DIRICHLET PROBLEM WITH $p(x)$-LAPLACIAN

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We give several sufficient conditions for the existence of weak solutions for the Dirichlet problem with $p(x)$-laplacian

$$\begin{cases} -\Delta_{p(x)}u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^{\mathbb{N}}$, $p(x)$ a continuous function defined on $\overline{\Omega}$ with $p(x) > 1$ for all $x \in \overline{\Omega}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function.

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1. INTRODUCTION

The study of Dirichlet problems with $p(x)$-laplacian is an interesting topic in recent years. Especially, the special $p(x) \equiv p$ (constant) is the well-known Dirichlet problem with p-laplacian. There have been a large numbers of papers on the existence of solutions for p-laplacian equations in a bounded domain. For example, Dinca *et al.* ([1], [2]) proved the existence of weak solutions for the Dirichlet problem with p-laplacian using variational and topological methods.

In this paper we consider the Dirichlet problem with $p(x)$-laplacian

$$\text{(P)} \qquad \begin{cases} -\Delta_{p(x)}u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, $p : \overline{\Omega} \to \mathbb{R}$ a continuous function with $p(x) > 1$ for any $x \in \overline{\Omega}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function which satisfies the growth condition inspired by the case $p(x) \equiv p$ (constant). We give sufficient conditions which allow to use variational and topological methods in the case of $p(x)$-laplacian. The results obtained are generalizations of well-known results for p-laplacian problems.

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2. THE SPACES $W_0^{1,p(x)}(\Omega)$

In order to discuss problem (P), we need some properties of the space $W_0^{1,p(x)}(\Omega)$, which we call generalized Lebesgue-Sobolev spaces. Define

$$C_+(\overline{\Omega}) = \left\{p \in C(\overline{\Omega}) \mid p(x) > 1 \text{ for any } x \in \overline{\Omega}\right\},$$

$$p_- = \min_{x\in\overline{\Omega}} p(x), \quad p_+ = \max_{x\in\overline{\Omega}} p(x), \quad p \in C_+(\overline{\Omega}),$$

$$M = \{u : \Omega \to \mathbb{R} \mid u \text{ is a measurable real-valued function}\},$$

$$L^{p(x)}(\Omega) = \left\{u \in M \mid \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x < \infty\right\}.$$

Let us introduce in $L^{p(x)}(\Omega)$ the norm

$$\|u\|_p = \inf\left\{\lambda > 0 \mid \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \le 1\right\}.$$

Then $\left(L^{p(x)}(\Omega), \|\ \|_p\right)$ is a reflexive Banach space, call it a generalized Lebesgue space. On $L^{p(x)}(\Omega)$ we also consider the function $\varphi_p : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_p(u) = \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x.$$

The connection between φ_p and $\|\ \|_p$ is established by the next result.

Proposition 2.1 (Fan and Zhao [3]). a) *We have the equivalences*

$$\|u\|_p < (>,=)1 \iff \varphi_p(u) < (>,=)1,$$
$$\|u\|_p = \alpha \iff \varphi_p(u) = \alpha \ \textit{when}\ \alpha \neq 0.$$

b) *If* $\|u\|_p > 1$, *then* $\|u\|_p^{p_-} \le \varphi_p(u) \le \|u\|_p^{p_+}$. *If* $\|u\|_p < 1$, *then* $\|u\|_p^{p_+} \le \varphi_p(u) \le \|u\|_p^{p_-}$.

c) $A \subseteq L^{p(x)}(\Omega)$ *is bounded if and only if* $\varphi_p(A) \subseteq \mathbb{R}$ *is bounded.*

d) *For a sequence* $(u_n)_{n\in} \subseteq L^{p(x)}(\Omega)$ *and an element* $u \in L^{p(x)}(\Omega)$ *the following statements are equivalent:*

(1) $\lim\limits_{n\to\infty} u_n = u$ *in* $L^{p(x)}(\Omega)$;

(2) $\lim\limits_{n\to\infty} \varphi_p(u_n - u) = 0$;

(3) $u_n \to u$ *in measure in* Ω *and* $\lim\limits_{n\to\infty} \varphi_p(u_n) = \varphi_p(u)$;

e) $\lim\limits_{n\to\infty} \|u_n\|_p = +\infty$ *if and only if* $\lim\limits_{n\to\infty} \varphi_p(u_n) = +\infty$.

Define the space $W^{1,p(x)}(\Omega)$ as

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) \mid \exists \frac{\partial u}{\partial x_i} \in L^{p(x)}(\Omega) \text{ for all } 1 \le i \le N\right\}$$

and equipp it with the norm $\|u\|_{W^{1,p(x)}} = \|u\|_p + \||\nabla u|\|_p$, where $|\nabla u| = \sqrt{\sum_{i=1}^{N}\left(\frac{\partial u}{\partial x_i}\right)^2}$.

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ and consider the function $p^\star : \overline{\Omega} \to \overline{\mathbb{R}}$ defined by

$$p^\star(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

Proposition 2.2 (Fan and Zhao [3]). a) *The spaces* $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ *and* $W_0^{1,p(x)}(\Omega)$ *are separable and reflexive Banach spaces.*

b) *If* $q \in C_+(\overline{\Omega})$ *and* $q(x) < p^\star(x)$ *for any* $x \in \overline{\Omega}$, *then the imbedding from* $W^{1,p(x)}(\Omega)$ *into* $L^{q(x)}(\Omega)$ *is compact.*

c) *There is a constant* $C > 0$ *such that*

$$\|u\|_p \leq C \||\nabla u|\|_p \quad \text{for all } u \in W_0^{1,p(x)}(\Omega).$$

Remark 2.1. By Proposition 2.2 (c), $\||\nabla u|\|_p$ and $\|u\|_{W^{1,p(x)}}$ are equivalents norms in $W_0^{1,p(x)}(\Omega)$. Hence from now on we will use the space $W_0^{1,p(x)}(\Omega)$ equipped with the norm $\|u\|_{1,p} = \||\nabla u|\|_p$ for all $u \in W_0^{1,p(x)}(\Omega)$.

Remark 2.2. If $q \in C_+(\overline{\Omega})$ and $q(x) < p^\star(x)$ for any $x \in \overline{\Omega}$, then the imbedding from $W_0^{1,p(x)}(\Omega)$ into $L^{q(x)}(\Omega)$ is compact.

3. PROPERTIES OF THE $p(x)$-LAPLACE AND NEMYTSKII OPERATOR

To simplify the notation, we consider the separable and reflexive Banach space $X = W_0^{1,p(x)}(\Omega)$ equipped with the norm $\|u\|_{1,p} = \||\nabla u|\|_p$.

As in the case $p(x) \equiv p$ (constant), we consider the $p(x)$-laplace operator $-\Delta_{p(x)} : X \to X^\star$ defined by

$$\left\langle -\Delta_{p(x)} u, v \right\rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v \mathrm{d}x, \quad u, v \in X.$$

Proposition 3.1 (Fan and Zhao [4]).

a) $-\Delta_{p(x)} : X \to X^\star$ *is a homeomorphism from* X *into* $X^\star$.

b) $-\Delta_{p(x)} : X \to X^\star$ *is a strictly monotone operator, that is,*

$$\left\langle (-\Delta_{p(x)})u - (-\Delta_{p(x)})v, u - v \right\rangle > 0, \quad u \neq v \in X.$$

c) $-\Delta_{p(x)} : X \to X^\star$ *is a mapping of type* (δ_+), i.e., *if* $u_n \rightharpoonup u$ *in* X *and* $\limsup_{n\to\infty} \left\langle -\Delta_{p(x)} u_n, u_n - u \right\rangle \leq 0$, *then* $u_n \to u$ *in* X.

PROPOSITION 3.2. *The functional* $\Psi : X \to \mathbb{R}$ *defined by*

$$\Psi(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, \mathrm{d}x$$

is continuously Fréchet differentiable and $\Psi'(u) = -\Delta_{p(x)} u$ *for all* $u \in X$.

In the last part of this section we recall the basic results on the Nemytskii operator. If $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $u \in M$, then the function $N_f u : \Omega \to \mathbb{R}$ defined by $(N_f u)(x) = f(x, u(x))$ is measurable in Ω. Thus, the Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ defines an operator $N_f : M \to M$, which is called the Nemytskii operator.

PROPOSITION 3.3 (Zhao and Fan [7]). *Suppose* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function and satisfies the growth condition*

$$|f(x,t)| \le c\,|t|^{\frac{\alpha(x)}{\beta(x)}} + h(x), \quad x \in \Omega,\ t \in \mathbb{R},$$

where $\alpha, \beta \in C_+(\overline{\Omega})$, $c \ge 0$ *is constant and* $h \in L^{\beta(x)}(\Omega)$. *Then* $N_f(L^{\alpha(x)}(\Omega)) \subseteq L^{\beta(x)}(\Omega)$. *Moreover,* N_f *is continuous from* $L^{\alpha(x)}(\Omega)$ *into* $L^{\beta(x)}(\Omega)$ *and maps bounded set into bounded set.*

For a function $\alpha \in C_+(\overline{\Omega})$, we recall that $\beta \in C_+(\overline{\Omega})$ is its conjugate function if $\frac{1}{\alpha(x)} + \frac{1}{\beta(x)} = 1$ for all $x \in \overline{\Omega}$.

Concerning the Nemytskii operator, we have

PROPOSITION 3.4. *Suppose* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function and satisfies the growth condition*

$$|f(x,t)| \le c\,|t|^{\alpha(x)-1} + h(x), \quad x \in \Omega,\ t \in \mathbb{R},$$

where $c \ge 0$ *is constant,* $\alpha \in C_+(\overline{\Omega})$, $h \in L^{\beta(x)}(\Omega)$ *and* $\beta \in C_+(\overline{\Omega})$ *is the conjugate function of* α. *Let* $F : \Omega \times \mathbb{R} \to \mathbb{R}$ *be defined by*

$$F(x,t) = \int_\Omega f(x,s)\mathrm{d}s.$$

Then

(i) *F is a Carathéodory function and there exist a constant* $c_1 \ge 0$ *and* $\sigma \in L^1(\Omega)$ *such that*

$$|F(x,t)| \le c_1\,|t|^{\alpha(x)} + \sigma(x), \quad x \in \Omega,\ t \in \mathbb{R};$$

(ii) *the functional* $\Phi : L^{\alpha(x)}(\Omega) \to \mathbb{R}$ *defined by* $\Phi(u) = \int_\Omega F(x, u(x))\mathrm{d}x$ *is continuously Fréchet differentiable and* $\Phi'(u) = N_f(u)$ *for all* $u \in L^{\alpha(x)}(\Omega)$.

Remark 3.1. If in the growth condition we take $\alpha \in C_+(\overline{\Omega})$ and $\alpha(x) < p^\bigstar(x)$ for any $x \in \overline{\Omega}$, the imbedding $X \hookrightarrow L^{\alpha(x)}(\Omega)$ is compact. Hence the diagram

$$X \overset{I}{\hookrightarrow} L^{\alpha(x)}(\Omega) \overset{N_f}{\to} L^{\beta(x)}(\Omega) \overset{I^\star}{\hookrightarrow} X^\star$$

shows that $N_f : X \to X^\star$ is strongly continuous on X.

Moreover, using the same argument, we can show that the functional $\Phi : X \to \mathbb{R}$ defined by $\Phi(u) = \int_\Omega F(x, u(x))\mathrm{d}x$ is strongly continuous on X and $\Phi'(u) = N_f(u)$ for all $u \in X$.

4. EXISTENCE OF SOLUTIONS BY A VARIATIONAL METHOD

Let the functional $H : X \to \mathbb{R}$ defined by

$$H(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \,\mathrm{d}x - \int_\Omega F(x, u(x))\mathrm{d}x.$$

The results from Section 3 show that H is a C^1 functional on X and

$$H'(u) = (-\Delta_{p(x)})(u) - N_f(u), \quad u \in X.$$

We recall that $u \in X$ is a weak solution for problem (P) if and only if

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v \mathrm{d}x = \int_\Omega f(x, u(x))\mathrm{d}x, \quad v \in X.$$

It is now obvious that $u \in X$ is a weak solution for problem (P) if and only if $H'(u) = 0$. The main tool in searching critical points of H is the "Symmetric Mountain Pass Lemma" (see Willem [6], Theorem 6.5) below.

THEOREM 4.1. *Suppose X is an infinite dimensional real Banach space such that $X = V \oplus W$, where V is a finite dimensional subspace of X and W is a subspace of X. Let $H \in C^1(X, \mathbb{R})$ be even and satisfy the (PS) condition and $H(0) = 0$. Assume that*

(i) *there are constants $\varphi, \gamma > 0$ such that $H(x) \geq \gamma$ for all $x \in V$ with $\|x\| = \varphi$;*

(ii) *for each finite dimensional subspace Y of X there is $R > 0$ such that $H(x) \leq 0$ for all $x \in Y$ with $\|x\| \geq R$.*

Then H has an unbounded sequence of positive critical values.

In this section we shall work under the hypotheses

$$p_+ = \max_{x \in \overline{\Omega}} p(x) < p_-^\bigstar = \inf_{x \in \overline{\Omega}} p^\bigstar(x), \tag{1.1}$$

$$f(x, -t) = -f(x, t), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{1.2}$$

$$|f(x, t)| \leq c\,|t|^{\alpha(x)-1} + h(x), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{1.3}$$

where $c \geq 0$ is constant, $\alpha \in C_+(\overline{\Omega})$ with $\alpha(x) < p^\bigstar(x)$ for all $x \in \overline{\Omega}$, $h \in L^{\beta(x)}(\Omega)$ and $\beta \in C_+(\overline{\Omega})$ is the conjugate function of α;

$$0 < \theta F(x, t) \leq t f(x, t) \tag{1.4}$$

for $x \in \Omega,\ t \in \mathbb{R}$ with $|t| \geq M$, where $M > 0$ and $\theta \in (\ p_+, p_-^{\bigstar})$;

$$\alpha_+ < p_-. \tag{1.5}$$

Definition 4.1. We say that the C^1-functional $H : X \to \mathbb{R}$ satisfies the (PS) condition if any sequence $(u_n)_{n\in\mathbb{N}} \subseteq X$ for which $(H(u_n))_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $H'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence.

We have the result below.

PROPOSITION 4.1. *Assume* (1.4). *Then the functional* $H : X \to \mathbb{R}$ *satisfies the* (*PS*) *condition.*

Proof. Let the sequence $(u_n)_{n\in\mathbb{N}} \subseteq X$ be such that $(H(u_n))_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $H'(u_n) \to 0$ as $n \to \infty$. Then there exists $d \in \mathbb{R}$ such that $H(u_n) \leq d$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ denote

$$\Omega_n = \{x \in \Omega \mid |u_n(x)| \geq M\}\,, \quad \Omega'_n = \Omega \diagdown \Omega_n.$$

Without any loss of generality, we can suppose that $M \geq 1$.

If $x \in \Omega'_n$ then $|u_n(x)| < M$ and, by Proposition 3.4 (i),

$$F(x, u_n) \leq c_1 \left|u_n(x)\right|^{\alpha(x)} + \sigma(x) \leq c_1 M^{\alpha_+} + \sigma(x),$$

hence

$$\int_{\Omega'_n} F(x, u_n)\mathrm{d}x \leq \int_{\Omega'_n} (c_1 M^{\alpha_+} + \sigma(x))\mathrm{d}x \leq \int_{\Omega} (c_1 M^{\alpha_+} + \sigma(x))\mathrm{d}x = \tag{1}$$

$$= c_1 M^{\alpha_+} \mathrm{meas}(\Omega) + \int_{\Omega} \sigma(x)\mathrm{d}x = k_1.$$

If $x \in \Omega_n$ then $|u_n(x)| \geq M$ and, by (1.4),

$$F(x, u_n) \leq \frac{1}{\theta} f(x, u_n(x))u_n(x)$$

which gives

$$\int_{\Omega_n} F(x, u_n)\mathrm{d}x \geq \frac{1}{\theta} \int_{\Omega_n} f(x, u_n(x))u_n(x)\mathrm{d}x = \tag{2}$$

$$= \frac{1}{\theta}\left(\int_{\Omega} f(x, u_n(x))u_n(x)\mathrm{d}x - \int_{\Omega'_n} f(x, u_n(x))u_n(x)\mathrm{d}x\right).$$

By the growth condition (1.4), we have

$$\left|\int_{\Omega'_n} f(x, u_n(x))u_n(x)\mathrm{d}x\right| \leq \int_{\Omega'_n} \left(c\left|u_n(x)\right|^{\alpha(x)} + h(x)\left|u_n(x)\right|\right)\mathrm{d}x \leq$$

$$\leq cM^{\alpha_+}\mathrm{meas}(\Omega'_n) + M\int_{\Omega'_n} h(x)\mathrm{d}x \leq cM^{\alpha_+}\mathrm{meas}(\Omega) + M\int_{\Omega} h(x)\mathrm{d}x = k_2,$$

which yields

$$-\frac{1}{\theta}\int_{\Omega_n'} f(x,u_n(x))u_n(x)\mathrm{d}x \le \frac{k_2}{\theta}. \tag{3}$$

We have

$$\int_\Omega \frac{1}{p(x)}\left|\nabla u_n\right|^{p(x)}\mathrm{d}x \le d+\int_{\Omega_n} F(x,u_n(x))\mathrm{d}x+\int_{\Omega_n'} F(x,u_n(x))\mathrm{d}x \le \tag{4}$$

$$\le d+k_1+\int_{\Omega_n} F(x,u_n(x))\mathrm{d}x.$$

By (1), (2), (3) and (4), we get

$$\int_\Omega \frac{1}{p(x)}\left|\nabla u_n\right|^{p(x)}\mathrm{d}x-\frac{1}{\theta}\int_\Omega f(x,u_n(x))u_n(x)\mathrm{d}x \le k, \tag{5}$$

where $k=d+k_1+\frac{k_2}{\theta}$.

On the other hand, because $H'(u_n)\to 0$ as $n\to\infty$, there is $n_0\in\mathbb{N}$ such that $\|H'(u_n)\|\le 1$ for $n\ge n_0$. Consequently, for all $n\ge n_0$ we have

$$\left\langle H'(u_n),u_n\right\rangle \le \|u_n\|_{1,p}$$

or

$$\left|\varphi_p(|\nabla u_n|)-\langle N_f u_n,u_n\rangle\right| \le \|u_n\|_{1,p},$$

which gives

$$-\frac{1}{\theta}\langle N_f u_n,u_n\rangle \ge -\frac{1}{\theta}\|u_n\|_{1,p}-\frac{1}{\theta}\varphi_p(|\nabla u_n|). \tag{6}$$

It follows from (5) and (6) that

$$\left(\frac{1}{p_+}-\frac{1}{\theta}\right)\varphi_p(|\nabla u_n|)-\frac{1}{\theta}\|u_n\|_{1,p}\le k,\quad n\ge n_0. \tag{7}$$

Consider the sets

$$A=\{n\in\mathbb{N}\mid n\ge n_0 \text{ and } \|u_n\|_{1,p}\le 1\}$$

and

$$B=\{n\in\mathbb{N}\mid n\ge n_0 \text{ and } \|u_n\|_{1,p}>1\}.$$

It is obvious that the sequence $(u_n)_{n\in A}\subseteq X$ is bounded. If $n\in B$, then $\|u_n\|_{1,p}>1$ and we have the inequality

$$\varphi_p(|\nabla u_n|)\ge \|u_n\|_{1,p}^{p_-}. \tag{8}$$

Finally, by (7) and (8) we have

$$\left(\frac{1}{p_+}-\frac{1}{\theta}\right)\|u_n\|_{1,p}^{p_-}-\frac{1}{\theta}\|u_n\|_{1,p}\le k,\quad n\in B.$$

We know that $\theta > p_-$ and the last inequality shows that the sequence $(u_n)_{n\in\mathbb{N}} \subseteq X$ is bounded. By the Smuljan theorem, we can extract a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ weakly convergent to some $u \in X$. As $H'(u_{n_k}) \to 0$, we get

$$\lim_{k\to\infty} \left\langle H'(u_{n_k}), u_{n_k} - u \right\rangle = 0. \tag{9}$$

The Nemytskii operator N_f is strongly continuous, so that $\lim\limits_{k\to\infty} N_f(u_{n_k}) = N_f(u)$ in X^* and the weak convergence $u_{n_k} \rightharpoonup u$ in X yields

$$\lim_{k\to\infty} \langle N_f u_{n_k}, u_{n_k} - u \rangle = 0. \tag{10}$$

From (9) and (10) we conclude that

$$\lim_{k\to\infty} \left\langle -\Delta_{p(x)} u_{n_k}, u_{n_k} - u \right\rangle = 0$$

which, together with Proposition 3.1(iii), shows that $u_{n_k} \to u$ in X. □

PROPOSITION 4.2. *Suppose that the Carathéodory function* $f : \Omega\times\mathbb{R} \to \mathbb{R}$ *satisfies hypotheses* (1.3), (1.4) *while the function* $p \in C_+(\overline{\Omega})$ *satisfies hypothesis* (1.1). *If* V *is a finite dimensional subspace of* X*, then the set* $S = \{u \in V \mid H(u) > 0\}$ *is bounded in* X.

Proof. We claim (see [1], Theorem 2.6) that there exists $\gamma \in L^\infty(\Omega)$, $\gamma > 0$, such that $F(x,t) \geq \gamma(x)\,|t|^\theta$ for $x \in \Omega, |t| \geq M$.

Consider the sets

$$\Omega_1 = \{x \in \Omega \mid |u(x)| < M\}, \quad \Omega_2 = \Omega \setminus \Omega_1.$$

Using an argument similar to that in the proof of Proposition 4.1, we get a constant $k_1 \geq 0$ such that $\left|\int_{\Omega_1} F(x,u)\mathrm{d}x\right| \leq k_1$. Then

$$H(u) = \int_\Omega \frac{1}{p(x)}\,|\nabla u|^{p(x)}\,\mathrm{d}x - \left(\int_{\Omega_1} F(x,u)\mathrm{d}x + \int_{\Omega_2} F(x,u)\mathrm{d}x\right) \leq \tag{11}$$

$$\leq \frac{1}{p_-}\varphi_p(|\nabla u|) - \int_{\Omega_2} F(x,u)\mathrm{d}x + k_1 \leq \frac{1}{p_-}\varphi_p(|\nabla u|) - \int_{\Omega_2} \gamma(x)\,|u(x)|^\theta\,\mathrm{d}x + k_1.$$

On the other hand,

$$\int_{\Omega_2} \gamma(x)\,|u(x)|^\theta\,\mathrm{d}x = \int_\Omega \gamma(x)\,|u(x)|^\theta\,\mathrm{d}x - \int_{\Omega_1} \gamma(x)\,|u(x)|^\theta\,\mathrm{d}x \geq \tag{12}$$

$$\geq \int_\Omega \gamma(x)\,|u(x)|^\theta\,\mathrm{d}x - M^\theta \int_{\Omega_1} \gamma(x)\mathrm{d}x \geq$$

$$\geq \int_\Omega \gamma(x)\,|u(x)|^\theta\,\mathrm{d}x - M^\theta\,\|\gamma\|_\infty\,\mathrm{meas}(\Omega).$$

From (11) and (12) we conclude that

$$H(u) \leq \frac{1}{p_-}\varphi_p(|\nabla u|) - \int_\Omega \gamma(x)\,|u(x)|^\theta \,\mathrm{d}x + k, \tag{13}$$

where $k = M^\theta \|\gamma\|_\infty \operatorname{meas}(\Omega) + k_1$.

The function $\| \ \|_\gamma : x \to \mathbb{R}$ defined by

$$\|u\|_\gamma = \left(\int_\Omega \gamma(x)\,|u(x)|^\theta \,\mathrm{d}x\right)^{\frac{1}{\theta}}$$

is a norm on X. On the finite dimensional subspace V the norms $\| \ \|_{1,p}$ and $\| \ \|_\gamma$ are equivalent, so there is a constant $\alpha > 0$ such that

$$\|u\|_{1,p} \leq \alpha \|u\|_\gamma, \quad u \in V. \tag{14}$$

It is obvious that $S \cap \{u \in V \mid \|u\|_{1,p} < 1\}$ is bounded in X. Choose $u \in S \cap \{u \in V \mid \|u\|_{1,p} \geq 1\}$. Then, by (13) and (14), we have

$$H(u) \leq \frac{1}{p_-}\varphi_p(|\nabla u|) - \|u\|_\gamma^\theta + k \leq$$

$$\leq \frac{1}{p_-}\|u\|_{1,p}^{p_+} - \|u\|_\gamma^\theta + k \leq \frac{\alpha^{p_+}}{p_-}\|u\|_\gamma^{p_+} - \|u\|_\gamma^\theta + k.$$

Therefore

$$\frac{\alpha^{p_+}}{p_-}\|u\|_\gamma^{p_+} - \|u\|_\gamma^\theta + k > 0$$

for all $u \in S \cap \{u \in V \mid \|u\|_{1,p} \geq 1\}$. Taking into account that $\theta > p_+$, we conclude that the set S is bounded in X. □

Proposition 4.3. *Assume* (1.3) *and* (1.5). *Then there exist constants* $\rho, \gamma > 0$ *such that* $H\,|_{\|u\|_{1,p}=\rho} \geq \gamma$.

Proof. In the present context we have the inequalities

$$H(u) \geq \frac{1}{p_+}\varphi_p(|\nabla u|) - \int_\Omega F(x,u)\mathrm{d}x$$

and

$$\left|\int_\Omega F(x,u)\mathrm{d}x\right| \leq a\varphi_\alpha(u) + b\|u\|_{1,p}$$

for any $u \in X$, where $a, b \geq 0$ are two constants depending on the numbers $c, \alpha_-, \|h\|_\beta, \beta_-$ which appear in (1.3). Clearly,

$$H(u) \geq \frac{1}{p_+}\varphi_p(|\nabla u|) - a\varphi_\alpha(u) - b\|u\|_{1,p}, \quad u \in X,$$

taking into account that $\varphi_p(|\nabla u|) \geq \|u\|_{1,p}^{p_-}$ for all $u \in X$ with $\|u\|_{1,p} > 1$, we obtain

$$H(u) \geq \frac{1}{p_+} \|u\|_{1,p}^{p_-} - a\varphi_\alpha(u) - b\,\|u\|_{1,p} \tag{15}$$

for all $u \in X$ with $\|u\|_{1,p} > 1$.

The imbedding $X \hookrightarrow L^{\alpha(x)}(\Omega)$ being compact, there is a constant $k \geq 0$ such that

$$\|u\|_\alpha \leq k\,\|u\|_{1,p}\,, \quad u \in X. \tag{16}$$

If $\|u\|_\alpha > 1$ and $\|u\|_{1,p} > 1$, then $\varphi_\alpha(u) \leq \|u\|_\alpha^{\alpha_+}$ and from (16) we have

$$H(u) \geq \frac{1}{p_+} \|u\|_{1,p}^{p_-} - ak^{\alpha_+}\,\|u\|_{1,p}^{\alpha_+} - b\,\|u\|_{1,p}\,. \tag{17}$$

If $\|u\|_\alpha \leq 1$ and $\|u\|_{1,p} > 1$, then $\varphi_\alpha(u) \leq \|u\|_\alpha^{\alpha_-}$ and from (16) we have

$$H(u) \geq \frac{1}{p_+} \|u\|_{1,p}^{p_-} - ak^{\alpha_-}\,\|u\|_{1,p}^{\alpha_-} - b\,\|u\|_{1,p}\,. \tag{18}$$

The functions $h_1, h_2 : (1, +\infty) \to \mathbb{R}$ defined by

$$h_1(t) = \frac{1}{p_+} t^{p_-} - ak^{\alpha_-} t^{\alpha_-} - bt,$$

$$h_2(t) = \frac{1}{p_+} t^{p_-} - ak^{\alpha_+} t^{\alpha_+} - bt$$

are continuous. Taking into account that $\alpha_+ < p_-$, we have $\lim\limits_{t\to\infty} h_1(t) = \lim\limits_{t\to\infty} h_1(t) = +\infty$. Consequently, for any $\gamma > 0$ there exists $\delta > 0$ such that $h_1(t) > \gamma$ and $h_2(t) > \gamma$ for all $t \in (1, +\infty)$ with $t > \delta$.

Choosing $\rho = \max\{1, \delta\} + 1$, it is obvious from (18) that $H(u) > \gamma$ for all $u \in X$ with $\|u\|_{1,p} = \rho$. □

At this stage we are in a position to prove the main result of this section.

THEOREM 4.2. *Assume that hypotheses* (1.1), (1.2), (1.3), (1.4), (1.5) *are satisfied . Then problem* (P) *has an unbounded sequence of weak solutions* $(u_n)_{n\in\mathbb{N}} \subseteq X$ *for which* $H(u_n) \geq 0$ *for any* $n \in \mathbb{N}$, *and* $\lim\limits_{n\to\infty} H(u_n) = +\infty$.

Proof. The function f being odd in the second argument, the functional H is even. It is obvious that $H(0) = 0$.

By Proposition 4.1, H satisfies the (PS) condition. Proposition 4.2 shows that the set $\{u \in Y \mid H(u) > 0\}$ is bounded in X whenever Y is a finite dimensional subspace of X. Now, it is obvious that there exists $R > 0$ such that $H(u) \leq 0$ for all $u \in Y$ with $\|u\|_{1,p} \geq R$.

The space X is a reflexive separable infinite dimensional real Banach space. A well-known result from the theory of Banach spaces ensures the existence of two subspaces V and W of X such that $X = V \oplus W$ and $\dim V < \infty$.

By Proposition 4.3, there are constants $\rho, \gamma > 0$ such that $H \mid_{\|u\|_{1,p}=\rho} \geq \gamma$. So, $H(u) \geq \gamma$ for all $u \in V$ with $\|u\|_{1,p} = \rho$. Applying Theorem 4.1, we conclude that H has an unbounded sequence of positive critical values. The proof is complete. □

5. EXISTENCE OF SOLUTIONS BY A TOPOLOGICAL METHOD

In this section we study the existence of weak solutions for problem (P) giving up part of hypotheses from Section 4. As we have seen is Section 4, problem (P), under assumptions (1.1)–(1.5), has an unbounded sequence of weak solutions in X. We shall see that less hypotheses results in fewer weak solutions. Moreover, we prove that in a special case problem (P) has a unique weak solution in X. The main tool in this section in searching solutions for problem (P) is a Fredholm-type result for a couple of nonlinear operators (see Dinca [2]).

Theorem 5.1. *Let X and Y be real Banach spaces and two nonlinear operators $T, S : X \to Y$ such that*

(i) *T is bijective and T^{-1} is continuous;*

(ii) *S is compact.*

Let $\lambda \neq 0$ be a real number such that

(iii) *$\|(\lambda T - S)(x)\| \to \infty$ as $\|x\| \to \infty$;*

(iv) *there is a constant $R > 0$ such that*

$$\|(\lambda T - S)(x)\| > 0 \text{ if } \|x\| \geq R, \quad d_{LS}\left(I - T^{-1}\left(\frac{1}{\lambda}S\right), B(0,R), 0\right) \neq 0.$$

Then $\lambda T - S$ is surjective from X onto Y.

Before stating the main result of this section, we recall that $u \in X$ is a weak solution for problem (P) if and only if $-\Delta_{p(x)} u = N_f u$ in X^*.

Theorem 5.2. *Suppose that the Carathéodory function f satisfies* (1.3) *and* (1.5). *Then problem (P) has at least a weak solution in X.*

Proof. In order to apply Theorem 5.1 we take $X = W_0^{1,p(x)}(\Omega)$, $Y = X^*$, $T = -\Delta_{p(x)}$ and $S = N_f$.

The real Banach space X being reflexive, every strongly continuous operator $U : X \to Y$ is compact (see [5], Theorem 1.1). Previous considerations show that the operator S is compact.

On the other hand, by Proposition 3.1(i), the operator T is a homeomorphism, so T is bijective and T^{-1} is continuous.

By Proposition 2.1(b),

$$\|T(u)\| \geq \|u\|_{1,p}^{p_- -1} \tag{19}$$

for all $u \in X$ with $\|u\|_{1,p} \geq 1$. Since

$$|\langle S(u), v\rangle| \leq \int_\Omega \left(c\,|u|^{\alpha(x)-1}\,|v| + h(x)\,|v|\right) \mathrm{d}x, \quad u, v \in X,$$

using Holder inequality and the compact imbedding $X \hookrightarrow L^{\alpha(x)}(\Omega)$, we get

$$\|S(u)\| \leq c_1 \big\|\,|u(x)|^{\alpha(x)-1}\,\big\|_\beta + c_2, \quad u \in X, \tag{20}$$

where $c_1, c_2 \geq 0$ are two constants. Moreover, we can choose $q \in [\alpha_- - 1, \alpha_+ - 1]$ and $c_3 \geq 0$ such that

$$\big\|\,|u(x)|^{\alpha(x)-1}\,\big\|_\beta \leq \|u\|_\alpha^q \leq c_3\,\|u\|_{1,p}^q\,, \quad u \in X. \tag{21}$$

From (19), (20) and (21), we deduce that

$$\|(T-S)(u)\| \geq \|T(u)\| - \|S(u)\| \geq \|u\|_{1,p}^{p_- -1} - c_1 c_3\,\|u\|_{1,p}^{\alpha_+ -1} - c_2 \tag{22}$$

for all $u \in X$ with $\|u\|_{1,p} \geq 1$. But

$$\lim_{t\to\infty} \left(t^{p_- -1} - c_1 c_3 t^{\alpha_+ -1} - c_2\right) = +\infty$$

and from (22) we conclude that $\|(T-S)(u)\| \to \infty$ as $\|u\|_{1,p} \to \infty$. Moreover, there exists $r_1 > 1$ such that $\|(T-S)(u)\| > 1$ for all $u \in X$ with $\|u\|_{1,p} > r_1$.

Denote

$$A = \left\{u \in X \mid u = tT^{-1}\left(S(u)\right) \text{ for some } t \in [0,1]\right\}$$

and let us prove that A is bounded in X. For $u \in A\setminus\{0\}$, i.e., $u = tT^{-1}\left(S(u)\right)$ with some $t \in [0,1]$, we have

$$\left\|T\left(\frac{u}{t}\right)\right\| = \|S(u)\| \leq c_1 c_3\,\|u\|_{1,p}^q + c_2. \tag{23}$$

There are two constants $a, b > 0$ such that

$$\|u\|_{1,p}^{p_+ -1} \leq a\,\|u\|_{1,p}^{\alpha_- -1} + b \quad \text{if } \|u\|_{1,p} \in (0,t),$$
$$\|u\|_{1,p}^{p_- -1} \leq a\,\|u\|_{1,p}^{\alpha_- -1} + b \quad \text{if } \|u\|_{1,p} \in [t,1],$$
$$\|u\|_{1,p}^{p_- -1} \leq a\,\|u\|_{1,p}^{\alpha_+ -1} + b \quad \text{if } \|u\|_{1,p} \in (1,+\infty).$$

Let $h_1, h_2 : [0,1] \to \mathbb{R}$ and $h_3 : (1,+\infty) \to \mathbb{R}$ be defined by

$$h_1(t) = t^{p_+ -1} - at^{\alpha_- -1} - b,\ h_2(t) = t^{p_- -1} - at^{\alpha_- -1} - b,\ h_3(t) = t^{p_- -1} - at^{\alpha_+ -1} - b.$$

The sets $\{t \in [0,1] \mid h_1(t) \leq 0\} \subseteq \mathbb{R}$, $\{t \in [0,1] \mid h_2(t) \leq 0\} \subseteq \mathbb{R}$ and $\{t \in (1,+\infty) \mid h_3(t) \leq 0\} \subseteq \mathbb{R}$ are bounded in $\mathbb{R}$.

It follows from (23) and previous inequalities that A is bounded in X. Then there exists $r_2 > 0$ such that $A \subseteq B(0, r_2)$. Choose $R = \max\{r_1, r_2\}$ and consider the homotopy of compact transforms $H_1 : [0,1] \times \overline{B}(0,R) \to X$ defined by $H_1(t,u) = tT^{-1}(S(u))$. It is obvious from the choice of R that, $H_1(t,u) \neq u$ for any $u \in \partial\overline{B}(0,R)$. Consequently,

$$d_{LS}\left(I - H_1(0,\cdot), B(0,R), 0\right) = d_{LS}\left(I - H_1(1,\cdot), B(0,R), 0\right),$$

that is,

$$d_{LS}\left(I - T^{-1}(S), B(0,R), 0\right) = d_{LS}\left(I, B(0,R), 0\right) = 1 \neq 0.$$

The couple of nonlinear operators (T, S) satisfies the hypotheses of Theorem 5.1 for $\lambda = 1$. We conclude that $T - S : X \to Y$ is surjective. Then there exists $u \in X$ such that $T(u) = S(u)$ and the proof is complete. □

Corollary 5.1. *Assume the hypotheses of Theorem* 5.2, *and assume also that the function $f_x : \mathbb{R} \to \mathbb{R}$ defined by $f_x(t) = f(x,t)$ is decreasing a.e. $x \in \Omega$. Then problem (P) has a unique weak solution in X.*

Proof. It is enough to prove that $T - S : X \to Y$ is injective. Let $u_1, u_2 \in X$ such that $(T-S)(u_1) = (T-S)(u_2)$. We have

$$\langle T(u_1) - T(u_2), u_1 - u_2 \rangle = \langle S(u_1) - S(u_2), u_1 - u_2 \rangle$$

and, since T is strictly monotone,

$$\langle S(u_1) - S(u_2), u_1 - u_2 \rangle \geq 0. \tag{24}$$

For a.e. $x \in \Omega$ we have $\left(f(x, u_1(x)) - f(x, u_2(x))\right)\left(u_1(x) - u_2(x)\right) \leq 0$. From (24) we deduce that

$$\langle S(u_1) - S(u_2), u_1 - u_2 \rangle = 0. \tag{25}$$

Proposition 3.1(ii), (24) and (25) show that $u_1 = u_2$.

The operator $T - S : X \to Y$ is bijective. Consequently, there exists a unique $u \in X$ such that $T(u) = S(u)$. □

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