j}}\Big((-1)^l\Big(\prod_{i=\overline{l+1,j}} a_i\Big)b_l\Big)\Big] + x_0 =$$

$$= (x_1 - x_0)\sum_{j=\overline{1,n}}(-1)^j\Big(\prod_{l=\overline{1,j}} a_l\Big) + \sum_{j=\overline{1,n}}\Big(\sum_{l=\overline{1,j}}\Big((-1)^{l+j}\Big(\prod_{i=\overline{l+1,j}} a_i\Big)b_l\Big)\Big) + x_0 =$$

$$= (x_1 - x_0)\sum_{j=\overline{1,n}}(-1)^j\Big(\prod_{l=\overline{1,j}} a_l\Big) + \sum_{l=\overline{1,n}}(-1)^l b_l\Big(\sum_{j=\overline{l,n}}(-1)^j\Big(\prod_{i=\overline{l+1,j}} a_i\Big)\Big) + x_0.$$

a) If the sequence $(x_n)_{n\geq 0}$ is convergent for every x_0 and every x_1 then the sequence $(y_n)_{n\geq 0}$ is convergent for every y_0 and from the case $k = 1$ it follows that the series $\sum_{l=\overline{1,n}}\Big((-1)^l\Big(\prod_{j=\overline{l+1,n}} a_j\Big)b_l\Big)$ is convergent and the product $\prod_{j\geq 1}(-a_j)$ is also convergent. Because $0<a_n<1$ we have $\prod_{j\geq 1} a_j = 0$. Taking $x_0 = x_1 = 0$ we obtain that the sequence $\Big(\sum_{l=\overline{1,n}}(-1)^l b_l\Big(\sum_{j=\overline{l,n}}(-1)^j\Big(\prod_{i=\overline{l+1,j}} a_i\Big)\Big)\Big)_{n\geq 1}$ is also convergent.

b) If the product $\prod_{j\geq 1} a_j$ is convergent to 0, then the sequence

$$\Big(\sum_{j=\overline{1,n}}(-1)^j\Big(\prod_{l=\overline{1,j}} a_l\Big)\Big)_{n\geq 1}$$

is also convergent. Taking account that the sequence

$$\Big(\sum_{l=\overline{1,n}}(-1)^l b_l\Big(\sum_{j=\overline{l,n}}\Big(\prod_{i=\overline{l+1,j}} a_i\Big)\Big)\Big)_{n\geq 1}$$

is convergent, we obtain that the sequence $(x_n)_{n\geq 0}$ is convergent for every x_0 and every x_1.

c) Let the sequence $\Big(\sum_{j\geq n}\Big(\prod_{i=\overline{n+2,j+1}} a_i\Big)\Big)_{n\geq 1}$ be bounded by $M > 0$. We have to prove that the sequence $\Big(\sum_{l=\overline{1,n}}(-1)^l b_l(-1)^j\Big(\sum_{j=\overline{l,n}}\Big(\prod_{i=\overline{l,j}} a_i\Big)\Big)\Big)_{n\geq 1}$ is convergent. Let $u_n = \sum\limits_{l=\overline{1,n}} (-1)^l b_l \Big(\sum\limits_{j=\overline{l,n}} (-1)^j \Big(\prod\limits_{i=\overline{l+1,j}} a_i \Big)\Big)$. Then

$$|u_{n+1}-u_n| = \Big|(-1)^{n+1} \sum_{l=\overline{1,n+1}} \Big((-1)^l \Big(\prod_{i=\overline{l+1,n+1}} a_i\Big) b_l\Big)\Big| \leq \sum_{l=\overline{1,n+1}} \Big(\prod_{i=\overline{l+1,n+1}} a_i\Big)|b_l|$$

and

$$\sum_{n\geq 1}|u_{n+1}-u_n| \leq \sum_{n\geq 1}\sum_{l=\overline{1,n+1}} \Big(\prod_{i=\overline{l+1,n+1}} a_i\Big)|b_l| =$$
$$= \sum_{l\geq 1}|b_l|\Big(\sum_{n\geq l-1} \Big(\prod_{i=\overline{l+1,n+1}} a_i\Big)\Big) \leq M\sum_{l\geq 1}|b_l|.$$

This imply that the sequence $\Big(\sum\limits_{l=\overline{1,n}} (-1)^l b_l \Big(\sum\limits_{j=\overline{l,n}} \Big(\prod\limits_{i=\overline{l+1,j}} a_i\Big)\Big)\Big)_{n\geq 1}$ is convergent.

d) results from c).

3. THE REAL HOMOGENOUS CASE WHEN THE SUM OF THE COEFFICIENTS IS 1

In this part we study real homogenous linear recurrence with positive coefficients when the sum of the coefficients is one.

For a real linear recurrence of order k as in Definition 1.1 let

$y_n = \min\{x_n, x_{n-1}, \ldots, x_{n-k+1}\},\ n \geq k,$

$z_n = \max\{x_n, x_{n-1}, \ldots, x_{n-k+1}\},\ n \geq k,$

$d_n = z_n - y_n,$

$m_n = \min\{a_n^1, a_n^2, \ldots, a_n^k\},\ n \geq k+1,$

$\underline{m}_n^l = \min\{m_n, m_{n-1}, \ldots, m_{n-l+1}\},\ n \geq k+l$

and

$\underline{m}_n = \underline{m}_n^{k-1},\ n \geq 2k-1.$

Lemma 3.1. *Let $(a_n)_{n\geq k+1}$ be a fixed sequence with $a_n \in \mathbb{R}_+^k$ such that $c_n = \sum\limits_{j=\overline{1,k}} a_n^j = 1$. Let $(x_n)_{n\geq 1}$ be the homogeneous linear recurrence of order k associated to the sequence $(a_n)_{n\geq k+1}$ with initial values $x_1, x_2, \ldots, x_k$, that is $x_{n+k} = a_{n+k}^1 x_{n+k-1} + a_{n+k}^2 x_{n+k-2} + \cdots + a_{n+k}^k x_n$. Then $d_{n+k-1} \leq d_n(1 - \underline{m}_{n+k-1})$ for every $n \geq k$.*

Proof. Let $y_n = \min\{x_n, x_{n-1}, \ldots, x_{n-k+1}\}$ and $z_n = \max\{x_n, x_{n-1}, \ldots, x_{n-k+1}\}$ for $n \geq k$. It is clear that $y_n \leq x_{n+1} \leq z_n$. It follows that $z_{n+1} \leq z_n$, $y_{n+1} \geq y_n$ and $d_{n+1} \leq d_n$.

If $\underline{m}_{n+k-1} = 0$ the result is obvious. So, we can suppose $\underline{m}_{n+k-1} > 0$.

We want to estimate $d_{n+k-1} = z_{n+k-1} - y_{n+k-1} = \max\limits_{i,j=\overline{n,n+k-1}} |x_i - x_j|$, where

$$x_{n+i} = a^1_{n+i} x_{n+i-1} + a^2_{n+i} x_{n+i-2} + \cdots + a^k_{n+i} x_{n+i-k}, \quad i = \overline{1, k-1},$$

under the conditions

$$a^1_{n+i} + a^2_{n+i} + \cdots + a^k_{n+i} = 1, \quad i = \overline{1, k-1},$$
$$m_{n+i} \le a^1_{n+i},\ m_{n+i} \le a^2_{n+i}, \ldots, m_{n+i} \le a^k_{n+i}, \quad i = \overline{1, k-1},$$

and

$$y_n \le x_{n+1-i} \le z_n, \quad i = \overline{1, k}.$$

Let us consider the function $\tilde{d}_{n+k-1} : \mathop{\times}\limits_{i=1}^{k} [y_n, z_n] \to \mathbb{R}$ defined by

$$\tilde{d}_{n+k-1}(t_1, \ldots, t_{k-1}, t_k) = \max_{i,j=\overline{n,n+k-1}} \left| \tilde{x}_i(t_1, \ldots, t_{k-1}, t_k) - \tilde{x}_j(t_1, \ldots, t_{k-1}, t_k) \right|,$$

where $\tilde{x}_i : \mathop{\times}\limits_{i=1}^{k} [y_n, z_n] \to \mathbb{R}$ for $i \ge n-k+1$ are the functions defined such that $\tilde{x}_i(t_1, \ldots, t_{k-1}, t_k)$ is the value of the ith term of the linear recurrence if $x_{n+1-i} = t_i$ for $i = \overline{1,k}$ that is $\tilde{x}_i$ are defined inductively by

$$\tilde{x}_i(t_1, \ldots, t_{k-1}, t_k) = a^1_i \tilde{x}_{i-1}(t_1, \ldots, t_{k-1}, t_k) +$$
$$+ a^2_i \tilde{x}_{i-2}(t_1, \ldots t_{k-1}, t_k) + \cdots + a^k_i \tilde{x}_{i-k}(t_1, \ldots, t_{k-1}, t_k), \quad i \ge n+1,$$

and by

$$\tilde{x}_i(t_1, \ldots, t_{k-1}, t_k) = t_{n+1-i}, \quad i = \overline{n-k+1, n}.$$

It is easy to see that $\tilde{x}_i$ are linear functions. Because $|x| = \max(x, -x)$, $\tilde{d}_{n+k-1}$ is the maximum of a finite family of linear functions defined on a compact convex set. After the linear programming theory it results that the maximum is taken into an extreme point of the convex set. So, we can suppose that $x_{n+1-i} = z_n$ or $x_{n+1-i} = y_n$ for $i = \overline{1,k}$.

If $\underline{m}_{n+k-1} > 0$ then $\underline{m}_n > 0$ and $y_n < x_{n+1} < z_n$. Also we have $y_n \le y_{n+i-1} < x_{n+i} < z_{n+i-1} \le z_n$ for $i = \overline{2, k-1}$. It follows that $d_{n+k-1} < d_n$. Let l be such that $d_{n+l} < d_n$ and $d_{n+l-1} = d_n$. This means that $z_{n+l-1} = z_{n+l-2} = \cdots = z_n$ and $y_{n+l-1} = y_{n+l-2} = \cdots = y_n$. Then $z_{n+l} < z_n$ or $y_{n+l} > y_n$. Let us suppose that $z_{n+l} < z_n$. In this case, for $j = \overline{1,l}$

$$x_{n+j} = a^1_{n+j} x_{n+j-1} + a^2_{n+j} x_{n+j-2} + \cdots + a^k_{n+j} x_{n+j-k} \le$$
$$\le (1 - m_{n+j}) z_{n+j-1} + m_{n+j} y_{n+j-1} = (1 - m_{n+j}) z_n + m_{n+j} y_n =$$
$$= z_n - m_{n+j} d_n \le z_n - \underline{m}_{n+k-1} d_n.$$

Also from the assumption that $x_{n+1-i} = z_n$ or $x_{n+1-i} = y_n$ for $i = \overline{1,k}$ it follows that $x_{n+1-i} = y_n$ for $i = \overline{1, k-l}$. If there is an $i \in \{1, 2, \ldots, k-l\}$ such that $x_{n+1-i} = z_n$ we should have $z_{n+l} = z_n$. Then $z_{n+l} \le z_n - \underline{m}_{n+k-1} d_n$. Finally,

$$d_{n+k-1} \le d_{n+l} = z_{n+l} - y_{n+l} \le z_{n+l} - y_n \le$$
$$\le z_n - \underline{m}_{n+k-1} d_n - y_n = d_n(1 - \underline{m}_{n+k-1}).$$

PROPOSITION 3.1. *Let $(a_n)_{n\ge k+1}$ be a fixed sequence with $a_n \in \mathbb{R}_+^k$ and $(x_n)_{n\ge1}$ be the homogeneous linear recurrence of order k associated to the sequence $(a_n)_{n\ge k+1}$ with initial values $x_1, x_2, \ldots, x_k$, that is $x_{n+k} = a^1_{n+k} x_{n+k-1} + a^2_{n+k} x_{n+k-2} + \cdots + a^k_{n+k} x_n$ such that $c_n = \sum_{j=\overline{1,k}} a^j_n = 1$. If $\sum_{n\ge k+1} \underline{m}_n = \infty$ then the sequence $(x_n)_{n\ge1}$ is convergent to a finite limit l and*

$$|x_n - l| \le d_k \prod_{l=\overline{2,\left[\frac{n-p}{k-1}\right]}} \left(1 - \underline{m}_{l(k-1)+p}\right),$$

where $p \in \mathbb{N}^*$ is fixed and $n \ge p + 2k - 2$.

Proof. It is clear that $y_n \le x_{n+1} = a^1_{n+k} x_{n+k-1} + a^2_{n+k} x_{n+k-2} + \cdots + a^k_{n+k} x_n \le z_n$. It follows that $z_{n+1} \le z_n$, $y_{n+1} \ge y_n$ and $d_{n+1} \le d_n$. To prove the convergence it is enough to prove that $d_n = z_n - y_n \to 0$ when $n \to \infty$ because $[y_{n+1}, z_{n+1}] \subset [y_n, z_n]$.

From Lemma 3.1, $d_{n+k-1} \le d_n(1 - \underline{m}_{n+k-1})$ for every $n \ge k$. It follows for $n \ge 2$ that

$$d_{n(k-1)+p} \le d_{k+p-1} \prod_{l=\overline{2,n}} \left(1 - \underline{m}_{l(k-1)+p}\right) \le d_k \prod_{l=\overline{2,n}} \left(1 - \underline{m}_{l(k-1)+p}\right).$$

Because $+\infty = \sum_{n\ge1} \underline{m}_n = \sum_{p=1}^{k} \left(\sum_{l\ge1} \underline{m}_{l(k-1)+p}\right)$ there is a $p \in \{1, 2, \ldots, k\}$ such that $\sum_{l\ge1} \underline{m}_{l(k-1)+p} = +\infty$. It follows that $d_n = z_n - y_n \to 0$ when $n \to \infty$.

Let $l = \lim_{n\to\infty} x_n$. Then $|x_n - l| \le d_n \le d_{\left[\frac{n-p}{k-1}\right](k-1)+p}$ and

$$|x_n - l| \le d_k \prod_{l=\overline{2,\left[\frac{n-p}{k-1}\right]}} \left(1 - \underline{m}_{l(k-1)+p}\right).$$

4. THE GENERAL HOMOGENOUS CASE WHEN THE SUM OF THE COEFFICIENTS IS 1

In this part we study real homogenous linear recurrence in Banach space with positive coefficients when the sum of the coefficients is one.

Theorem 4.1. *Let $(X, \| \|)$ be a normed space. Let $(a_n)_{n\geq k+1}$ be a fixed sequence with $a_n \in \mathbb{R}_+^k$ and $(x_n)_{n\geq 1} \subset X$ be the homogeneous linear recurrence of order k associated to the sequence $(a_n)_{n\geq k+1}$ with initial values $x_1, x_2, \ldots, x_k \in X$, such that $c_n = \sum\limits_{j=\overline{1,k}} a_n^j = 1$. If $\sum\limits_{n\geq 1} \underline{m}_n = \infty$ then $(x_n)_{n\geq 1}$ is convergent to a limit $l \in X$ and $\|x_n - l\| \leq d_k \prod\limits_{l=\overline{2,\left[\frac{n-p}{k-1}\right]}} \big(1 - \underline{m}_{l(k-1)+p}\big)$, where $p \in \mathbb{N}^*$ is fixed, $n \geq p + 2k - 2$ and $d_k = \max\limits_{i,j=\overline{1,k}} \|x_i - x_j\|$.*

Proof. Let us suppose first that X is finite dimensional. Let $\varphi : X \to \mathbb{R}$ be a linear function. Then $\big(\varphi(x_n)\big)_{n\geq 1}$ is the homogeneous linear recurrence of order k associated to the sequence $(a_n)_{n\geq k+1}$ with initial values $\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_k) \in \mathbb{R}$. From Proposition 3.1 $\big(\varphi(x_n)\big)_{n\geq 1}$ is convergent to a finite limit l_φ and

$$|\varphi(x_n) - l_\varphi| \leq \max_{i,j=\overline{1,k}} |\varphi(x_i) - \varphi(x_j)| \prod_{l=\overline{2,\left[\frac{n-p}{k-1}\right]}} \big(1 - \underline{m}_{l(k-1)+p}\big).$$

Since $\max\limits_{i,j=\overline{1,k}} |\varphi(x_i) - \varphi(x_j)| \leq \|\varphi\| \max\limits_{i,j=\overline{1,k}} \|x_i - x_j\| = \|\varphi\|\, d_k$ we have

$$|\varphi(x_n) - l_\varphi| \leq \|\varphi\|\, d_k \prod_{l=\overline{2,\left[\frac{n-p}{k-1}\right]}} \big(1 - \underline{m}_{l(k-1)+p}\big).$$

Since X is finite dimensional, $(x_n)_{n\geq 1}$ is convergent to a limit $l \in X$ and $\varphi(l) = l_\varphi$ for every linear function $\varphi : X \to \mathbb{R}$. It follows that

$$\|x_n - l\| \leq d_k \prod_{l=\overline{2,\left[\frac{n-p}{k-1}\right]}} \big(1 - \underline{m}_{l(k-1)+p}\big).$$

This ends the proof in the case of finite dimensional spaces. The case when X is not finite dimensional can be reduced to the case when X is finite dimensional because $x_n \in \langle x_1, x_2, \ldots, x_k \rangle$ for every $n \geq 1$.

Corollary 4.1. *Let $(X, \| \|)$ be a normed space. Let $(a_n)_{n\geq k+1}$ be a fixed sequence with $a_n \in \mathbb{R}_+^k$ and $(x_n)_{n\geq 1} \subset X$ be the homogeneous linear recurrence of order k associated to the sequence $(a_n)_{n\geq k+1}$ with initial values $x_1, x_2, \ldots, x_k \in X$, such that $c_n = \sum\limits_{j=\overline{1,k}} a_n^j = 1$. Let $K_n = \mathbf{conv}\langle\{x_n, x_{n-1}, \ldots, x_{n-k+1}\}\rangle$ for $n \geq k$ and l be the limit of the sequence $(x_n)_{n\geq 1}$. Then $K_{n+1} \subset K_n$ and $\bigcap\limits_{n\geq 1} K_n = \{l\}$. In particular, we have*

$$\|l\| \leq \max\{\|x_1\|, \|x_2\|, \ldots, \|x_k\|\}.$$

Proof. We have $x_{n+1} \in K_n$ and so $K_{n+1} \subset K_n$. The rest result from the convergence of the sequence $(x_n)_{n\geq 1}$.

5. THE GENERAL INHOMOGENEOUS CASE WHEN THE SUM OF THE COEFFICIENTS IS 1

The following two lemmas are technical results that give us the possibility to reduce the inhomogeneous case to the homogenous one.

LEMMA 5.1. *Let $(X, \| \|)$ be a Banach space. Let $f : \mathbb{N} \to \mathbb{N}$ be an increasing function such that $\lim\limits_{n\to\infty} f(n) = \infty$. Let $(x(m)_n)_{n\geq 1} \subset X$, $(l_m)_{m\geq 1} \subset X$ for $m \in \mathbb{N}$ be sequences of vectors and $(d_{n,m})_{n\geq 1}$ for $m \in \mathbb{N}$, $(t_m)_{m\geq 0}$ be sequences of positive numbers such that*

1) *$\big(x(m)_n\big)_{n\geq 1}$ is convergent to l_m;*
2) *$\sum\limits_{m\geq 0} \|l_m\|$ is convergent;*
3) *$\|x(m)_n - l_m\| \leq d_{n,m} t_m$ for $m \leq f(n)$;*
4) *$\sum\limits_{m\geq 0} t_m$ is convergent;*
5) *$d_{n,m} \to 0$ when $n \to \infty$ for every $m \in \mathbb{N}$;*
6) *there exists a $M > 0$ such that $d_{n,m} < M$ for every $n, m \in \mathbb{N}$ with $m \leq f(n)$.*

Then the sequence $\big(y_n = \sum\limits_{l=\overline{0,f(n)}} x(l)_n\big)_{n\geq 1}$ is convergent to $\sum\limits_{m\geq 0} l_m$.

Proof. Let $\varepsilon > 0$ be fixed. From 2) and 4) there exists $m_0 \in \mathbb{N}$ and $m_1 \in \mathbb{N}$ such that $m_0 \leq m_1$, $\sum\limits_{m\geq m_0} t_m < \frac{\varepsilon}{3M}$ and $\sum\limits_{m\geq m_1} \|l_m\| < \frac{\varepsilon}{3}$. There also exists $n_\varepsilon \geq m_1 + 1$ such that $f(n_\varepsilon) \geq m_1 + 1$ and for every $n \geq n_\varepsilon$, $d_{n,m} \leq \frac{\varepsilon}{3\sum_{m\geq 0} t_m}$ for $m = \overline{0, m_0}$. Then for $n \geq n_\varepsilon$ we have

$$\|y_n - \sum_{m\geq 0} l_m\| \leq \sum_{m\geq f(n)+1} \|l_m\| + \sum_{m=\overline{0,f(n)}} \|x(m)_n - l_m\| \leq$$

$$\leq \sum_{m\geq f(n)+1} \|l_m\| + \sum_{m=\overline{0,f(n)}} d_{n,m} t_m \leq$$

$$\leq \sum_{m\geq f(n)+1} \|l_m\| + \sum_{m=\overline{0,m_0}} d_{n,m} t_m + \sum_{m=\overline{m_0+1,f(n)}} d_{n,m} t_m \leq$$

$$\leq \sum_{m\geq m_1} \|l_m\| + \sum_{m=\overline{0,m_0}} d_{n,m} \Big(\sum_{m=\overline{0,m_0}} t_m \Big) + M \sum_{m=\overline{m_0+1,f(n)}} t_m \leq$$

$$\le \sum_{m\ge m_1} \|l_m\| + \frac{\varepsilon}{3\sum\limits_{m\ge 0} t_m} \sum_{m=\overline{0,m_0}} t_m + M \sum_{m\ge m_0+1} t_m \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Notation. $\sum\limits_{l=\overline{m,n}} a_l = 0$ and $\prod\limits_{l=\overline{m,n}} a_l = 1$ if $m > n$.

LEMMA 5.2. *Let $(X, \| \|)$ be a Banach space. Let $f : \mathbb{N} \to \mathbb{N}$ be an increasing function such that $\lim\limits_{n\to\infty} f(n) = \infty$ and $f(n) \le n$. Let $(x(m)_n)_{n\ge 1} \subset X$ for $m \in \mathbb{N}$, $(l_m)_{m\ge 0} \subset X$ be sequences of vectors and $(d_{n,m})_{n\ge 1}$ for $m \in \mathbb{N}$, $(t_m)_{m\ge 0}$, $(c_m)_{m\ge 1}$ and $(s_m)_{m\ge 0}$ be sequences of positive numbers such that*

1) $\big(x(m)_n\big)_{n\ge 1}$ *is convergent to* l_m;
2) $\sum\limits_{m\ge 0} s_m$ *is convergent*;
3) $\|l_m\| \le s_m$ *and* $t_m \le s_m$;
4) $\|x(m)_n - l_m\| \le d_{n,m} t_m$ *for* $m \le f(n)$;
5) $d_{n,m} = \prod\limits_{l=\overline{m+1,f(n)}} c_l$ *for* $m \le f(n) - 1$;
6) $c_n \in (0, 1]$.

Let $(y_n)_{n\ge 1}$ be the sequence $\Big(\sum\limits_{l=\overline{0,f(n)}} x(l)_n\Big)_{n\ge 1}$. Then the series $\sum\limits_{m\ge 0} l_m$ is convergent and

$$\|y_n - \sum_{m\ge 0} l_m\| \le \min_{m_0=\overline{0,f(n)}} \Big(\sum_{m\ge m_0+1} s_m + \Big(\prod_{l=\overline{m_0+1,f(n)}} c_l \Big) \sum_{m=\overline{0,m_0}} \Big(t_m \prod_{l=\overline{m+1,m_0}} c_l \Big)\Big) \le$$

$$\le \min_{m_0=\overline{0,f(n)}} \Big(\sum_{m\ge m_0+1} s_m + \Big(\prod_{l=\overline{m_0+1,f(n)}} c_l \Big) \sum_{m=\overline{0,m_0}} t_m \Big) \le$$

$$\le \min_{m_0=\overline{0,f(n)}} \Big(\sum_{m\ge m_0+1} s_m + \Big(\prod_{l=\overline{m_0+1,f(n)}} c_l \Big) \sum_{m\ge 0} t_m \Big).$$

Proof. Since $\sum\limits_{m\ge 0} \|l_m\| \le \sum\limits_{m\ge 0} s_m$ and the series $\sum\limits_{m\ge 0} s_m$ is convergent it results that the series $\sum\limits_{m\ge 0} l_m$ is convergent.

Let n be fixed and m_0 be a natural number such that $m_0 \le f(n)$. As in the proof of Lemma 5.1, we have

$$\|y_n - \sum_{m\ge 0} l_m\| \le \sum_{m\ge f(n)+1} \|l_m\| + \sum_{m=\overline{0,m_0}} d_{n,m} t_m + \sum_{m=\overline{m_0+1,f(n)}} d_{n,m} t_m.$$

Then

$$\|y_n - \sum_{m\geq 0} l_m\| \leq \sum_{m\geq f(n)+1} s_m + \sum_{m=\overline{0,m_0}} \Bigg(\prod_{l=\overline{m+1,f(n)}} c_l\Bigg) t_m + \sum_{m=\overline{m_0+1,f(n)}} s_m \leq$$

$$\leq \sum_{m\geq m_0+1} s_m + \Bigg(\prod_{l=\overline{m_0+1,f(n)}} c_l\Bigg) \sum_{m=\overline{0,m_0}} \Bigg(t_m \prod_{l=\overline{m+1,m_0}} c_l\Bigg) \leq$$

$$\leq \sum_{m\geq m_0+1} s_m + \Bigg(\prod_{l=\overline{m_0+1,f(n)}} c_l\Bigg) \sum_{m=\overline{0,m_0}} t_m \leq$$

$$\leq \sum_{m\geq m_0+1} s_m + \Bigg(\prod_{l=\overline{m_0+1,f(n)}} c_l\Bigg) \sum_{m\geq 0} t_m.$$

Because m_0 is an arbitrary natural number such that $m_0 \leq f(n)$ we obtain the desired conclusion.

Remark 5.1. If the conditions from Lemma 5.2 are fulfilled and $\prod_{l\geq 1} c_l = 0$ then the conditions from Lemma 5.1 are also fulfilled.

The following lemma is obvious.

LEMMA 5.3. *Let $(a_n)_{n\geq k+1}$ be a fixed sequence with $a_n \in \mathbb{R}_+^k$ and $(x_n)_{n\geq 1}$ be the homogeneous linear recurrence of order k of real numbers associated to the sequence $(a_n)_{n\geq k+1}$ with initial values $x_1, x_2, \ldots, x_k \in \mathbb{R}$, that is $x_{n+k} = a_{n+k}^1 x_{n+k-1} + a_{n+k}^2 x_{n+k-2} + \cdots + a_{n+k}^k x_n$ such that $c_n = \sum_{j=\overline{1,k}} a_n^j = 1$. Then $x_n \in [\min\{x_1, x_2, \ldots, x_k\}, \max\{x_1, x_2, \ldots, x_k\}]$.*

LEMMA 5.4. *Let $(X, \|\ \|)$ be a Banach space. Let $(a_n)_{n\geq k+1}$ be a fixed sequence with $a_n \in \mathbb{R}_+^k$ and $(b_n)_{n\geq k+1}$ be a sequence of elements from X and $(x_n)_{n\geq 1}$ be the linear recurrence of order k associated to the sequence $(a_n)_{n\geq 1}$ with initial values $x_1 = x_2 = \cdots = x_k = 0$, that is $x_{n+k} = a_{n+k}^1 x_{n+k-1} + a_{n+k}^2 x_{n+k-2} + \cdots + a_{n+k}^k x_n + b_n$. We suppose that $c_n = \sum_{j=\overline{1,k}} a_n^j = 1$ and $b_n = 0$ for $n \geq k+1$. Then*

$$\sup_{i,j\geq 1} \|x_i - x_j\| \leq \sum_{j=\overline{1,k}} \|b_j\|.$$

Proof. Let us consider the following linear recurrences $(x(l)_n)_{n\geq 1}$ for $l \in \{1, 2, \ldots, k\}$ defined by $x(l)_1 = 0$, $x(l)_2 = 0, \ldots, x(l)_k = 0$ for $l \in \{1, 2, \ldots, k\}$ and $x(l)_{n+k} = a_{n+k}^1 x(l)_{n+k-1} + a_{n+k}^2 x(l)_{n+k-2} + \cdots + a_{n+k}^k x(l)_n + b_{n+k}\delta_{n+k}^l$.

Then $x_n = x(1)_n + x(2)_n + \cdots + x(k)_n$.

Also $x(l)_n = y(l)_n b_l$, where $\big(y(l)_n\big)_{n\geq 1}$ are the real linear recurrences defined by $y(l)_1 = 0$, $y(l)_2 = 0, \ldots, y(l)_k = 0$ for $l \in \{1,2,\ldots,k\}$ and $y(l)_{n+k} = a^1_{n+k} y(l)_{n+k-1} + a^2_{n+k} y(l)_{n+k-2} + \cdots + a^k_{n+k} y(l)_n + \delta^l_{n+k}$.

It follows that $x_n = \sum\limits_{l=\overline{1,k}} y(l)_n b_l$. From Lemma 5.3, $y(l)_n \in [0,1]$.

The rest is obvious.

Definition 5.1. We denote by $\mathcal{A}$ the set of all sequences $(a_n)_{n\geq 1}$ with $a_n \in \mathbb{R}^k_+$ and $c_n = \sum\limits_{j=\overline{1,k}} a^j_n = 1$ for $n \geq k+1$.

Definition 5.2. Let $(X, \| \ \|)$ be a Banach space. Let $b_1, b_2, \ldots, b_k$ be elements from X. Let $(a_n)_{n\geq 1}$ be an arbitrary sequence with $a_n \in \mathbb{R}^k_+$ and $(x_n)_{n\geq 1}$ be the linear recurrence of order k associated to the sequence $(a_n)_{n\geq 1}$ with initial values $x_1 = x_2 = \cdots = x_k = 0$, defined by $x_{n+k} = a^1_{n+k} x_{n+k-1} + a^2_{n+k} x_{n+k-2} + \cdots + a^k_{n+k} x_n + (\delta^1_n b_1 + \delta^2_n b_2 + \cdots + \delta^k_n b_k)$ such that $c_n = \sum\limits_{j=\overline{1,k}} a^j_n = 1$ for $n \geq k$. x_n depends on the sequence $(a_n)_n$ so we can write $x_n = x_n\big((a_n)_n\big)$. Let

$$v_m(b_1, b_2, \ldots, b_k) = \sup_{(a_n)_n \in \mathcal{A}} \Big(\max_{i,j=\overline{k+1,m}} \big\| x_i\big((a_n)_n\big) - x_j\big((a_n)_n\big) \big\| \Big)$$

for $m \in \mathbb{N}^*$ and $m \geq k+1$.

LEMMA 5.5. *With the notations from the above Definition* (5.2) *we have*

1)

$$v_m(b_1, b_2, \ldots, b_k) = \sup_{(a_n)_n \in \mathcal{A}} \Bigg(\max_{i,j=\overline{k+1,m}} \Big\| \sum_{l=\overline{1,k}} \big[y_i\big(l, (a_n)_n\big) - y_j\big(l, (a_n)_n\big) \big] b_l \Big\| \Bigg),$$

where $\big(y_m(l,(a_n)_n)\big)_{m\geq 1}$ *are the real linear recurrences defined by*

$$y_1\big(l, (a_m)_m\big) = y_2\big(l, (a_m)_m\big) = \cdots = y_k\big(l, (a_m)_m\big) = 0$$

for $l \in \{1,2,\ldots,k\}$ *and*

$$y_{n+k}\big(l, (a_m)_m\big) = a^1_{n+k} y_{n+k-1}\big(l, (a_m)_m\big) + \\ + a^2_{n+k} y_{n+k-2}\big(l, (a_m)_m\big) + \cdots + a^k_{n+k} y_n\big(l, (a_m)_m\big) + \delta^l_{n+k};$$

2) $v_m(b_1, b_2, \ldots, b_k) \leq \sum\limits_{j=\overline{1,k}} \|b_j\|$;

3) $v_m(b_1 + e_1, b_2 + e_2, \ldots, b_k + e_k) \leq v_m(b_1, b_2, \ldots, b_k) + \sum\limits_{j=\overline{1,k}} \|e_j\|$;

4) $v_m(b_1, b_2, \ldots, b_k) \leq v_{2k}(b_1, b_2, \ldots, b_k)$ *for* $m \geq 2k$.

Proof. 1) It is easy to see that $x_i\big((a_n)_n\big) = \sum\limits_{l=\overline{1,k}} y_i\big(l, (a_n)_n\big) b_l$.

2) and 3) results from the fact that $y_i\big(l,(a_n)_n\big) \in [0,1]$ (see Lemma 5.3).

4) For $m \geq 2k$ the recurrence is homogenous and we have

$$x_{m+p} \in \mathbf{conv}\{x_m, x_{m-1}, \ldots, x_{m-k+1}\} \subset \mathbf{conv}\{x_{2k}, x_{2k-1}, \ldots, x_k\} \text{ if } p \in \mathbb{N}^*.$$

LEMMA 5.6. *Let $(X, \|\ \|)$ be a Banach space. Let $(a_n)_{n\geq k+1}$ be a fixed sequence with $a_n \in \mathbb{R}_+^k$ and $(b_n)_{n\geq k+1}$ be a sequence of elements from X. Let $(x_n)_{n\geq 1}$ be the linear recurrence of order k associated to the sequence $(a_n)_{n\geq k+1}$ with initial values $x_1, x_2, \ldots, x_k$, that is $x_{n+k} = a_{n+k}^1 x_{n+k-1} + \cdots + a_{n+k}^k x_n + b_{n+k}$ such that $c_n = \sum_{j=\overline{1,k}} a_n^j = 1$. Let $d_n = \sup_{i,j=\overline{1,n}} \|x_i - x_j\|$. Then $d_k + \sum_{l=\overline{k+1,k+p}} \|b_l\| \geq d_{k+p}$, where $p \in \mathbb{N}^*$.*

Proof. It is enough to give the proof in the case $p = 1$. The general case results by induction. We have

$$d_{k+1} = \max\Big(d_k, \sup_{i=\overline{1,n}} \|x_{n+1} - x_i\|\Big)$$

and

$$\|x_{n+1} - x_i\| \leq a_{n+1}^1 \|x_n - x_i\| + a_{n+1}^2 \|x_{n-1} - x_i\| + \cdots +$$
$$+ a_{n+1}^k \|x_1 - x_i\| + \|b_{n+1}\| \leq d_k + \|b_{n+1}\|.$$

It follows that $d_{k+1} \leq d_k + \|b_{n+1}\|$.

The following theorem extends the results of the homogenous case to the inhomogeneous case when the sum of the coefficients of the linear recurrence is 1 and it is the main result of this section.

THEOREM 5.1. *Let $(X, \|\ \|)$ be a Banach space. Let $(a_n)_{n\geq k+1}$ be a fixed sequence with $a_n \in \mathbb{R}_+^k$, $(b_n)_{n\geq k+1}$ be a sequence of elements from X and $(x_n)_{n\geq 1}$ be the linear recurrence of order k associated to the sequences $(a_n)_{n\geq k+1}$ and $(b_n)_{n\geq k+1}$ with initial values $x_1, x_2, \ldots, x_k$, that is $x_{n+k} = a_{n+k}^1 x_{n+k-1} + a_{n+k}^2 x_{n+k-2} + \cdots + a_{n+k}^k x_n + b_{n+k}$, such that $c_n = \sum_{j=\overline{1,k}} a_n^j = 1$. If $\sum_{n\geq 2k-1} \underline{m}_n = \infty$ and $\sum_{n\geq k+1} \|b_n\| < \infty$ then $(x_n)_{n\geq 1}$ is convergent to a limit $a \in X$ and*

$$\|x_n - a\| \leq \min_{m_0=0}^{\left[\frac{n-p}{k-1}\right]} \left(\sum_{l\geq m_0(k-1)+p+1} \|b_l\| + \Big(d_k + \sum_{l\geq 1} r_{l,p}\Big) \prod_{l=m_0+1}^{\left[\frac{n-p}{k-1}\right]} \big(1 - \underline{m}_{l(k-1)+p}\big) \right)$$

$$\leq \min_{m_0=0}^{\left[\frac{n-p}{k-1}\right]} \left(\sum_{l\geq m_0(k-1)+p+1} \|b_l\| + \Big(d_k + \sum_{l\geq k+1} \|b_l\|\Big) \prod_{l=m_0+1}^{\left[\frac{n-p}{k-1}\right]} \big(1 - \underline{m}_{l(k-1)+p}\big) \right),$$

where $r_{l,p} = v_{2k}\big(0_X, b_{l(k-1)+p+1}, b_{l(k-1)+p+2}, \dots, b_{l(k-1)+p+k-1}\big)$, $p \in \mathbb{N}^*$ *is fixed,* $n \geq p+2k-2$ *and* $d_k = \max_{i,j=\overline{1,k}} \|x_i - x_j\|$.

Proof. The idea of the proof is to decompose the inhomogeneous linear recurrence $(x_n)_{n\geq 1}$ into a countable sum of homogenous linear recurrences. Then using Theorem 4.1 we can estimate the speed of the convergence of these recurrences. From Lemma 5.1 we obtain the convergence and from Lemma 5.2 we can estimate the speed of the convergence of the inhomogeneous linear recurrence $(x_n)_{n\geq 1}$.

It is enough to make the proof only in the case $p = 1$.

For the general case let us consider the linear recurrence of order k, $(\overline{x}_n)_{n\geq 1}$, associated to the sequences $(\overline{a}_n)_{n\geq k+1}$ and $(\overline{b}_n)_{n\geq k+1}$ with initial values $\overline{x}_1, \overline{x}_2, \dots, \overline{x}_k$, that is $\overline{x}_{n+k} = \overline{a}^1_{n+k}\overline{x}_{n+k-1} + \overline{a}^2_{n+k}\overline{x}_{n+k-2} + \dots + \overline{a}^k_{n+k}\overline{x}_n + \overline{b}_{n+k}$, where $\overline{x}_n = x_{n+p-1}$, $\overline{a}^j_n = a^j_{n+p-1}$ and $\overline{b}_n = b_{n+p-1}$. We have $\overline{c}_n = \sum_{j=\overline{1,k}} \overline{a}^j_n = c_{n+p-1} = 1$. Applying the results of the Theorem to the linear recurrence $(\overline{x}_n)_{n\geq 1}$ for $p = 1$ and taking account that

$$d_k + \sum_{l=\overline{k+1,k+p-1}} \|b_l\| \geq d_{k+p-1} = \sup_{i,j=\overline{1,k+p-1}} \|x_i - x_j\|$$

(from Lemma 5.6) we can obtain the results of the Theorem for the linear recurrence $(x_n)_{n\geq 1}$ for a general p.

Let us consider the sequences $(x(l)_n)_{n\geq 1}$ for $l \in \mathbb{N}$ defined by

$$x(0)_{n+k} = a^1_{n+k}x(0)_{n+k-1} + a^2_{n+k}x(0)_{n+k-2} + \dots + a^k_{n+k}x(0)_n,$$
$$x(0)_1 = x_1,\ x(0)_2 = x_2,\ \dots,\ x(0)_k = x_k \text{ for } l = 0$$

and

$$x(l)_{n+k} = a^1_{n+k}x(l)_{n+k-1} + a^2_{n+k}x(l)_{n+k-2} + \dots + a^k_{n+k}x(l)_n +$$
$$+ b_{n+k}\Big(\delta^{l(k-1)+2}_{n+k} + \delta^{l(k-1)+3}_{n+k} + \dots + \delta^{(l+1)(k-1)+1}_{n+k}\Big),$$
$$x(l)_1 = 0,\ x(l)_2 = 0,\ \dots,\ x(l)_k = 0 \text{ for } l > 0.$$

Then $x_n = x(0)_n + x(1)_n + \dots + x\left(\left[\frac{n-2}{k-1}\right]\right)_n$.

From Theorem 4.1 and Corollary 4.1 the sequences $\big(x(l)_n\big)_{n\geq 1}$ for $l \in \mathbb{N}$ are convergent to the limits $a(l)$ such that

$$\|a(0)\| \leq \sum_{i=\overline{1,k}} \|x_j\|,$$

$$\|a(l)\| \le \sum_{j=\overline{l(k-1)+2,(l+1)(k-1)+1}} \|b_j\| \text{ for } l > 0,$$
$$\|x(0)_n - a(0)\| \le d_k \prod_{j=\overline{2,\left[\frac{n-1}{k-1}\right]}} \left(1 - \underline{m}_{j(k-1)+1}\right) \text{ for } l = 0$$

and

$$\|x(l)_n - a(l)\| \le v_{2k}\left(0_X, b_{l(k-1)+2}, \dots, b_{l(k-1)+k}\right) \prod_{j=\overline{l+1,\left[\frac{n-1}{k-1}\right]}} \left(1 - \underline{m}_{j(k-1)+1}\right)$$

for $l > 0$ and $n \ge (l+1)(k-1)+1$.

We can apply Lemma 5.2 with x_n as y_n, $\left(x(m)_n\right)_{n\ge1}$ as $\left(x(m)_n\right)_{n\ge1}$, $a(m)$ as l_m, $r_{l,1} = v_{2k}\left(0_X, b_{l(k-1)+2}, b_{l(k-1)+3}, \dots, b_{(l+1)(k-1)+1}\right)$ as t_l for $l \ge 1$, $\left(1 - \underline{m}_{l(k-1)+1}\right)$ as c_l, $s_l = \sum\limits_{j=\overline{l(k-1)+2,(l+1)(k-1)+1}} \|b_j\|$ as s_l for $l \ge 1$, $\sum\limits_{i=\overline{1,k}} \|x_j\|$ as s_0, d_k as t_0 and $\left[\frac{n-1}{k-1}\right]$ for $f(n)$. It results that the series $\sum\limits_{n\ge0} a(n)$ is convergent and

$$\|x_n - a\| \le \min_{m_0=\overline{0,\left[\frac{n-1}{k-1}\right]}} \left(\sum_{l\ge m_0+1} s_l + \left(d_k + \sum_{l\ge1} r_{l,1}\right) \prod_{l=\overline{m_0+1,\left[\frac{n-1}{k-1}\right]}} \left(1 - \underline{m}_{l(k-1)+1}\right) \right),$$

where $a = \sum\limits_{n\ge0} a(n)$.

The last inequality follows from the fact that (see Lemma 5.5.2))

$$v_{2k}\left(0_X, b_{l(k-1)+2}, b_{l(k-1)+3}, \dots, b_{(l+1)(k-1)+1}\right) \le \sum_{j=\overline{l(k-1)+2,(l+1)(k-1)+1}} \|b_j\|.$$

It remains us to prove the convergence of the sequence $(x_n)_{n\ge1}$. Since $+\infty = \sum\limits_{n\ge k} \underline{m}_n = \sum\limits_{p=1}^{k} \left(\sum\limits_{l\ge1} \underline{m}_{l(k-1)+p}\right)$ there is a $p \in \{1, 2, \dots, k\}$ such that $\sum\limits_{l\ge1} \underline{m}_{l(k-1)+p} = +\infty$, and so $\prod\limits_{l\ge m_0+1} \left(1 - \underline{m}_{l(k-1)+p}\right) = 0$ for every m_0. From Remark 5.1 conditions from Lemma 5.1 are also fulfilled. Therefore the convergence results from Lemma 5.1.

6. THE GENERAL INHOMOGENEOUS CASE

The next result is obvious.

LEMMA 6.1. *Let $n \in \mathbb{N}^*$, $\varepsilon > 0$ and $a_1, a_2, \dots, a_n, a$ be such that, $n\varepsilon \le 1$, $a_i \ge \varepsilon$ for $i \in \overline{1,n}$, $a \ne 0$ and $a_1 + a_2 + \cdots + a_n = 1 + a$. Then there exists*

$b_1, b_2, \ldots, b_n$ such that $b_1 + b_2 + \cdots + b_n = 1$ and $b_i \geq a_i \geq \varepsilon$ if $a < 0$ or $a_i \geq b_i \geq \varepsilon$ if $a > 0$.

THEOREM 6.1. *Let $(X, \| \, \|)$ be a Banach space. Let $(a_n)_{n \geq k+1}$ be a fixed sequence with $a_n \in \mathbb{R}_+^k$, $(b_n)_{n \geq k+1}$ be a sequence of elements from X and $(x_n)_{n \geq 1}$ be the linear recurrence of order k associated to the sequences $(a_n)_{n \geq k+1}$ and $(b_n)_{n \geq k+1}$ with initial values $x_1, x_2, \ldots, x_k \in X$, that is $x_{n+k} = a^1_{n+k} x_{n+k-1} + a^2_{n+k} x_{n+k-2} + \cdots + a^k_{n+k} x_n + b_{n+k}$. We suppose that $c_n = \sum_{j=\overline{1,k}} a^j_n = 1 + a_n$ and $\sum_{n \geq k+1} |a_n| < \infty$. If $\sum_{n \geq k+1} \underline{m}_n = \infty$ and $\sum_{n \geq k+1} \|b_n\| < \infty$ then $(x_n)_{n \geq 1}$ is convergent to a limit $l \in X$ and*

$$\|x_n - l\| \leq \min_{m_0 = \overline{0, \left[\frac{n-p}{k-1}\right]}} \left(A_p^{m_0} + B_p \prod_{l = \overline{m_0, \left[\frac{n-p}{k-1}\right]}} \left(1 - \widetilde{m}_{l(k-1)+p}\right) \right) \leq$$

$$\leq \min_{m_0 = \overline{0, \left[\frac{n-p}{k-1}\right]}} \left(A_p^{m_0} + \left(d_k + \sum_{l \geq k+1} \left(\|b_l\| + M \, |a_l| \right) \right) \prod_{l = \overline{m_0, \left[\frac{n-p}{k-1}\right]}} \left(1 - \widetilde{m}_{l(k-1)+p}\right) \right),$$

where $p \in \mathbb{N}^*$ is fixed, $n \geq p + 2k - 2$,

$$A_p^{m_0} = \sum_{l \geq m_0(k-1)+p+1} \left(\|b_l\| + M \, |a_l| \right),$$

$$B_p = \left(d_{k+p-1} + \sum_{l \geq 1} \left(r_{l,p}(b) + M s_{l,p} \right) \right),$$

$$M = \prod_{j \geq k+1} \max\{1, c_j\} \left(\max_{i = \overline{1,k}} \|x_{ij}\| + \sum_{l \geq 1} \|b_l\| \right), \quad \widetilde{m}_n = \min\left(\frac{1}{k}, \underline{m}_n \right),$$

$$r_{l,p}(b) = v_{2k}\left(0_X, b_{l(k-1)+p+1}, b_{l(k-1)+p+2}, \ldots, b_{l(k-1)+p+k-1}\right),$$

$$d_{k+p-1} = \sup_{i,j = \overline{1, k+p-1}} \|x_i - x_j\|$$

and

$$s_{l,p} = \sum_{j = \overline{l(k-1)+p+1, (l+1)(k-1)+p}} |a_j| \, .$$

Proof. As in the proof of Theorem 5.1 we can make the proof of the inequality only in the case $p = 1$.

We prove first that $(x_n)_{n \geq 1}$ is bounded.

Let us suppose that $X = \mathbb{R}$ and $(x_n)_{n \geq 1} \subset \mathbb{R}_+$. Let

$$y_n = \min\{x_n, x_{n-1}, \ldots, x_{n-k+1}\} \quad \text{for } n \geq k$$

and

$$z_n = \max\{x_n, x_{n-1}, \ldots, x_{n-k+1}\} \quad \text{for } n \geq k.$$

Then

$$x_{n+k} = a^1_{n+k}x_{n+k-1} + a^2_{n+k}x_{n+k-2} + \cdots + a^k_{n+k}x_n + b_{n+k} \le$$
$$\le a^1_{n+k}z_{n+k-1} + a^2_{n+k}z_{n+k-1} + \cdots + a^k_{n+k}z_{n+k-1} + |b_{n+k}| = c_{n+k}z_{n+k-1} + |b_{n+k}|$$

and

$$z_{n+k} \le \max\{x_{n+k}, z_{n+k-1}\} \le$$
$$\le \max\{c_{n+k}z_{n+k-1} + |b_{n+k}|\,, z_{n+k-1}\} \le z_{n+k-1}\max\{1, c_{n+k}\} + |b_{n+k}|.$$

Now, it is easy to see that

$$z_n \le z_k \prod_{j=\overline{k+1,n}} \max\{1, c_j\} + \sum_{i=\overline{k+1,n}} |b_{i+1}| \left(\prod_{j=\overline{i+1,n}} \max\{1, c_j\} \right) \le$$
$$\le \left(z_k + \sum_{l\ge 1} |b_l| \right) \prod_{j=\overline{k+1,n}} \max\{1, c_j\} \le \left(z_k + \sum_{l\ge 1} |b_l| \right) \prod_{j\ge k+1} \max\{1, c_j\},$$

where $\prod_{j=\overline{n+1,n}} \max\{1, c_j\} = 1$.

This proves that $(x_n)_{n\ge 1}$ is bounded by $\left(z_k + \sum_{l\ge 1} |b_l| \right) \prod_{j\ge k+1} \max\{1, c_j\}$ when $X = \mathbb{R}$ and $(x_n)_{n\ge 1} \subset \mathbb{R}_+$.

If $X = \mathbb{R}$ and $(x_n)_{n\ge 1} \subset \mathbb{R}$, let $(\bar{x}_n)_{n\ge 1}$ be the homogeneous linear recurrence of order k associated to the sequences $(a_n)_{n\ge k+1} \subset \mathbb{R}^k$ and $(b_n)_{n\ge k+1} \subset \mathbb{R}$ with initial values $\bar{x}_1 = |x_1|$, $\bar{x}_2 = |x_2|, \ldots, \bar{x}_k = |x_k|$ defined by $\bar{x}_{n+k} = a^1_{n+k}\bar{x}_{n+k-1} + a^2_{n+k}\bar{x}_{n+k-2} + \cdots + a^k_{n+k}\bar{x}_n + |b_{n+k}|$.

It follows that $|x_n| \le \bar{x}_n \le \left(\max_{j=\overline{1,k}}(|x_j|) + \sum_{l\ge 1} |b_l| \right) \prod_{j\ge k+1} \max\{1, c_j\}$.

Let X be a Banach space and let $\varphi : X \to \mathbb{R}$ be a linear and continuous function with $\|\varphi\| \le 1$. Then $\big(\varphi(x_n)\big)_{n\ge 1}$ is the homogeneous linear recurrence of order k associated to the sequence $(a_n)_{n\ge k+1}$ with initial values $\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_k) \in \mathbb{R}$; that is $\varphi(x_{n+k}) = a^1_{n+k}\varphi(x_{n+k-1}) + a^2_{n+k}\varphi(x_{n+k-2}) + \cdots + a^k_{n+k}\varphi(x_n) + \varphi(b_{n+k})$.

From the previous, $(\varphi(x_n))_{n\ge 1}$ is bounded by

$$\left(\max_{j=\overline{1,k}}\big(|\varphi(x_j)|\big) + \sum_{l\ge 1} |\varphi(b_l)| \right) \prod_{j\ge k} \max\{1, c_j\} \le$$
$$\le \left(\max_{j=\overline{1,k}}\big(\|x_j\|\big) + \sum_{l\ge 1} \|b_l\| \right) \prod_{j\ge k} \max\{1, c_j\} = M.$$

From the Hahn-Banach theorem, it follows that the sequence $(x_n)_{n\ge 1}$ is bounded in norm by M. Indeed, if there exists x_m such that $\|x_m\| > M$ there also exists $\varphi : X \to \mathbb{R}$ a linear and continuous function such that $\|\varphi\| = 1$ and $\|\varphi(x_m)\| = \|x_m\| > M$, which is a contradiction.

Using Lemma 6.1 when $a_{n+k} \neq 0$ with $\widehat{m}_{n+k} = \min\left(\frac{1}{k}, m_{n+k}\right)$ for ε, k for n, $a^1_{n+k}, a^2_{n+k}, \ldots, a^k_{n+k}, a_{n+k}$ for $a_1, a_2, \ldots, a_n, a$ we obtain $e^1_{n+k}, e^2_{n+k}, \ldots, e^k_{n+k}$ such that $e^1_{n+k} + e^2_{n+k} + \cdots + e^k_{n+k} = 1$ and $e^i_{n+k} \geq a^i_{n+k} \geq \widehat{m}_{n+k}$ if $a_{n+k} < 0$ or $a^i_{n+k} \geq e^i_{n+k} \geq \widehat{m}_{n+k}$ if $a_{n+k} > 0$. Then

$$\begin{aligned} x_{n+k} &= a^1_{n+k} x_{n+k-1} + a^2_{n+k} x_{n+k-2} + \cdots + a^k_{n+k} x_n + b_{n+k} = \\ &= e^1_{n+k} x_{n+k-1} + e^2_{n+k} x_{n+k-2} + \cdots + e^k_{n+k} x_n + d_{n+k}, \end{aligned}$$

where

$$d_{n+k} = b_{n+k} + \left(a^1_{n+k} - e^1_{n+k}\right) x_{n+k-1} + \cdots + \left(a^k_{n+k} - e^k_{n+k}\right) x_n.$$

If $a_{n+k} = 0$ we take $a^i_{n+k} = e^i_{n+k}$ and $d_{n+k} = b_{n+k}$.

In this way we have founded two sequences $(e_n)_{n \geq k+1}$ with $e_n \in \mathbb{R}^k_+$ and $(d_n)_{n \geq k+1}$ with $d_n \in X$ such that

$$\sum_{j=\overline{1,k}} e^j_n = 1, \quad e^j_n \geq \widehat{m}_n = \min\left(\tfrac{1}{k}, \underline{m}_n\right), \quad \|d_{n+k} - b_{n+k}\| \leq M\,|a_{n+k}|$$

and

$$x_{n+k} = e^1_{n+k} x_{n+k-1} + e^2_{n+k} x_{n+k-2} + \cdots + e^k_{n+k} x_n + d_{n+k}.$$

We also have $\widetilde{m}_{n+k} = \min\left(\frac{1}{k}, \underline{m}_{n+k}\right) = \min(\widehat{m}_{n+k}, \widehat{m}_{n+k-1}, \ldots, \widehat{m}_{n+1})$.

The new recurrence fulfills the conditions from Theorem 5.1. Indeed $\sum\limits_{n \geq 2k-1} \underline{m}_n = \infty$ if and only if $\sum\limits_{n \geq 2k-1} \widetilde{m}_n = \infty$ and $\sum\limits_{n \geq k+1} \|d_n\| < \infty$ since $\sum\limits_{n \geq k+1} \|b_n\| < \infty$ and $\sum\limits_{n \geq 1} |a_{n+k}| < \infty$.

Let $r_l(b) = r_{l,1}(b) = v_{2k}(0_X, b_{l(k-1)+2}, b_{l(k-1)+3}, \ldots, b_{(l+1)(k-1)+1})$, $s_l = s_{l,1} = \sum\limits_{j=\overline{l(k-1)+2, l(k-1)+k}} |a_j|$ and $r_l(d) = r_{l,1}(d) = v_{2k}(d_{l(k-1)+2}, d_{l(k-1)+3}, \ldots, d_{l(k-1)+k})$.

From Lemma 5.5.3) we have $r_l(d) \leq r_l(b) + M s_l$ and from Lemma 5.5.2) we have

$$r_l(d) \leq \max_{j=\overline{l(k-1)+2,(l+1)(k-1)+1}} (\|d_j\|) \leq \max_{j=\overline{l(k-1)+2,(l+1)(k-1)+1}} (\|b_j\| + M s_l).$$

From the Theorem 5.1 the sequence $(x_n)_{n\geq 1}$ is convergent to a limit $l \in X$ and

$$\|x_n - l\| \leq \min_{m_0=0,\left[\frac{n-1}{k-1}\right]} \left(\sum_{l\geq m_0(k-1)+2} \|d_l\| + \left(d_k + \sum_{l\geq 1} r_l(d)\right) \prod_{l=m_0+1}^{\left[\frac{n-p}{k-1}\right]} \left(1 - \underline{m}_{l(k-1)+1}\right)\right)$$

$$\leq \min_{m_0=0,\left[\frac{n-1}{k-1}\right]} \left(\sum_{l\geq m_0(k-1)+2} \left(\|b_l\| + M\,|a_l|\right) + \right.$$

$$\left. + \left(d_k + \sum_{l\geq 1} \left(r_l(b) + M s_l\right)\right) \prod_{l=m_0+1,\left[\frac{n-1}{k-1}\right]} \left(1 - \underline{m}_{l(k-1)+1}\right)\right) \leq$$

$$\leq \min_{m_0=0,\left[\frac{n-1}{k-1}\right]} \left(A_1^{m_0} + \left(d_k + \sum_{l\geq k+1} \left(\|b_l\| + M\,|a_l|\right)\right) \prod_{l=m_0+1,\left[\frac{n-1}{k-1}\right]} \left(1 - \underline{m}_{l(k-1)+1}\right)\right).$$

REFERENCES

[1] Tom Apostol, *Mathematical Analysis.* Addison-Wesley Publishing, Massachusetts, USA, 1973.

[2] Alexandru Mihail and Alexandru-Petre Tache, *On linear recurrence with variable coefficients.* Anal. Univ. Bucureşti Mat. **2** (2004), 239–252.

[3] Gheorghe Siretchi, *Calcul diferential şi integral.* Vol. 1, 2. Editura Ştiinţifică şi Enciclopedică, Bucureşti, 1985.

Received 21 October 2008

University of Bucharest
Faculty of Mathematics and Computer Science
Str. Academiei 14
010014 Bucharest, Romania
mihail_alex@yahoo.com