A DECOMPOSITION OF SUB-FRACTIONAL BROWNIAN MOTION

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We obtain a decomposition in distribution of the sub-fractional Brownian motion as a sum of independent fractional Brownian motion and a centered Gaussian process with absolutely continuous paths. Applications to the domain of the Wiener integral and the variation and strong variation of sub-fractional Brownian motion are given.

AMS 2000 Subject Classification: 60H05, 60G18.

Key words: Brownian motion, fractional Brownian motion, sub-fractional Brownian motion, Wiener integral, variation.

1. INTRODUCTION

The *sub-fractional Brownian motion* (*sfBm* for short) is a centered continuous Gaussian process $\left(S_t^k\right)_{t\in[0,T]}$, starting from zero, with covariance

$$C_{S^k}(s,t) = s^{2k+1} + t^{2k+1} - \frac{1}{2}\left[(s+t)^{2k+1} + |t-s|^{2k+1}\right], \quad s,t \in [0,T], \tag{1.1}$$

where $k \in \left(-\frac{1}{2}, \frac{1}{2}\right)$; $H = k + \frac{1}{2}$ is called the Hurst parameter. The case $k = 0$ corresponds to the Brownian motion.

For $k > 0$, sfBm arises from occupation time fluctuations of branching particle systems (see [2]). It has properties analogous to those of fBm (self-similarity, long-range dependence, Hölder paths, variation and renormalized variation and it is neither a Markov processes nor a semimartingale). Moreover, sfBm has non-stationary increments and the increments over non-overlaping intervals are more weakly correlated and their covariance decays polynomially at a higher rate in comparison with fBm (for this reason in [2] it is called sfBm). The above mentioned properties make sfBm a possible candidate for models which involve long-dependence, self-similarity and non-stationarity. Some basic properties of sub-fBm are given in [2], [3], [15].

It is well known that in the Brownian case the space of integrands for the Wiener integral is $L^2(\mathbf{R})$. In the case of fBm the same problem is addressed in [9], [10]. For $k \in \left(-\frac{1}{2}, 0\right)$, in [9] is found the entire domain while for

MATH. REPORTS **11(61)**, *1* (2009), 67–74

$k \in \left(0, \frac{1}{2}\right)$ an enough large class of functions as integrands is determined. The corresponding families of integrands are defined in terms of fractional integrals and derivatives. In the case $k \in \left(0, \frac{1}{2}\right)$, it was not clear for some time what is the whole domain of the Wiener integral for fBm. Recently, in [5] the problem of the domain of the Wiener integral for fBm is completely solved for every k. In the case $k \in \left(0, \frac{1}{2}\right)$, the domain is a space of tempered distributions in some Sobolev space.

For sfBm, the domain of the corresponding Wiener integral is described in [16] for an infinite time interval and in [17] for a finite time interval by using the fractional integrals of deterministic functions and a transfer idea.

In the present paper we obtain a decomposition in law of sfBm as a sum of independent fBm and a Gaussian process with absolutely continuous paths (Theorem 3.5). A similar result has been recently obtained in [8] for the so-called bifractional Brownian motion. Such a decomposition is useful in order to obtain an alternative form of the domain of the Wiener integral, to derive easier proofs for different properties of sfBm (like variation, strong variation) and to develop the corresponding stochastic calculus.

2. PRELIMINARIES

For $k \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, $k \neq 0$, we consider a sfBm $\left(S_t^k\right)_{t\in[0,T]}$. Recall that a fractional Brownian motion (fBm for short) is a centered continuous Gaussian process $\left(B_t^k\right)_{t\in[0,T]}$ starting from zero, with covariance

$$C_{B^k}(s,t) = \frac{1}{2}\left[s^{2k+1} + t^{2k+1} - |s-t|^{2k+1}\right], \quad s,t \in [0,T], \tag{2.1}$$

where $k \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ ($H = k + \frac{1}{2}$ is called the Hurst parameter). The case $k = 0$ corresponds to the Brownian motion.

If $(Z_t)_{t\in[0,T]}$ is a continuous centered Gaussian process and $\mathcal{E}$ is the set of elementary functions on $[0,T]$, then the equation

$$\left\langle 1_{(0,s)}, 1_{(0,t)}\right\rangle_{\Lambda_Z} = E\left(Z_s Z_t\right), \quad s,t \in [0,T], \tag{2.2}$$

defines a scalar product on $\mathcal{E}$ and the mapping $t \to Z_t$ can be extended to an isometry $\varphi \to \int_0^T \varphi(t)\,\mathrm{d}Z_t$ between the closure Λ_Z of $\mathcal{E}$ with respect to the above scalar product and the Gaussian space associated with Z. The space $(\Lambda_Z, \langle\cdot\,,\cdot\rangle_{\Lambda_Z})$ is called the *domain of the Wiener integral* and $\int_0^T \varphi(t)\,\mathrm{d}Z_t$ is the *Wiener integral*.

An important problem is to describe $(\Lambda_Z, \langle\cdot\,,\cdot\rangle_{\Lambda_Z})$ and $\int_0^T \varphi(t)\,\mathrm{d}Z_t$ for particular processes Z. The solution is well known in the case of Bm, where the domain is $L^2([0,T])$. In the case of fBm, it is known that Λ_{B^k} is the fractional Sobolev space $I_{0+}^{-k}(L^2([0,T])) = I_{T-}^{-k}(L^2([0,T]))$ if $k \in \left(-\frac{1}{2}, 0\right)$ (see

[9]), where $I_{0+}^{\alpha} f$ is the Riemann-Liouville fractional integral (see [13]). An alternative form of the domain of the Wiener integral for B^k is given in [1]. In the case $k \in \left(0, \frac{1}{2}\right)$, the space Λ_{B^k} is a set of distributions (see [5]).

Recently, the same problem has been discussed in [16], [17] for the case of sfBm. In both cases (of B^k, S^k) the corresponding Wiener integral is obtained as a Wiener integral with respect to a Bm defined on the same probability space and a deterministic kernel expressed in terms of fractional derivatives and integrals.

Recall that a continuous process $(X_t)_{t\in[0,T]}$ admits *α-variation* (resp. *α-strong variation*) if the limit in probability

$$\lim_{\varepsilon\to\infty} \sum_{i=0}^{n-1} \left| X_{\frac{(i+1)t}{n}} - X_{\frac{it}{n}} \right|^{\alpha}$$

(resp.

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^t \left| X_{s+\varepsilon} - X_s \right|^{\alpha} \mathrm{d}s).$$

exists for every $t \in [0, T]$.

Remark 2.1. It is known for fBm that $\frac{2}{2k+1}$-variation (or $\frac{2}{2k+1}$-strong variation) is $\rho_{\frac{2}{2k+1}} t$, where $\rho_p = E\left(|N(0,1)|^p\right)$ (see [11] for the case of variation: the case of strong variation follows using the same arguments as for variation). For sfBm the same results (see [15]), obtained by using linear regression, do hold.

Remark 2.2. A Chung's law of iterated logaritihm is obtained in [6] for fBm: for every $t_0 > 0$ there exists a positive constant $c_k(t_0)$ such that

$$\lim_{t\to 0} \frac{\max\limits_{0\le r\le t} \left| S^k_{r+t_0} - S^k_{t_0} \right|}{t^{\frac{2k+1}{2}} \left(\log|\log t|\right)^{\frac{2k+1}{2}}} = c_k(t_0) \quad \text{a.s.}$$

3. MAIN RESULT

Consider the function

$$K(s,t) = \frac{1}{2}\left[s^{2k+1} + t^{2k+1} - (s+t)^{2k+1}\right], \quad s,t \in [0,T]. \tag{3.1}$$

If $(W_t)_{t\in[0,T]}$ is a Brownian motion, we define the process $\left(X^k_t\right)_{t\in[0,T]}$ as the Wiener integral

$$X^k_t = \int_0^{\infty} \left(1 - \mathrm{e}^{-\theta t}\right) \theta^{-k-1} \mathrm{d}W_\theta. \tag{3.2}$$

Remark 3.1 ([8]). The centered Gaussian process $\left(X_t^k\right)_{t\in[0,T]}$ has the covariance

$$C_{X^k}(s,t) = -\frac{\Gamma(1-2k)}{k(2k+1)}K(s,t) \tag{3.3}$$

and the representation

$$X_t^k = \int_0^t Y_s^k \mathrm{d}s \tag{3.4}$$

with

$$Y_t^k = \int_0^\infty \mathrm{e}^{-\theta t}\theta^{-k}\mathrm{d}W_\theta. \tag{3.5}$$

In particular, X^k has a version with infinitely differentiable paths on $(0,\infty)$ and absolutely continuous paths on $\mathbf{R}_+$.

Remark 3.2. Note that K is a covariance function if $k \in \left(-\frac{1}{2},0\right)$ and $-K$ is also a covariance function if $k \in \left(0,\frac{1}{2}\right)$.

Denote

$$|\Lambda|_{X^k} = \left\{ f:[0,T]\to\mathbf{R} : \int_0^T\int_0^T |f(s)f(t)|\,(s+t)^{2k-1}\mathrm{d}s\mathrm{d}t < \infty \right\}.$$

LEMMA 3.3. *We have the inclusion* $|\Lambda|_{X^k} \subset \Lambda_{X^k}$ *and the relation*

$$\|f\|^2_{\Lambda_{X^k}} = \Gamma(1-2k)\int_0^T\int_0^T f(s)f(t)(s+t)^{2k-1}\mathrm{d}s\mathrm{d}t, \quad f\in|\Lambda|_{X^k}. \tag{3.6}$$

Moreover, if $f \in L^1\left([0,T],t^{k-\frac{1}{2}}\mathrm{d}t\right)$ *then* $f\in|\Lambda|_{X^k}$ *and*

$$\int_0^T f(t)\mathrm{d}X_t^k = \int_0^T f(t)Y_t^k\mathrm{d}t. \tag{3.7}$$

Proof. First, we show that for each $f\in|\Lambda|_{X^k}$ there exists a sequence $(f_n)_n$ of elementary functions such that

$$\int_0^T\int_0^T \left(f_n(s)-f(s)\right)\left(f_n(t)-f(t)\right)(s+t)^{2k-1}\mathrm{d}s\mathrm{d}t \to 0. \tag{3.8}$$

Indeed, let $(g_n)_n$ be a sequence of simple functions such that $g_n\to f$, $|g_n|\le|f|$. By the dominated convergence theorem *we* have

$$\int_0^T\int_0^T \left(g_n(s)-g(s)\right)\left(g_n(t)-g(t)\right)(s+t)^{2k-1}\mathrm{d}s\mathrm{d}t \to 0.$$

For fixed n, choose a sequence of elementary functions $(g_{m,n})_m$ such that

$$g_{m,n} \underset{m\to\infty}{\longrightarrow} g_n, \quad \sup_m |g_{m,n}| \le c_n.$$

Again, the dominated convergence theorem implies the convergence

$$\int_0^T\int_0^T (g_{m,n}(s)-g_n(s))\,(g_{m,n}(t)-g_n(t))\,(s+t)^{2k-1}\mathrm{d}s\mathrm{d}t \underset{m\to\infty}{\longrightarrow} 0.$$

It is easily seen that (3.6) holds if f is elementary. Then it follows from (3.8) that $|\Lambda|_{X^k}\subset\Lambda_{X^k}$ and (3.6) holds for $f\in|\Lambda|_{X^k}$.

Assume now that $f\in L^1\big([0,T],t^{k-\frac12}\mathrm{d}t\big)$. Then

$$\begin{aligned}\int_0^T\int_0^T |f(s)f(t)|\,(s+t)^{2k-1}\mathrm{d}s\mathrm{d}t &\le 2^{1-2k}\int_0^T\int_0^T |f(s)f(t)|\,(st)^{k-\frac12}\,\mathrm{d}s\mathrm{d}t\\ &= 2^{1-2k}\left(\int_0^T |f(s)|\,s^{k-\frac12}\mathrm{d}s\right)^2<\infty.\end{aligned}$$

Also,

$$E\left(\int_0^T |f(t)Y_t^k|\mathrm{d}t\right)=\int_0^T |f(t)|\,E\big(|Y_t^k|\big)\mathrm{d}t$$

$$=\sqrt{\frac{2}{\pi}}\int_0^T |f(t)|\left[E\big(|Y_t^k|^2\big)\right]^{\frac12}\mathrm{d}t = C\int_0^T |f(t)|s^{k-\frac12}\mathrm{d}t<\infty,$$

thus the integral $\int_0^T f(t)Y_t^k\mathrm{d}t$ is well defined.

Equation (3.7) is obvious if f is elementary (by (3.4)) while for $f\in L^1\big([0,T],t^{k-\frac12}\mathrm{d}t\big)$ it follows from (3.8). □

Remark 3.4. It follows from Hölder's inequality that $L^p\big([0,T]\big)\subset L^1\big([0,T],t^{k-\frac12}\mathrm{d}t\big)$ if $p>\frac{2}{2k+1}$.

Our main result is

THEOREM 3.5 (a) *Let* $k\in\big(-\frac12,0\big)$ *and let* $\big(B_t^k\big)_{t\in[0,T]}$ *be a fBm independent of the Bm* $(W_t)_{t\in[0,T]}$. *Then the process*

$$S_t^k=\sqrt{-\frac{k(2k+1)}{\Gamma(1-2k)}}X_t^k+B_t^k,\quad t\in[0,T], \tag{3.9}$$

is a sfBm. In particular,

$$\Lambda_{X^k}\cap\Lambda_{B^k}=\Lambda_{S^k}. \tag{3.10}$$

Moreover, if

$$f\in I_{T-}^{-k}\left(L^2([0,T])\right)\cap L^1\left([0,T],t^{k-\frac12}\mathrm{d}t\right),$$

then $f\in\Lambda_{S^k}$ *and*

$$\int_0^T f(t)\mathrm{d}S_t^k=\sqrt{-\frac{k(2k+1)}{\Gamma(1-2k)}}\int_0^T f(t)Y_t^k\mathrm{d}t+\int_0^T f(t)\mathrm{d}B_t^k, \tag{3.11}$$

$$\left\| \int_0^T f(t)\,\mathrm{d}S_t^k \right\|^2_{L^2(\Omega,\mathcal{F},P)} = \Gamma(1-2k) \int_0^T \int_0^T f(s)f(t)(s+t)^{2k-1}\mathrm{d}s\mathrm{d}t + \frac{\pi k(2k+1)}{\Gamma(1-2k)\sin \pi k} \left\|\varphi\right\|^2_{L^2([0,T])}, \tag{3.12}$$

where $I_{T-}^{-k}(s^k\varphi)(u) = u^k f(u)$.

(b) *Let* $k \in \left(0, \frac{1}{2}\right)$ *and let* $\left(S_t^k\right)_{t\in[0,T]}$ *be a sfBm independent of the Bm* $(W_t)_{t\in[0,T]}$. *Then the process*

$$B_t^k = \sqrt{\frac{k(2k+1)}{\Gamma(1-2k)}} X_t^k + S_t^k, \quad t \in [0,T], \tag{3.13}$$

is a fBm. In particular,

$$\Lambda_{X^k} \cap \Lambda_{S^k} = \Lambda_{B^k}. \tag{3.14}$$

Moreover, if

$$\int_0^T \int_0^T |f(s)f(t)|\,|s-t|^{2k-1}\,\mathrm{d}s\mathrm{d}t < \infty \tag{3.6}$$

then $f \in \Lambda_{B^k}$ *and*

$$\int_0^T f(t)\mathrm{d}S_t^k = \int_0^T f(t)\mathrm{d}B_t^k - \sqrt{\frac{k(2k+1)}{\Gamma(1-2k)}} \int_0^T f(t)Y_t^k \mathrm{d}t, \tag{3.16}$$

$$\left\| \int_0^T f(t)\mathrm{d}S_t^k \right\|^2_{L^2(\Omega,\mathcal{F},P)} = k(2k+1) \int_0^T \int_0^T f(s)f(t)\,|s-t|^{2k-1}\,\mathrm{d}s\mathrm{d}t - \Gamma(1-2k) \int_0^T \int_0^T f(s)f(t)\,(s+t)^{2k-1}\,\mathrm{d}s\mathrm{d}t. \tag{3.17}$$

Proof. (a) Equations (3.9), (3.10) follow from (3.3) and the independence of X^k, B^k.

Equation (3.11) is a consequence of (3.7) and the independence of X^k, B^k while (3.12) follows from (3.11), Lemma 3.3 and equation (1.10) in [10], namely,

$$\left\| \int_0^T f(t)\mathrm{d}B_t^k \right\|^2_{L^2(\Omega,\mathcal{F},P)} = \frac{\pi k(2k+1)}{\Gamma(1-2k)\sin \pi k} \left\|\varphi\right\|^2_{L^2([0,T])}.$$

(b) For f which satisfies (3.15) we also have $f \in \Lambda_{B^k}$ by Theorem 4.1 of [10]. Equations (3.13), and (3.14) follow from (3.3) and the independence of X^k, S^k.

Equation (3.17) is a consequence of (3.16) and equation (4.6) in [10]. □

Theorem 3.5 provides simple proof of the results below.

PROPOSITION 3.6. (a) *The α-variation* (*resp. α-strong variation*) *of sfBm is* $\rho_{\frac{2}{2k+1}}t$.

(b) (*Chung's law of iterated logarithm for sfBm*). *We have*

$$\lim_{t\to 0}\frac{\max\limits_{0\le r\le t}\left|S_r^k\right|}{t^{\frac{2k+1}{2}}\left(\log|\log t|\right)^{\frac{2k+1}{2}}}=c_k(t_0)\quad a.s.$$

Proof. (a) The result follows easily from the decomposition of sfBm and the fact that the corresponding variations of X^k are 0, since X^k is absolutely continuous.

(b) It is a consequence of the decomposition of sfBm and Remark 2.2. □

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Received 23 September 2008

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