Dedicated to Ion Cuculescu
on the occasion of his seventieth birthday

Δ-ERGODIC THEORY AND SIMULATED ANNEALING

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We give a probabilistic interpretation of strong [Δ]-ergodicity at time 0. Then we define the breaking up of a weakly Δ-ergodic Markov chain at time 0 as being the cardinal |Δ| of Δ. For a strongly Δ-ergodic Markov chain at time 0 this notion is a measure of the dependence on the initial state of the limit probability distribution; clearly, it is useful for the design and analysis of some Markovian algorithms, such as, the simulated annealing. Also we give a probabilistic interpretation of strong ergodicity on a nonempty subset of state space at time 0. We show some theorems on strong or uniform strong ergodicity on a nonempty subset of state space at time 0 or at all times, some theorems on weak or strong ergodicity, and a theorem on uniform weak Δ-ergodicity. Since the case $P_n \to P$ as $n \to \infty$ is met with the simulated annealing, some of these results refer to this case or, in particular, they can be applied to it. We also make a connection with reliability theory (see [6] and the references therein).

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1. PROBABILISTIC INTERPRETATION OF STRONG [Δ]-ERGODICITY AT TIME 0 AND BREAKING UP

In this section we give a probabilistic interpretation of strong [Δ]-ergodicity at time 0. Further, this probabilistic interpretation leads us to define the breaking up of the state space of a weakly Δ-ergodic Markov chain at time 0. The breaking up is useful, e.g., for the design and analysis of some Markovian algorithms, such as, the simulated annealing.

In this paper, a vector x is a row vector and x' denotes its transpose.

Consider a finite Markov chain $(X_n)_{n\geq 1}$ with state space $S=\{1,2,\ldots,r\}$, initial distribution p_0, and transition matrices $(P_n)_{n\geq 1}$. Frequently, we shall refer to it as the (finite) Markov chain $(P_n)_{n\geq 1}$. For all integers $m \geq 0$, $n > m$, define

$$P_{m,n} = P_{m+1}P_{m+2}\ldots P_n = ((P_{m,n})_{ij})_{i,j\in S}\,.$$

(The entries of a matrix Z will be denoted Z_{ij}.)

Set

$$\mathrm{Par}(E) = \{\Delta \mid \Delta \text{ is a partition of } E\},$$

where E is a nonempty set. We shall agree that the partitions do not contain the empty set, except for the case from Theorem 3.6 and that of bases from Remark 3.16 (like in [7], [18], and [19]).

Definition 1.1. Let $\Delta_1, \Delta_2 \in \mathrm{Par}(E)$. We say that Δ_1 *is finer than* Δ_2 if $\forall V \in \Delta_1$, $\exists W \in \Delta_2$ such that $V \subseteq W$.

Write $\Delta_1 \preceq \Delta_2$ when Δ_1 is finer than Δ_2.

In Δ-ergodic theory the natural space is $S \times \mathbf{N}$, called *state-time space.* Let $\emptyset \neq A \subseteq S$ and $\emptyset \neq B \subseteq \mathbf{N}$.

Definition 1.2 ([18]). Let $i, j \in S$. We say that i and j are in the same *weakly ergodic class on* $A \times B$ if $\forall\,(k, m) \in A \times B$ we have

$$\lim_{n\to\infty} [(P_{m,n})_{ik} - (P_{m,n})_{jk}] = 0.$$

Write $i \overset{A\times B}{\sim} j$ when i and j are in the same weakly ergodic class on $A \times B$. Then $\overset{A\times B}{\sim}$ is an equivalence relation and determines a partition $\Delta = \Delta\,(A \times B) = (C_1, C_2, \ldots, C_s)$ of S. The sets $C_1, C_2, \ldots, C_s$ are called *weakly ergodic classes on* $A \times B$.

Definition 1.3 ([18]). Let $\Delta = (C_1, C_2, \ldots, C_s)$ be the partition of weakly ergodic classes on $A \times B$ of a Markov chain. We say that the chain is *weakly* Δ-*ergodic on* $A \times B$. In particular, a weakly (S)-ergodic chain on $A \times B$ is called *weakly ergodic on* $A \times B$ for short.

Definition 1.4 ([18]). Let $(C_1, C_2, \ldots, C_s)$ be the partition of weakly ergodic classes on $A\times B$ of a Markov chain with state space S and $\Delta \in \mathrm{Par}\,(S)$. We say that the chain is *weakly* $[\Delta]$-*ergodic on* $A \times B$ if $\Delta \preceq (C_1, C_2, \ldots, C_s)$.

According to [18] (see also [10], [14], and [15]), in connection with the above notions and notation we mention some special cases.

1. $A \times B = S \times \mathbf{N}$. In this case we write $\sim$ instead of $\overset{S\times\mathbf{N}}{\sim}$ and omit '*on* $S \times \mathbf{N}$' in Definitions 1.2, 1.3, and 1.4.

2. $A = S$. In this case we write $\overset{B}{\sim}$ instead of $\overset{S\times B}{\sim}$ and replace '$S \times B$' by '(*time set*) B' in Definitions 1.2, 1.3, and 1.4. A special subcase is $B = \{m\}$ $(m \geq 0)$; in this situation we can write $\overset{m}{\sim}$ instead of $\overset{\{m\}}{\sim}$ and can replace '*on* (*time set*) $\{m\}$' by '*at time* m' in Definitions 1.2, 1.3, and 1.4.

3. $B = \mathbf{N}$. In this case we set $\overset{A}{\sim}$ instead of $\overset{A\times\mathbf{N}}{\sim}$ and replace '$A \times \mathbf{N}$' by '(*state set*) A' in Definitions 1.2, 1.3, and 1.4.

Definition 1.5 ([18]). Let C be a weakly ergodic class on $A \times B$. We say that C is a *strongly ergodic class on* $A \times B$ if $\forall i \in C$, $\forall (j, m) \in A \times B$ the limit

$$\lim_{n \to \infty} (P_{m,n})_{ij} := \pi_{m,j} = \pi_{m,j}(C)$$

exists and does not depend on i.

Definition 1.6 ([18]). Consider a weakly Δ-ergodic chain on $A \times B$. We say that the chain is *strongly* Δ*-ergodic on* $A \times B$ if any $C \in \Delta$ is a strongly ergodic class on $A \times B$. In particular, a strongly (S)-ergodic chain on $A \times B$ is called *strongly ergodic on* $A \times B$ for short.

Definition 1.7 ([18]). Consider a weakly $[\Delta]$-ergodic chain on $A \times B$. We say that the chain is *strongly* $[\Delta]$*-ergodic on* $A \times B$ if any $C \in \Delta$ is included in a strongly ergodic class on $A \times B$.

Also, according to [18] (see also [9] and [11]), in connection with the notions from Definitions 1.5, 1.6, and 1.7 we can simplify the language as in the above cases 1, 2, and 3.

Let $T = (T_{ij})$ be a real $m \times n$ matrix. Let $\emptyset \neq U \subseteq \{1, 2, \ldots, m\}$ and $\emptyset \neq V \subseteq \{1, 2, \ldots, n\}$. Define

$$T_U = (T_{ij})_{i \in U,\ j \in \{1,2,\ldots,n\}}, \quad T^V = (T_{ij})_{i \in \{1,2,\ldots,m\},\ j \in V},$$

$$T_U^V = (T_{ij})_{i \in U,\ j \in V}, \quad \alpha(T) = \min_{1 \leq i,j \leq m} \sum_{k=1}^{n} \min(T_{ik}, T_{jk})$$

(if T is a stochastic matrix, then $\alpha(T)$ is called *the ergodicity coefficient of Dobrushin of* T (see, e.g., [4, p. 56] or [5, p. 143])),

$$\bar{\alpha}(T) = \frac{1}{2} \max_{1 \leq i,j \leq m} \sum_{k=1}^{n} |T_{ik} - T_{jk}|,$$

$$\gamma_\Delta(T) = \min_{K \in \Delta} \alpha(T_K), \quad \bar{\gamma}_\Delta(T) = \max_{K \in \Delta} \bar{\alpha}(T_K),$$

where $\Delta \in \mathrm{Par}(\{1, 2, \ldots, m\})$ (see [14] for γ_Δ and $\bar{\gamma}_\Delta$), and

$$|||T|||_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |T_{ij}|$$

(*the* ∞*-norm of* T).

Let

$$R_{m,n} = \{P \mid P \text{ is a real } m \times n \text{ matrix}\},$$
$$N_{m,n} = \{P \mid P \text{ is a nonnegative } m \times n \text{ matrix}\},$$
$$S_{m,n} = \{P \mid P \text{ is a stochastic } m \times n \text{ matrix}\},$$

and for $m = n := r$

$$R_r = R_{r,r}, \quad N_r = N_{r,r}, \quad S_r = S_{r,r}.$$

Definition 1.8 ([15]). Let $\Delta \in \text{Par}(\{1, 2, \ldots, m\})$. We say that a matrix $P \in R_{m,n}$ is $[\Delta]$-*stable* if P_K is a stable matrix (i.e., a real matrix whose rows are identical), $\forall K \in \Delta$.

Definition 1.9 ([14]). Let $\Delta \in \text{Par}(\{1, 2, \ldots, m\})$. We say that a matrix $P \in R_{m,n}$ is Δ-*stable* if Δ is the least fine partition for which P is a $[\Delta]$-stable matrix. In particular, a $(\{1, 2, \ldots, m\})$-stable matrix is called *stable* for short.

Definition 1.10. Let $\emptyset \neq K \subseteq S$. Let p be a probability distribution on S. We say that p is *concentrated on* K if $p^{S-K} = 0$.

THEOREM 1.11. *Let* $(X_n)_{n\geq 1}$ *be a strongly* $[\Delta]$-*ergodic Markov chain at time* 0 *with state space* S, *initial distribution* p_0, *and transition matrices* $(P_n)_{n\geq 1}$. *If* $\exists K \in \Delta$ *such that* p_0 *is concentrated on* K *and* $\lim_{n\to\infty} P_{0,n} := \Pi_0$, *then*

$$\lim_{n\to\infty} P(X_n = j) = (\Pi_0)_{ij} := a_{0,K,j}, \quad \forall j \in S, \ \forall i \in K. \tag{1.1}$$

Proof. Since the chain is strongly $[\Delta]$-ergodic at time 0, Π_0 is a $[\Delta]$-stable (stochastic) matrix. Now, from the fact that p_0 is concentrated on K and $P(X_n = j) = (p_0 P_{0,n})_j$, we obtain the conclusion by letting $n \to \infty$. □

Theorem 1.11 (1.1) yields a probabilistic interpretation of strong $[\Delta]$-ergodicity at time 0. We see that the limit distribution depend on $K \in \Delta$ and does not depend on the initial distribution p_0, if it is concentrated on K. Therefore, all strongly $[\Delta]$-ergodic chains at time 0 with the same state space S, initial distribution concentrated on the same set $K \in \Delta$, and the same transition matrices $(P_n)_{n\geq 1}$ have the same limit distribution. This result generalizes a well-known result for strong ergodicity (see, e.g., [4, p. 223] or [5, p. 157]). Also, this result holds for any strongly $[\Delta]$-ergodic chain on (time set) B with $0 \in B$ (because this implies that the chain is strongly $[\Delta]$-ergodic at time 0) and for any strongly Δ-ergodic chain on B with $0 \in B$ (because this implies that the chain is strongly $[\Delta]$-ergodic on B with $0 \in B$). Moreover, we remark that if a chain is strongly $[\Delta]$- or Δ-ergodic on B with $0 \in B$, then $\exists \Delta' \in \text{Par}(S)$ with $\Delta \preceq \Delta'$ such that the chain is strongly Δ'-ergodic at time 0. Therefore, we can replace '$[\Delta]$-' with 'Δ-' in Theorem 1.11 (this is the maximal case). As concerns some Markovian algorithms, such as, simulated annealing, which are strongly $[\Delta]$- or Δ-ergodic at time 0 (Niemiro [8] asserts that under some conditions simulated annealing is strongly Δ-ergodic at any time $m \geq 0$ (he used the term 'convergent', therefore he did not determine Δ)), we infer that it does not matter the starting point belonging to a given set $K \in \Delta$. This means that Δ and $|\Delta|$ give us information about the dependence on the initial state of the asymptotic behaviour of the algorithm. Moreover, for strongly Δ-ergodic Markov chains at time 0, the importance of Δ and $|\Delta|$ is supported, too, by the following basic fact. Consider the collection of strongly Δ-ergodic Markov chains at time 0 with the same state space S and the same transition matrices $(P_n)_{n\geq 1}$. For any $K \in \Delta$, all chains of the collection with the initial distribution concentrated on K have the same limit distribution, say $p(K)$. If a chain belonging to the considered collection has initial distribution p_0 and limit distribution π_0, then

$$\pi_0 = \sum_{K\in\Delta}\sum_{i\in K} (p_0)_i \, p(K),$$

i.e., π_0 is a convex combination of the $p(K)$, $K \in \Delta$. Accordingly, we define the following notion in a more general context, namely, for weakly Δ-ergodic Markov chains at time 0.

Definition 1.12. Consider a weakly Δ-ergodic Markov chains at time 0. We say that the number $b_0 := |\Delta|$ is *the breaking up* of the state space S at time 0 of the Markov chain. (In general, if a Markov chain is weakly Δ-ergodic on $A \times B$, then we say that the number $b_{A\times B} := |\Delta|$ is *the breaking up* of A on B of the Markov chain. In particular, if $A \times B = S \times \mathbf{N}$, then we use b in place of $b_{S\times\mathbf{N}}$ and say that b is *the breaking up* of the Markov chain for short.)

We can say that the breaking up (of the state space S at time 0) is a measure of the dependence on the initial state of the limit probability distribution of a strongly Δ-ergodic Markov chain at time 0. This statement is supported by the above considerations and the following result.

Proposition 1.13. *We have*

(i) $\Delta \preceq \Delta'$ *implies* $|\Delta| \geq |\Delta'|$;

(ii) $b_0 \geq 1$; $b_0 = 1$ *if and only if the chain is weakly ergodic at time* 0;

(iii) $b_0 \leq |S| = r$; $b_0 = r$ *if and only if the chain is weakly* $(\{i\})_{i\in S}$ *ergodic at time* 0.

Proof. Obvious. □

Remark 1.14. (a) Proposition 1.13 (i) can be used to obtain lower bounds for breaking up (for b_0 we cannot use Theorems 1.30, 1.31, 1.44, and 1.45 from [20], but for b we can use them).

(b) $b_0 = 1$ means that we do not have dependence on the initial state (this is the best case); $b_0 = r$ means that we have maximum dependence on the initial state (this is the worst case).

Let $H : S \to \mathbf{R}$ be a function. We want to find $\min_{y\in S} H(y)$. A stochastic optimization technique for solving this problem approximately when S is very large is the simulated annealing (see, e.g., [8] and the references therein). For this, consider a sequence $(\beta_n)_{n\geq 1}$ of positive real numbers with $\beta_n \to \infty$ as $n \to \infty$ ($(\beta_n)_{n\geq 1}$ is called *the cooling schedule*), a irreducible stochastic matrix $G = (G_{ij})_{i,j\in S}$ (G is called *the generation matrix*) and the Markov chain $(X_n)_{n\geq 1}$ with state space S and transition matrices $(P_n)_{n\geq 1}$, where

$$(P_n)_{ij} = \begin{cases} G_{ij}\mathrm{e}^{-\beta_n (H(j)-H(i))^+} & \text{if } i \neq j, \\ 1 - \sum\limits_{k\neq i} (P_n)_{ik} & \text{if } i = j, \end{cases}$$

$\forall i, j \in S$. $(P_n)_{n\geq 1}$ is the simulated annealing chain; it determines the simulated annealing algorithm (this is the classical case).

We have

$$\lim_{n\to\infty} (P_n)_{ij} = \begin{cases} 0 & \text{if } i \neq j,\ H(j) > H(i), \\ G_{ij} & \text{if } i \neq j,\ H(j) \leq H(i), \\ 1 - \sum\limits_{k\neq i, H(k)\leq H(i)} G_{ik} & \text{if } i = j, \end{cases}$$

$\forall i, j \in S$. Therefore, this chain belongs to the class of Markov chain $(P_n)_{n\geq 1}$ with the property that there exists a (stochastic) matrix P such that $P_n \to P$ as $n \to \infty$. It follows that it is important to study the collection of Markov chains $(P_n)_{n\geq 1}$ for which $P_n \to P$ as $n \to \infty$ (related to this see, e.g., [1], [4, pp. 226–228], [5, p. 170 and pp. 176–178], [13], [16], [20], and, in this paper, Theorem 2.7 (from Section 2) and Sections 3 and 4).

The breaking up for the simulated annealing is an open problem. This is a hard problem, but for some examples it can be easy as shows the following case.

Example 1.15. Let $H : \{1, 2, 3, 4\} \to \mathbf{R}$, $H(1) = H(4) = 4$, $H(2) = H(3) = 2$ and

$$G = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

We have

$$P_n = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3}\mathrm{e}^{-2\beta_n} & 1 - \frac{1}{3}\left(1 + \mathrm{e}^{-2\beta_n}\right) & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 1 - \frac{1}{3}\left(1 + \mathrm{e}^{-2\beta_n}\right) & \frac{1}{3}\mathrm{e}^{-2\beta_n} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad \forall n \geq 1,$$

and

$$P_n \to \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} := P \quad \text{as } n \to \infty.$$

Since P is a mixing matrix (see Definition 4.7), the chain is strongly ergodic (this is a theorem of J.L. Mott (see, e.g., [4, p. 226] or [22, p. 150]). Therefore, $b_0 = 1, \forall\, (\beta_n)_{n\geq 1}$ with $\beta_n \to \infty$ as $n \to \infty$, i.e., as for breaking up, this is the best case.

Now, we give an example for which the breaking up problem is not easy even if the state space S is very small.

Example 1.16. Let $H : \{1,2,3,4\} \to \mathbf{R}$, $H(1) = 2$, $H(2) = H(3) = 4$, $H(4) = 0$ and

$$G = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

(here $G_{12} > 0$ and $G_{21} = 0$). We have

$$P_n = \begin{pmatrix} 1 - \frac{1}{2}\mathrm{e}^{-2\beta_n} & \frac{1}{2}\mathrm{e}^{-2\beta_n} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3}\mathrm{e}^{-2\beta_n} & 0 & \frac{1}{3}\mathrm{e}^{-4\beta_n} & 1 - \frac{1}{3}\left(\mathrm{e}^{-2\beta_n} + \mathrm{e}^{-4\beta_n}\right) \end{pmatrix}, \quad \forall n \geq 1,$$

and

$$P_n \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix} := P \quad \text{as } n \to \infty.$$

Since P is a reducible matrix we cannot use the theorem of J.L. Mott. Some open problems here:

1) $\forall\left(\beta_n\right)_{n\geq 1}$ with $\sum_{n\geq 1} \mathrm{e}^{-2\beta_n} < \infty$, do we have $b_0 \geq 2$? (Obviously, using [1] or [20], we have $b \geq 2, \forall\left(\beta_n\right)_{n\geq 1}$ with $\sum_{n\geq 1} \mathrm{e}^{-2\beta_n} < \infty$.)

2) $\exists\left(\beta_n\right)_{n\geq 1}$ with $\sum_{n\geq 1} \mathrm{e}^{-2\beta_n} < \infty$ such that $b_0 = 2$ (i.e., is the chain weakly (or strongly) $(\{1\}, \{2,3,4\})$-ergodic at time 0)? ($\exists\left(\beta_n\right)_{n\geq 1}$ with $\sum_{n\geq 1} \mathrm{e}^{-2\beta_n} < \infty$ such that $b = 2$?)

From this section we infer that for the design and analysis of some Markovian algorithms (in particular the simulated annealing) we should have in view breaking up, too; obviously, an algorithm with a breaking up as small as possible is preferable.

2. PROBABILISTIC INTERPRETATION OF STRONG ERGODICITY ON A NONEMPTY SUBSET OF STATE SPACE AT TIME 0

In this section we give a probabilistic interpretation of strong ergodicity on a state set A, $\emptyset \neq A \subseteq S$, at time 0, i.e., on $A \times \{0\}$ (see Definition 1.6). Also we prove some results related to strong ergodicity on A (see also [9] for some results and examples).

In the next result we give a probabilistic interpretation of strong ergodicity on $A \times \{0\}$. In particular, it holds for a strongly Markov chain on A.

THEOREM 2.1. *Let* $\left(X_n\right)_{n\geq 1}$ *be a strongly ergodic Markov chain on* $A \times \{0\}$ *with state space* S*, initial distribution* p_0*, and transition matrices* $(P_n)_{n\geq 1}$*. If* $\lim_{n\to\infty} \left(P_{0,n}\right)^A := \Pi_0$*, then*

$$\lim_{n\to\infty} P\left(X_n = j\right) = \left(\Pi_0\right)_{ij} := a_{0,j}, \quad \forall j \in A,\ \forall i \in S.$$

(*Since* Π_0 *is an* $r \times |A|$ *matrix, we agree that* $\left(\Pi_0\right)_{ij}$ *is the entry of* Π_0 *in its* ith *row and the column corresponding to state* j.)

Proof. Since Π_0 is a stable $r \times |A|$ matrix, we have

$$\lim_{n\to\infty} P\left(X_n = j\right) = \lim_{n\to\infty} \left(p_0 P_{0,n}\right)_j = p_0 \lim_{n\to\infty} \left(P_{0,n}\right)^{\{j\}} =$$

$$= p_0 \left(\Pi_0\right)^{\{j\}} = \left(\Pi_0\right)_{ij}, \quad \forall j \in A,\ \forall i \in S$$

($\left(\Pi_0\right)^{\{j\}}$ is the column of Π_0 corresponding to state j), i.e., the conclusion. □

In Theorem 2.1 we obtained a formula for the limit distribution on A of the chain; it also give us a probabilistic interpretation of strong ergodicity on $A \times \{0\}$.

We know that a strongly ergodic chain has a unique limit, i.e., $\exists \Pi \in S_r$ (moreover, Π is a stable matrix) such that $\lim_{n\to\infty} P_{m,n} = \Pi, \forall m \geq 0$ (see, e.g., [4, p. 223] or [5, p. 157]). In the next theorem we show that this result can be generalized for strong ergodicity on any nonempty subset of the state space.

THEOREM 2.2. *Consider a Markov chain* $(P_n)_{n\geq 1}$. *If it is strongly ergodic on* A *at time* m *with limit* Π_m, $\forall m \geq 0$, *then there exists a stable* (*substochastic*) *matrix* Π *such that* $\Pi_m = \Pi$, $\forall m \geq 0$.

Proof. We show that $\Pi_m = \Pi_{m+1}$, $\forall m \geq 0$. First, remark that if P is an $r \times r$ stochastic matrix and Q is a stable real matrix, then $PQ = Q$. Next,

$$\Pi_m = \lim_{n\to\infty} (P_{m,n})^A = \lim_{n\to\infty} P_{m+1} (P_{m+1,n})^A =$$

$$= P_{m+1} \lim_{n\to\infty} (P_{m+1,n})^A = P_{m+1}\Pi_{m+1} = \Pi_{m+1}, \quad \forall m \geq 0.$$

Therefore, $\Pi_m = \Pi_0 := \Pi$, $\forall m \geq 0$. □

A criterion for strong ergodicity on a nonempty subset of state space S is the following result.

THEOREM 2.3. *Consider a Markov chain* $(X_n)_{n\geq 1}$ *with state space* S, *initial distribution* p_0, *and transition matrices* $(P_n)_{n\geq 1}$.

(i) *If* $\exists j \in S$ *such that* $\lim_{n\to\infty} (P_n)^{\{j\}} = 0$, *then*

(i1) *the chain is strongly ergodic on* $\{j\}$ *with limit* 0;

(i2) $\lim_{n\to\infty} P(X_n = j) = 0$.

(ii) (*A generalization of* (i)) *Let*

$$A = \left\{ j \mid j \in S \text{ and } \lim_{n\to\infty} (P_n)^{\{j\}} = 0 \right\}.$$

If $A \neq \emptyset$, *then*

(ii1) *the chain is strongly ergodic on* A *with limit* 0;

(ii2) $\lim_{n\to\infty} P(X_n = j) = 0$, $\forall j \in A$ (*therefore,* $\lim_{n\to\infty} P(X_n \in A) = 0$).

Proof. (i) (i2) follows from (i1) and Theorem 2.1. To prove (i1) let

$$v_n = v_n(j) = \max_{i\in S} (P_n)_{ij}, \quad \forall n \geq 1.$$

Then

$$(P_{m,n})_{ij} = \sum_{k\in S} (P_{m,n-1})_{ik} (P_n)_{kj} \leq \sum_{k\in S} (P_{m,n-1})_{ik} v_n =$$

$$= v_n \sum_{k\in S} (P_{m,n-1})_{ik} = v_n \to 0 \text{ as } n \to \infty, \quad \forall m \geq 0, \ \forall i \in S,$$

i.e., the chain is strongly ergodic on $\{j\}$ with limit 0 (see also Theorem 2.2).

(ii) It follows from (i) because a chain is strongly ergodic on A with limit 0 if and only if it is strongly ergodic on $\{j\}$, $\forall j \in A$, with limit 0. □

A first generalization of Theorem 2.3 is

THEOREM 2.4. *Consider a Markov chain* $(X_n)_{n\geq 1}$ *with state space* S, *initial distribution* p_0, *and transition matrices* $(P_n)_{n\geq 1}$. *If there exist* A, $\emptyset \neq A \subseteq S$, *and a stable matrix* Π *such that* $\lim_{n\to\infty} (P_n)^A = \Pi$, *then the chain is strongly ergodic on* A *with limit* Π *and* $\lim_{n\to\infty} P(X_n \in A) = \sum_{j\in A} \Pi_{ij}$, $\forall i \in S$.

Proof. The chain is strongly ergodic on A with limit Π if and only if it is strongly ergodic on $\{j\}$ with limit $\Pi^{\{j\}}$, $\forall j \in A$ ($\Pi^{\{j\}}$ is the column of Π corresponding to state j). Let $j \in A$. Let

$$u_n = u_n(j) = \min_{i\in S}(P_n)_{ij} \quad \text{and} \quad v_n = v_n(j) = \max_{i\in S}(P_n)_{ij}, \quad \forall n \geq 1.$$

By hypothesis we have

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = \Pi_{ij} := a_j, \quad \forall i \in S$$

(Π_{ij} is the entry of Π in its ith row and the column corresponding to state j). Using the proof of Theorem 2.3 it is easy to see that

$$u_n \leq (P_{m,n})_{ij} \leq v_n, \quad \forall m, n,\ 0 \leq m < n,\ \forall i \in S.$$

Hence

$$\lim_{n\to\infty} (P_{m,n})_{ij} = \lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = a_j.$$

The last assertion is now obvious. □

A second generalization of Theorem 2.3 is

THEOREM 2.5. *Consider a Markov chain* $(X_n)_{n\geq 1}$ *with state space* S, *initial distribution* p_0, *and transition matrices* $(P_n)_{n\geq 1}$. *If* $\exists A$, $\emptyset \neq A \subseteq S$, *such that* $\limsup_{l\to\infty} \limsup_{n\to\infty} (P_{n-l,n})^A = 0$, *then the chain is strongly ergodic on* A *with limit* 0 *and* $\lim_{n\to\infty} P(X_n \in A) = 0$.

Proof. We first show that the chain is strongly ergodic on A with limit 0. Let $j \in A$. It follows from

$$(P_{m,n})_{ij} = \sum_{k\in S} (P_{m,n-l})_{ik} (P_{n-l,n})_{kj} \leq$$

$$\leq \sum_{k\in S} (P_{n-l,n})_{kj}, \quad \forall l, m, n,\ 0 \leq m < n-l < n,\ \forall i \in S,$$

that

$$\limsup_{n\to\infty} (P_{m,n})_{ij} \leq \sum_{k\in S} \limsup_{n\to\infty} (P_{n-l,n})_{kj}, \quad \forall l \geq 1,\ \forall m \geq 0,\ \forall i \in S.$$

So that

$$\limsup_{n\to\infty} (P_{m,n})_{ij} \leq \sum_{k\in S} \limsup_{l\to\infty} \limsup_{n\to\infty} (P_{n-l,n})_{kj} = 0, \quad \forall m \geq 0,\ \forall i \in S.$$

Therefore,

$$\lim_{n\to\infty} (P_{m,n})_{ij} = 0, \quad \forall m \geq 0,\ \forall i \in S,$$

i.e., the chain is strongly ergodic on $\{j\}$ with limit 0. Thus it is strongly ergodic on A with limit 0. The second part follows from the first part and Theorem 2.1. $\square$

Theorem 2.5 has a related result which is in fact a generalization of Theorem 1.35 from [20] (see also Theorem 2.27 from [18]). For this, we should remember that a chain $(P_n)_{n\geq 1}$ is limit strongly ergodic on A ($\emptyset \neq A \subseteq S$) if there exists a stable (substochastic) matrix Π such that

$$\lim_{m\to\infty} \lim_{n\to\infty} (P_{m,n})^A = \Pi$$

(see Definition 2.5 from [18]; the equivalence follows from $\exists \lim_{n\to\infty} (P_{m,n})^A$, $\forall m \geq m_0$ ($m_0 \geq 0$), because $|S| < \infty$ and, as in [18], we agree that when writing $\lim_{u\to\infty} \lim_{v\to\infty} a_{u,v}$, where $a_{u,v} \in \mathbf{R}$, $\forall u, v \in \mathbf{N}$ with $u \geq u_1$, $v \geq v_1(u)$, we assume that $\exists u_0 \geq u_1$ such that $\exists \lim_{v\to\infty} a_{u,v}$, $\forall u \geq u_0$).

Theorem 2.6. *Consider a Markov chain $(P_n)_{n\geq 1}$. Then the chain is strongly ergodic on A with limit Π if and only if it is limit strongly ergodic on A with limit Π.*

Proof. "⇒" If the chain is strongly ergodic on A with limit Π, then $\lim_{n\to\infty} (P_{m,n})^A = \Pi$, $\forall m \geq 0$. It follows that $\lim_{m\to\infty} \lim_{n\to\infty} (P_{m,n})^A = \Pi$, i.e., the chain is limit strongly ergodic on A with limit Π.

"⇐" If the chain is limit strongly ergodic on A with limit Π, then $\exists m_0 \geq 0$ such that $\exists \lim_{n\to\infty} (P_{m,n})^A$, $\forall m \geq m_0$, and

$$\lim_{m\to\infty} \lim_{n\to\infty} (P_{m,n})^A = \Pi.$$

Further, it follows that $\exists \lim_{n\to\infty} (P_{m,n})^A$, $\forall m \geq 0$. Now, we show that $\lim_{n\to\infty} (P_{m,n})^A = \Pi$, $\forall m \geq 0$. Setting $Q_m = \lim_{n\to\infty} (P_{m,n})^A$, $\forall m \geq 0$, we have

$$\left|\left|\left|(P_{m,n})^A - \Pi\right|\right|\right|_\infty = \left|\left|\left|P_{m,k}(P_{k,n})^A - P_{m,k}\Pi\right|\right|\right|_\infty \leq$$

$$\leq |||P_{m,k}|||_\infty \left|\left|\left|(P_{k,n})^A - \Pi\right|\right|\right|_\infty = \left|\left|\left|(P_{k,n})^A - \Pi\right|\right|\right|_\infty \leq$$

$$\leq \left|\left|\left|(P_{k,n})^A - Q_k\right|\right|\right|_\infty + |||Q_k - \Pi|||_\infty\,, \quad \forall k, m, n,\ 0 \leq m < k < n,$$

which implies

$$\limsup_{n\to\infty} \left|\left|\left|(P_{m,n})^A - \Pi\right|\right|\right|_\infty \leq |||Q_k - \Pi|||_\infty\,, \quad \forall k, m,\ 0 \leq m < k.$$

Since $\lim_{m\to\infty} Q_m = \Pi$, we have

$$\limsup_{n\to\infty} \left|\left|\left|(P_{m,n})^A - \Pi\right|\right|\right|_\infty \leq \inf_{k>m} |||Q_k - \Pi|||_\infty = 0, \quad \forall m \geq 0.$$

Therefore, $\lim_{n\to\infty} (P_{m,n})^A = \Pi$, $\forall m \geq 0$, i.e., the chain is strongly ergodic on A with limit Π. $\square$

In the case $P_n \to P$ as $n \to \infty$ with P having transient states (i.e., the homogeneous Markov chain with transition matrix P has transient states) we have more than Theorem 2.5. For this, we should remember that a Markov chain is uniformly strongly ergodic on A with limit Π if $\lim_{n\to\infty} (P_{m,n})^A = \Pi$ uniformly with respect to $m \geq 0$ (see Definition 1.13 from [18] and Theorem 2.2).

THEOREM 2.7. *Consider a Markov chain* $(P_n)_{n\geq 1}$ *with* $P_n \to P$ *as* $n \to \infty$. *Let* T *be the set of transient states of* P (P *has one or more recurrent classes*). *Suppose that* $T \neq \emptyset$. *Then the chain is uniformly strongly ergodic on* T *with limit* 0.

Proof. It is known that $(P^n)^T \to 0$ as $n \to \infty$ (see, e.g., [4, p. 91]). Let $\varepsilon > 0$. It follows from $|S| < \infty$ that $\exists n_0 \geq 1$ such that $(P^{n_0})_{ij} < \varepsilon$, $\forall i \in S$, $\forall j \in T$. Further, because $|S| < \infty$ and

$$\lim_{n\to\infty} (P_{n,n+n_0})^T = (P^{n_0})^T,$$

$\exists n_1 \geq 0$ such that

$$(P_{n,n+n_0})_{ij} < \varepsilon, \quad \forall n \geq n_1, \ \forall i \in S, \ \forall j \in T.$$

From

$$(P_{m,m+n+n_1+n_0})^T = P_{m,m+n+n_1} (P_{m+n+n_1,m+n+n_1+n_0})^T, \quad \forall m, n \geq 0,$$

we have

$$(P_{m,m+n+n_1+n_0})_{ij} < \varepsilon, \quad \forall m, n \geq 0, \ \forall i \in S, \ \forall j \in T,$$

since $P_{m,m+n+n_1}$ is a stochastic matrix, $\forall m, n \geq 0$. Therefore, the chain is uniformly strongly ergodic on T with limit 0. $\square$

To apply Theorem 2.7 to the simulated annealing chain $(P_n)_{n\geq 1}$ (see Section 1) we need to determine the set of transient states of P, where $P = \lim_{n\to\infty} P_n$. This is $T = \Big\{i \mid i \in S$ and $\exists p \geq 2$, $\exists i_1, i_2, \ldots, i_p \in S$ such that $i_1 = i$, $G_{i_1 i_2}, G_{i_2 i_3}, \ldots, G_{i_{p-1} i_p} > 0$, and $H(i_1) \geq H(i_2) \geq \cdots \geq H(i_{p-1}) > H(i_p)\Big\} \cup \Big\{i \mid i \in S$ and $\exists j \in S$, $j \neq i$, for which $\exists p \geq 2$, $\exists i_1, i_2, \ldots, i_p \in S$ such that $i_1 = i$, $i_p = j$, $G_{i_1 i_2}, G_{i_2 i_3}, \ldots, G_{i_{p-1} i_p} > 0$, and $H(i_1) = H(i_2) = \cdots = H(i_p)$ and $\forall q \geq 2$, $\forall j_1, j_2, \ldots, j_q \in S$ such that $j_1 = j$, $j_q = i$, and $H(j_1) = H(j_2) =$

$\cdots = H(j_q),\ \exists u \in \{1, 2, \ldots, q-1\}$ with $G_{j_u j_{u+1}} = 0\Big\}$. Further, by Theorem 2.7, the simulated annealing chain is uniformly strongly ergodic on T with limit 0 and $\lim\limits_{n\to\infty} P(X_n = j) = 0,\ \forall j \in T$ (see also Theorem 4.2 from [8] which has something in common with this result). Setting $R = S - T$ (R is the set of recurrent states of P), we note also that

(i) $P_C^C \geq G_C^C$ if C is a recurrent class;

(ii) if P_R^R is irreducible and aperiodic (this happens if R is a recurrent class and, e.g., G_R^R is irreducible and aperiodic), then P is mixing (hence, using a theorem of J.L. Mott (see, e.g., [4, p. 226] or [22, p. 150]), it follows that the simulated annealing chain is strongly ergodic in this case (therefore, $b_0 = 1$ for any cooling schedule $(\beta_n)_{n\geq 1}$)).

3. WEAK AND STRONG ERGODICITY

In this section we continue the study of special chains $(P_n)_{n\geq 1}$ with $P_n \to P$ as $n \to \infty$. Here we give some weak or strong ergodicity results both related to such chains and results which can be applied to some special cases.

The following theorem is a well-known result on weak ergodicity.

Theorem 3.1 (J. Hajnal). *Consider a Markov chain* $(P_n)_{n\geq 1}$. *Then it is weakly ergodic if and only if there exists a strictly increasing sequence* $0 \leq n_1 < n_2 < \cdots$ *of natural numbers such that* $\sum\limits_{s\geq 1} \alpha\left(P_{n_s, n_{s+1}}\right) = \infty$.

Proof. See, e.g., [4, p. 219] or [5, p. 151]. □

Closely related to Theorem 3.1 is the following result.

Theorem 3.2. *Consider a Markov chain* $(P_n)_{n\geq 1}$. *Then it is weakly ergodic if and only if there exist two strictly increasing sequences* $(n_s)_{s\geq 1}$ *and* $(n'_s)_{s\geq 1}$ *of natural numbers with* $0 \leq n'_1 < n_1 \leq n'_2 < n_2 \leq \cdots$ *such that* $\sum\limits_{s\geq 1} \alpha(P_{n'_s, n_s}) = \infty$.

Proof. "$\Rightarrow$" If the chain is weakly ergodic, then by Theorem 3.1 there exists a strictly increasing sequence $0 \leq m_1 < m_2 < \cdots$ of natural numbers such that $\sum\limits_{s\geq 1} \alpha(P_{m_s, m_{s+1}}) = \infty$. Now, the conclusion follows with $(n_s)_{s\geq 1} := (m_s)_{s\geq 2}$ and $(n'_s)_{s\geq 1} := (m_s)_{s\geq 1}$.

"$\Leftarrow$" Using the well-known properties $\bar{\alpha}\,(A) = 1 - \alpha(A)$ and $\bar{\alpha}\,(AB) \leq \bar{\alpha}\,(A)\,\bar{\alpha}\,(B)$, where $A, B \in S_r$ (see, e.g., [4, pp. 57–58] or [5, pp. 144–145]), we have

$$\alpha\left(P_{n_{s-1}, n_s}\right) \geq \alpha\left(P_{n'_s, n_s}\right), \quad \forall s \geq 2.$$

This implies

$$\sum_{s\geq 1} \alpha\left(P_{n_s,n_{s+1}}\right) = \infty,$$

i.e., by Theorem 3.1, the chain is weakly ergodic. □

Remark 3.3. (a) Theorem 3.1 was generalized for [Δ]-groupable Markov chains in [15]. The same thing can be made for Theorem 3.2 using γ_Δ in place of α.

(b) The application of Theorems 3.1 and 3.2 can fail in some cases, e.g., in the case when they require only unbounded sequences of the block lengths (the length of a block $P_{m,n}$ is $n-m$; for other things about blocks see [16] and [20]). An example where they can be applied and the sequences of the block lengths can be taken unbounded is

$$P_n = \begin{cases} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} & \text{if } \exists k \geq 0 \text{ such that } n = 2^k, \\ I_2 & \text{if } n \neq 2^k,\ \forall k \geq 0. \end{cases}$$

We can apply Theorems 3.1 and 3.2, e.g., with $P_{n_s,n_{s+1}} = P_{2^s,2^{s+1}}$ and $P_{n'_s,n_s} = P_{2^s-1,2^s}$, respectively, $\forall s \geq 1$. Also, Theorems 3.1 and 3.2 can be applied here with all blocks of length 1.

Proposition 3.4. *Let $(a_n)_{n\geq 1}$ be a sequence of nonnegative real numbers and $k \geq 1$. If $\sum_{n\geq 1} a_n = \infty$, then there exists a strictly increasing sequence $1 \leq n_1 < n_2 < \cdots$ of natural numbers with $n_{s+1} - n_s = k$, $\forall s \geq 1$, such that $\sum_{s\geq 1} a_{n_s} = \infty$. Moreover, $\exists u \in \{0, 1, \ldots, k-1\}$ such that $\sum_{n\geq 1} a_{kn+u} = \infty$.*

Proof. Case 1. $k = 1$. Obvious.

Case 2. $k \geq 2$. Suppose that for any strictly increasing sequence $1 \leq n_1 < n_2 < \cdots$ of natural numbers with $n_{s+1} - n_s = k$, $\forall s \geq 1$, we have $\sum_{s\geq 1} a_{n_s} < \infty$. Then

$$\sum_{n\geq 1} a_{kn} < \infty,\ \sum_{n\geq 1} a_{kn+1} < \infty, \ldots,\ \sum_{n\geq 1} a_{kn+(k-1)} < \infty,$$

so that $\sum_{n\geq 1} a_n < \infty$. Contradiction. The last part is now obvious. □

Remark 3.5. It follows from Proposition 3.4 that $\sum_{s\geq 1} a_{n_s} = \infty$ still holds if $n_{s+1} - n_s \geq k$, $\forall s \geq 1$.

Clearly, we also need simple results on weak ergodicity obtained directly from entries of matrices without using the ergodicity coefficients as in Theorems 3.1 and 3.2. An example is the following result.

THEOREM 3.6. *Consider a Markov chain* $(P_n)_{n\geq 1}$ *and* $\Delta = (K_1, K_2, \ldots, K_p, K_{p+1}) \in \mathrm{Par}\,(S)$, *where* $K_{p+1} \neq \emptyset$ *or* $K_{p+1} = \emptyset$ *(*K_{p+1} *is the set of transient states of* P *below). Let* $K = \bigcup_{u=1}^{p} K_u$. *If*

(i) $P_n \to P$ *as* $n \to \infty$;

(ii) $P_{K_u}^{K_u}$ *is an irreducible and aperiodic stochastic matrix,* $\forall u \in \{1, 2, \ldots, p\}$ *(therefore,* $P_{K_u}^{K_v} = 0$, $\forall u \in \{1, 2, \ldots, p\}$, $\forall v \in \{1, 2, \ldots, p+1\}$ *with* $u \neq v$ *(if* $K_{p+1} = \emptyset$, *then we set* $P_{K_u}^{\emptyset} = 0$, $\forall u \in \{1, 2, \ldots, p\}$*));*

(iii) $\exists v \in \{1, 2, \ldots, p+1\}$, $\exists j \in K_v$, $\exists i_1 \in K_{u_1}$, $\exists i_2 \in K_{u_2}, \ldots,$ $\exists i_w \in K_{u_w}$ *such that*

$$\sum_{n\geq 1} \min_{i\in\{i_1,i_2,\ldots,i_w\}} (P_n)_{ij} = \infty,$$

where w *is the smallest number with the property* $K - K_v = \bigcup_{t=1}^{w} K_{u_t}$, *then the chain is weakly ergodic.*

Proof. By (i), we have $\lim_{n\to\infty} P_{n,n+k} = P^k$, $\forall k \geq 1$. Further, by (ii), $\exists k_0 \geq 1$, $\exists n_0' \geq 1$ such that

$$(P_{n,n+k_0})_{K_u}^{K_u} > 0, \quad \forall n \geq n_0',\ \forall u \in \{1, 2, \ldots, p\}.$$

It follows that $\exists a' > 0$ such that

$$(P_{n,n+k_0})_{gh} > a', \quad \forall n \geq n_0',\ \forall g, h \in K_u,\ \forall u \in \{1, 2, \ldots, p\}.$$

By (i), (ii), and $|S| < \infty$, $\exists a'' > 0$, $\exists n_0'' \geq 1$ such that $\forall u \in \{1, 2, \ldots, p\}$, $\forall h \in K_u$, $\exists g \in K_u$ such that

$$(P_n)_{gh} \geq a'', \quad \forall n \geq n_0''.$$

Let $l \geq 0$, $a = \min(a', a'')$, and $n_0 = \max(n_0', n_0'')$. By (iii) and Proposition 3.4, there exists a strictly increasing sequence $1 \leq n_1 < n_2 < \cdots$ of natural numbers with $n_{s+1} - n_s = l + k_0 + 1$, $\forall s \geq 1$, such that

$$\sum_{n\geq 1} \min_{i\in\{i_1,i_2,\ldots,i_w\}} (P_{n_s})_{ij} = \infty$$

Let $a_{n_s} = \min_{i\in\{i_1,i_2,\ldots,i_w\}} (P_{n_s})_{ij}$, $\forall s \geq 1$. Let $s \geq 1$. If $n_s + l \geq n_0$, then the matrix

$$\left(P_{n_s+l,n_{s+1}-1} P_{n_{s+1}}\right)_K^{\{j\}}$$

has all entries greater or equal to $\min\left(a^2, aa_{n_{s+1}}\right)$ for $j \in K_v$, where $v \in \{1, 2, \ldots, p\}$, and it has all entries greater or equal to $aa_{n_{s+1}}$ for $j \in K_{p+1}$, when $K_{p+1} \neq \emptyset$ (therefore in both cases $\left(P_{n_s+l,n_{s+1}}\right)_K$ is a Markov matrix (see, e.g., [4, p. 57] or [22, p. 140]), if $a_{n_{s+1}} > 0$).

Case 1. $K_{p+1} = \emptyset$. It follows that P does not have transient states. By the above considerations we have

$$\alpha\left(P_{n_s+l,n_{s+1}}\right) \geq a \min\left(a, a_{n_{s+1}}\right), \quad \forall s \geq 1,\ n_s + l \geq n_0,$$

so that

$$\sum_{s\geq 1, n_s+l\geq n_0} \alpha\left(P_{n_s+l,n_{s+1}}\right) \geq a \sum_{s\geq 1, n_s+l\geq n_0} \min\left(a, a_{n_{s+1}}\right) = \infty$$

because a is a constant, $a > 0$, and

$$\sum_{s\geq 1} a_{n_s} = \infty$$

(we use the cases:

1) $|\{s\,|s \geq 1,\ n_s + l \geq n_0,\ \text{and}\ \min\left(a, a_{n_s}\right) = a\}| < \infty$;

2) $|\{s\,|s \geq 1,\ n_s + l \geq n_0,\ \text{and}\ \min\left(a, a_{n_s}\right) = a\}| = \aleph_0$).

Now, by Theorem 3.2 (or Theorem 3.1, if we take $l = 0$), it follows that the chain is weakly ergodic.

Case 2. $K_{p+1} \neq \emptyset$. Now, P have transient states. Let $0 < \varepsilon < \frac{1}{|K_{p+1}|}$. Let $\varepsilon' = |K_{p+1}|\,\varepsilon$. By Theorem 2.7, $\exists m_0 \geq 1$ such that

$$\left(P_{m,m+m_0}\right)_{gh} < \varepsilon, \quad \forall m \geq 0,\ \forall g \in S,\ \forall h \in K_{p+1}.$$

This implies that

$$\sum_{h\in K} \left(P_{m,m+m_0}\right)_{gh} > 1 - \varepsilon', \quad \forall m \geq 0,\ \forall g \in S.$$

First, suppose that $j \in K_v$, where $v \in \{1, 2, \ldots, p\}$. Take $l = m_0$. Because $\left(P_{n_s+l,n_{s+1}}\right)_K^{\{j\}}$ has all entries greater or equal to $\min\left(a^2, aa_{n_{s+1}}\right)$ for $n_s + l \geq n_0$, it follows that $\left(P_{n_s,n_{s+1}}\right)^{\{j\}}$ has all entries greater or equal to $(1 - \varepsilon') \min\left(a^2, aa_{n_{s+1}}\right)$ for $n_s + l \geq n_0$. Therefore,

$$\alpha\left(P_{n_s,n_{s+1}}\right) \geq \left(1 - \varepsilon'\right) a \min\left(a, a_{n_{s+1}}\right), \quad \forall s \geq 1,\ n_s + l \geq n_0.$$

So that, by Theorem 3.1, the chain is weakly ergodic. The subcase $j \in K_{p+1}$ is similar. $\square$

Problem 3.7. Under the assumptions of Theorem 3.6 is the chain even strongly ergodic? (For the simulated annealing if the weak ergodicity is proved, then under some conditions the strong ergodicity follows by a result of R.W. Madsen and D.L. Isaacson (see Theorem V.4.3 in [5, p. 160]).)

Example 3.8. Let

$$P_n = \begin{pmatrix} 1-\frac{1}{2n} & 0 & 0 & \frac{1}{4n} & 0 & \frac{1}{4n} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{4n} & 0 & \frac{3}{4}-\frac{1}{2n} & \frac{1}{4} & \frac{1}{4n} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2n} & \frac{1}{2} & \frac{1}{2}-\frac{1}{2n} \\ 0 & \frac{1}{4n} & \frac{1}{4n} & 0 & \frac{1}{4} & \frac{3}{4}-\frac{1}{2n} \end{pmatrix}, \quad \forall n \geq 1.$$

This chain satisfies the assumptions of Theorem 3.6 with $j = 4$, $i_1 = 1$, and $i_2 = 5$, therefore it is weakly ergodic.

Remark 3.9. Behind the reasoning from the proof of Theorem 3.6 there are some types of matrices. Take (in the case $K_{p+1} = \emptyset$), e.g.,

$$A = \begin{pmatrix} \times & \times & 0 & 0 \\ \times & \times & 0 & 0 \\ 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \times & \times & 0 & \times \\ \times & \times & 0 & 0 \\ 0 & 0 & \times & 0 \\ 0 & 0 & \times & \times \end{pmatrix},$$

where $\times$ stands for nonzero entries. We have

$$AB = \begin{pmatrix} \times & \times & 0 & \times \\ \times & \times & 0 & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \end{pmatrix}.$$

Note that A is a diagonal $[(\{1,2\}, \{3,4\})]$-simple matrix (see [20]) with $A_{\{1,2\}}^{\{1,2\}} > 0$ and $A_{\{3,4\}}^{\{3,4\}} > 0$, B is a Sarymsakov matrix, but it is not an almost scrambling matrix (see, e.g., [2, pp. 64–69], [3], [21], and [22, p. 146]), and the product AB is a Markov matrix (see, e.g., [4, p. 57] or [22, p. 140]).

In particular, the notions and results from general Δ-ergodic theory can be used to obtain results on weak or strong ergodicity (see also Section 2 and [18]). An example in this section is Theorem 3.6. Another one is the following theorem which, in particular, can be used in the case $P_n \to P$ as $n \to \infty$ with P having one or more recurrent classes and transient states.

Theorem 3.10. *Let*

$$P_n = \begin{pmatrix} Q_n & 0 \\ R_n & T_n \end{pmatrix}, \quad \forall n \geq 1,$$

be a Markov chain and $(K_1, K_2) \in \mathrm{Par}\,(S)$, *where* $(P_n)_{K_1}^{K_1} = Q_n,\ \forall n \geq 1$. *If*

(i) K_1 *is included in a weakly (respectively, strongly) ergodic class*;

(ii) $(P_n)_{n\geq 1}$ *is weakly ergodic on* K_2 *(equivalently,* $\prod\limits_{n\geq t} T_n = 0$, $\forall t \geq 1$),

then $(P_n)_{n\geq 1}$ *is weakly (respectively, strongly) ergodic.*

Proof. First, we suppose that K_1 is included in a weakly ergodic class. By (ii), $(P_n)_{n\geq 1}$ is strongly ergodic on K_2 with limit 0. Further, it follows from $|S| < \infty$ and

$$(P_{m,n})^{K_2} \to 0 \text{ as } n \to \infty, \quad \forall m \geq 0,$$

that $\forall \varepsilon,\ 0 < \varepsilon < 1,\ \forall m \geq 0,\ \forall i \in S,\ \exists n_{m,\varepsilon} > m$ such that

$$\sum_{j \in K_1} (P_{m,n})_{ij} \geq 1 - \varepsilon, \quad \forall n \geq n_{m,\varepsilon}.$$

Since $|K_1| < \infty$ and K_1 is included in a weakly ergodic class and $(P_n)_{K_1}^{K_2} = 0$, $\forall n \geq 1$, it follows that $\exists a > 0$ with property that $\forall m \geq 0$, $\exists n_m > m$ such that $\forall n \geq n_m$, $\exists j_{m,n} \in K_1$ for which

$$(Q_{m,n})_{ij_{m,n}} \geq a, \quad \forall i \in K_1$$

(obviously, we use also the fact that a stochastic $u \times u$ matrix has in every row at least one entry greater or equal to $\frac{1}{u}$).

Let $m \geq 0$ and $0 < \varepsilon < 1$. Then the matrix

$$P_{m,n_{m,\varepsilon}} P_{n_{m,\varepsilon},n_{n_{m,\varepsilon}}}$$

has a column with all entries greater or equal to $(1 - \varepsilon)\,a$ (therefore, it is a Markov matrix (see, e.g., [4, p. 57] or [22, p. 140]). So that

$$\alpha\left(P_{m,n_{m,\varepsilon}} P_{n_{m,\varepsilon},n_{n_{m,\varepsilon}}}\right) \geq (1 - \varepsilon)\,a.$$

Hence we can apply Theorem 3.1. Therefore, the chain $(P_n)_{n\geq 1}$ is weakly ergodic.

Second, we suppose that K_1 is included in a strongly ergodic class. It follows that it is included in a weakly ergodic class. Hence, using the first part, $(P_n)_{n\geq 1}$ is weakly ergodic. Now, the chain $(P_n)_{n\geq 1}$ being weakly ergodic and K_1 being included in a strongly ergodic class, it follows that $(P_n)_{n\geq 1}$ is strongly ergodic (see Theorem 2.1 in [13]). □

Remark 3.11. The special case $T_n = 0$, $\forall n \geq 1$, of the above theorem can be given another proof. For this, consider the ergodicity coefficient (see [17] (correctly, here, $X = \{1, 2, \ldots, m\}^2$))

$$\bar{\zeta}\,(A, B) = \frac{1}{2} \max_{i,j \in \{1,2,\ldots,m\}} \sum_{k=1}^{n} |a_{ik} - b_{jk}|\,,$$

where A and B are two nonnegative $m \times n$ matrices. Further,

$$P_{m,n} = \begin{pmatrix} Q_{m,n} & 0 \\ R_{m+1} Q_{m+1,n} & 0 \end{pmatrix}, \quad \forall m, n, 0 \leq m < n.$$

Using the inequality

$$\bar{\zeta}\,(AC, BC) \leq \bar{\zeta}\,(A, B)\,\bar{\alpha}\,(C)\,, \quad \forall A, B \in S_{p,q},\ \forall C \in S_{q,r},$$

from [17], we have

$$\bar{\zeta}\,(Q_{m,n}, R_{m+1} Q_{m+1,n}) \leq \bar{\zeta}\,(Q_{m+1}, R_{m+1})\,\bar{\alpha}\,(Q_{m+1,n}) \leq$$

$$\leq \bar{\alpha}\,(Q_{m+1,n}) \to 0 \text{ as } n \to \infty, \quad \forall m \geq 0,$$

because K_1 is included in a weakly ergodic class. Therefore, $(P_n)_{n \geq 1}$ is weakly (respectively, strongly) ergodic.

A special case of Theorem 3.10 (which, in particular, can be also applied in the case $P_n \to P$ as $n \to \infty$) is the following result.

THEOREM 3.12. *Let*

$$P_n = \begin{pmatrix} Q_n & 0 \\ R_n & T_n \end{pmatrix}, \quad \forall n \geq 1,$$

be a Markov chain and $(K_1, K_2) \in \operatorname{Par}(S)$, *where* $(P_n)_{K_1}^{K_1} = Q_n$, $\forall n \geq 1$. *If*

(i) K_1 *is included in a weakly (respectively, strongly) ergodic class*;

(ii) T_n *is lower triangular,* $\forall n \geq 1$, *or it is upper triangular,* $\forall n \geq 1$;

(iii) $\prod_{n \geq t} (P_n)_{ii} = 0$, $\forall t \geq 1$, $\forall i \in K_2$,

then $(P_n)_{n \geq 1}$ *is weakly (respectively, strongly) ergodic.*

Proof. It is sufficient to prove that $(P_n)_{n \geq 1}$ is strongly ergodic on K_2 since, by Theorem 3.10, we obtain weak (respectively, strong) ergodicity. Obviously,

$$\lim_{n \to \infty} (P_{m,n})_{K_1}^{K_2} = 0, \quad \forall m \geq 0.$$

It remains to prove that

$$\lim_{n \to \infty} (P_{m,n})_{K_2}^{K_2} = 0, \quad \forall m \geq 0.$$

Let $m \geq 0$. Then for $m_1, m_2, \ldots, m_u \geq 1$, where $u = |K_2|$, setting $q_t = m + m_1 + \cdots + m_t$, $\forall t \in \{1, 2, \ldots, u\}$, we have

$$(P_{m,q_u})^{K_2}_{K_2} = (P_{m,q_1} P_{q_1,q_2} \ldots P_{q_{u-1},q_u})^{K_2}_{K_2} = (P_{m,q_1})^{K_2}_{K_2} (P_{q_1,q_2})^{K_2}_{K_2} \ldots (P_{q_{u-1},q_u})^{K_2}_{K_2}.$$

Further, by (ii) and (iii), the matrices

$$\limsup_{m_1 \to \infty} (P_{m,q_1})^{K_2}_{K_2}, \ \limsup_{m_2 \to \infty} (P_{q_1,q_2})^{K_2}_{K_2}, \ldots, \ \limsup_{m_u \to \infty} \left(P_{q_{u-1},q_u}\right)^{K_2}_{K_2}$$

are strictly lower triangular, if T_n is lower triangular, $\forall n \geq 1$, or they are strictly upper triangular, if T_n is upper triangular, $\forall n \geq 1$. It follows that

$$\limsup_{m_1 \to \infty} \limsup_{m_2 \to \infty} \ldots \limsup_{m_u \to \infty} (P_{m,q_u})^{K_2}_{K_2} = 0.$$

This yields

$$\lim_{n \to \infty} (P_{m,n})^{K_2}_{K_2} = 0. \quad \square$$

Remark 3.13. Clearly, (ii) and (iii) from Theorem 3.12 imply that $\exists i \in K_2$, $\exists j \in K_1$ such that $\sum\limits_{n \geq 1} (P_n)_{ij} = \infty$.

Remark 3.14. As to reliability theory we mention that all chains from [6] verify (i) from Theorem 3.12 with $K_1 = \{r\}$ (here $P_n = \begin{pmatrix} T_n & R_n \\ 0 & Q_n \end{pmatrix}$, $\forall n \geq 1$) in the 'strongly' case. Moreover, the chain given by (2.2) from [6] also verifies (ii); if (iii) also holds, then it is strongly ergodic.

Remark 3.15. Theorem 3.10 or Theorem 3.12 can be used for each Markov chain $(P'_n)_{n \geq 1}$ whose perturbation of the first type is $(P_n)_{n \geq 1}$ there, i.e., $\sum\limits_{n \geq 1} |||P_n - P'_n|||_\infty < \infty$ (see [1] or [20]). E.g., if

$$P'_n = \begin{pmatrix} 1 - \frac{1}{n^2} & 0 & \frac{1}{n^2} \\ \frac{1}{4n^2} & 1 - \frac{1}{2n} - \frac{1}{4n^2} & \frac{1}{2n} \\ \frac{1}{2n} & 0 & 1 - \frac{1}{2n} \end{pmatrix}, \quad \forall n \geq 1,$$

then we can apply Theorem 3.12 to the chain

$$P_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \frac{1}{2n} & \frac{1}{2n} \\ \frac{1}{2n} & 0 & 1 - \frac{1}{2n} \end{pmatrix}, \quad \forall n \geq 1.$$

Further, since, by Theorem 3.12, $(P_n)_{n \geq 1}$ is strongly ergodic, it follows that $(P'_n)_{n \geq 1}$ is strongly ergodic (see [1] or [20]). Moreover, they have the same limit (see [1] or [20]), namely,

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Remark 3.16. Theorems 2.7 and 3.12 can be also used to determine T from a basis $(K_1, K_2, \ldots, K_p, T)$ of a strongly Δ-ergodic Markov chain or, more generally, of a tail indempotent (see [7], [18], and [19]).

Exercise 3.17. Let

$$P_n = \begin{pmatrix} 1-\frac{1}{n} & 0 & \frac{1}{n} \\ \frac{1}{n} & 1-\frac{1}{n} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \forall n \geq 1.$$

Prove that the chain is weakly ergodic (weak ergodicity and $\{3\}$ included in a strongly ergodic class imply strong ergodicity) using:

(i) Theorem 3.1 (of J. Hajnal);

(ii) Theorem 3.12.

Then compare the sizes of the computations.

4. UNIFORM WEAK Δ-ERGODICITY

In this section we give some results on uniform weak $[\Delta]$- or Δ- ergodicity for a chain $(P_n)_{n\geq 1}$ with $P_n \to P$ as $n \to \infty$.

Definition 4.1 ([12]). Let $i, j \in S$. We say that i and j are in the same *uniformly weakly ergodic class* if $\forall k \in S$ we have

$$\lim_{n\to\infty} \left[(P_{m,m+n})_{ik} - (P_{m,m+n})_{jk} \right] = 0$$

uniformly with respect to $m \geq 0$.

Write $i \overset{u}{\sim} j$ when i and j are in the same uniformly weakly ergodic class. Then $\overset{u}{\sim}$ is an equivalence relation and determines a partition $(U_1, U_2, \ldots, U_t)$ of S. The sets $U_1, U_2, \ldots, U_t$ are called *uniformly weakly ergodic classes.*

Definition 4.2 ([14]). Let $\Delta = (U_1, U_2, \ldots, U_t)$ be the partition of uniformly weakly ergodic classes of a Markov chain. We say that the chain is *uniformly weakly* Δ*-ergodic.* In particular, a uniformly weakly (S)-ergodic chain is called *uniformly weakly ergodic* for short.

Definition 4.3 ([15]). Let $(U_1, U_2, \ldots, U_t)$ be the partition of uniformly weakly ergodic classes of a Markov chain with state space S and $\Delta \in \text{Par}\,(S)$. We say that the chain is *uniformly weakly* $[\Delta]$*-ergodic* if $\Delta \preceq (U_1, U_2, \ldots, U_t)$.

Definition 4.4 ([14]). Let $A \in N_m$ and $\Delta \in \operatorname{Par}(\{1, 2, \ldots, m\})$. We say that A is a $[\Delta]$-*scrambling matrix* if $\gamma_\Delta(A) > 0$.

Definition 4.5. Let $A \in N_m$ and $\Delta \in \operatorname{Par}(\{1, 2, \ldots, m\})$. We say that A is a Δ-*scrambling matrix* if Δ is the least fine partition for which A is a $[\Delta]$-scrambling matrix. In particular, an (S)-scrambling matrix is called *scrambling* for short.

Definition 4.6 ([14]). Let $A \in N_m$ and $\Delta \in \operatorname{Par}(\{1, 2, \ldots, m\})$. We say that A is a $[\Delta]$-*mixing matrix* if $\exists n \geq 1$ such that $\gamma_\Delta(A^n) > 0$.

Definition 4.7. Let $A \in N_m$ and $\Delta \in \operatorname{Par}(\{1, 2, \ldots, m\})$. We say that A is a Δ-*mixing matrix* if Δ is the least fine partition for which A is a $[\Delta]$-mixing matrix. In particular, an (S)-mixing matrix is called *mixing* for short.

Definition 4.8. Let $A \in N_{m,n}$. We say that A is a *generalized stochastic matrix* if $\exists a \geq 0,\ \exists B \in S_{m,n}$ such that $A = aB$.

Let

$$G_\Delta = \left\{ P \mid P \in S_r \text{ and } \forall K, L \in \Delta,\ P_K^L \text{ is a generalized stochastic matrix} \right\},$$

where $\Delta \in \operatorname{Par}(S)$.

Definition 4.9 ([15]). We say that a Markov chain $(P_n)_{n\geq 1}$ is $[\Delta]$-*groupable* if $P_n \in G_\Delta, \forall n \geq 1$.

A uniform weak $[\Delta]$-ergodicity result in the case $P_n \to P$ as $n \to \infty$ is as follows.

THEOREM 4.10 ([15]). *Consider a $[\Delta]$-groupable (finite) Markov chain $(P_n)_{n\geq 1}$ such that $\lim_{n\to\infty} P_n = P$. Then the chain is uniformly weakly $[\Delta]$-ergodic if and only if P is a $[\Delta]$-mixing matrix. More generally, this equivalence still holds if $\exists k \geq 1$ such that $\lim_{n\to\infty} P_{n,n+k} = P$.*

Proof. See [15]. □

As to uniform weak Δ-ergodicity in the case $P_n \to P$ as $n \to \infty$ is the following result.

THEOREM 4.11 ([15]). *Consider a $[\Delta]$-groupable Markov chain $(P_n)_{n\geq 1}$ such that $\lim_{n\to\infty} P_n = P$ and $|\Delta| \leq 2$. Then the chain is uniformly weakly Δ-ergodic if and only if P is a Δ-mixing matrix. More generally, this equivalence still holds if $\exists k \geq 1$ such that $\lim_{n\to\infty} P_{n,n+k} = P$ and $|\Delta| \leq 2$.*

Proof. See [15]. □

The condition $|\Delta| \leq 2$ from Theorem 4.11 can be removed for one of the implications. This is showed in the following result.

THEOREM 4.12. *Consider a* $[\Delta]$*-groupable Markov chain* $(P_n)_{n\geq 1}$ *such that* $\lim_{n\to\infty} P_n = P$. *If* P *is a* Δ*-mixing matrix, then the chain is uniformly weakly* Δ*-ergodic. More generally, this implication still holds if* $\exists k \geq 1$ *such that* $\lim_{n\to\infty} P_{n,n+k} = P$.

Proof. Let $k \geq 1$ such that $\lim_{n\to\infty} P_{n,n+k} = P$. To prove that the chain $(P_n)_{n\geq 1}$ is uniformly weakly Δ-ergodic we use the following result. If we have a $[\Delta]$-groupable Markov chain $(P_n)_{n\geq 1}$ and Δ is the least fine partition for which $\exists a > 0,\ \exists u_0 \geq 1$ such that

$$\gamma_\Delta\left(P_{m,m+u_0}\right) \geq a, \quad \forall m \geq 0,$$

then the chain is uniformly weakly Δ-ergodic (see [15] or [19]).

Let Δ' be the least fine partition for which $\exists a > 0,\ \exists u_0 \geq 1$ such that

$$\gamma_{\Delta'}\left(P_{m,m+u_0}\right) \geq a, \quad \forall m \geq 0.$$

We show that $\Delta' = \Delta$. Note that if $Q \in S_r$ is a Δ_1-mixing matrix and $\gamma_{\Delta_2}\left(Q^n\right) > 0$, then $\Delta_2 \preceq \Delta_1$, where $\Delta_1, \Delta_2 \in \mathrm{Par}\,(S)$. Further, the continuity of $\gamma_{\Delta'}$, $\lim_{m\to\infty} P_{m,m+nk} = P^n$, $\forall n \geq 1$, and P is a Δ-mixing matrix (see Definition 4.7) implies that

$$a \leq \limsup_{m\to\infty} \gamma_{\Delta'}\left(P_{m,m+u_0}\right) \leq \limsup_{m\to\infty} \gamma_{\Delta'}\left(P_{m,m+nk}\right) =$$
$$= \lim_{m\to\infty} \gamma_{\Delta'}\left(P_{m,m+nk}\right) = \gamma_{\Delta'}\left(P^n\right), \quad \forall n \geq 1,\ nk \geq u_0$$

(we also used the inequality $\gamma_{\Delta'}\left(P_{m,m+nk}\right) \geq \gamma_{\Delta'}\left(P_{m,m+u_0}\right)$; this follows from $\bar{\gamma}_\Delta\,(AB) \leq \bar{\gamma}_\Delta\,(A)\,\bar{\alpha}\,(B)\,, \bar{\alpha}\,(A) \leq 1$, and $\bar{\gamma}_\Delta\,(A) = 1 - \gamma_\Delta(A)$, $\forall A, B \in S_r$, $\forall \Delta \in \mathrm{Par}\,(S)$ (see [14])). Therefore, $\Delta' \preceq \Delta$. If we prove that $\exists a > 0$, $\exists u_0 \geq 1$ such that

$$\gamma_\Delta\left(P_{m,m+u_0}\right) \geq a, \quad \forall m \geq 0,$$

then $\Delta' = \Delta$. Since P is a Δ-mixing matrix, $\exists n_0 \geq 1$ such that $\gamma_\Delta\left(P^{n_0}\right) > 0$. Then

$$\lim_{m\to\infty} \gamma_\Delta\left(P_{m,m+n_0k}\right) = \gamma_\Delta\left(P^{n_0}\right) > 0.$$

Hence $\exists a > 0,\ \exists m_0 \geq 0$ such that

$$\gamma_\Delta\left(P_{m,m+n_0k}\right) \geq a, \quad \forall m \geq m_0.$$

Since $\bar{\gamma}_\Delta\,(A) = 1 - \gamma_\Delta(A)$, $\forall A \in S_r$, and $\bar{\gamma}_\Delta\,(AB) \leq \bar{\gamma}_\Delta\,(A)\,\bar{\gamma}_\Delta\,(B)$, $\forall A \in G_\Delta$, $\forall B \in S_r$, where $\Delta \in \mathrm{Par}\,(S)$ (see [15]), we obtain

$$\bar{\gamma}_\Delta\left(P_{m,m+m_0+n_0k}\right) \leq \bar{\gamma}_\Delta\left(P_{m,m+m_0}\right)\bar{\gamma}_\Delta\left(P_{m+m_0,m+m_0+n_0k}\right) \leq$$
$$\leq \bar{\gamma}_\Delta\left(P_{m+m_0,m+m_0+n_0k}\right) \leq 1 - a, \quad \forall m \geq 0.$$

This yields

$$\gamma_\Delta\left(P_{m,m+m_0+n_0k}\right) \geq a, \quad \forall m \geq 0,$$

i.e., $\exists a > 0$, $\exists u_0 \geq 1$ (e.g., $u_0 = m_0 + n_0 k$) such that

$$\gamma_\Delta \left(P_{m,m+u_0}\right) \geq a, \quad \forall m \geq 0.$$

Therefore, $\Delta' = \Delta$. It follows that the chain $(P_n)_{n\geq 1}$ is uniformly weakly Δ-ergodic. $\square$

Remark 4.13. The statement 'the chain $(P_n)_{n\geq 1}$ is uniformly weakly Δ-ergodic if and only if P is a Δ-mixing matrix' is false. Indeed, let (as in [19])

$$P_n = P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \quad \forall n \geq 1.$$

This chain is uniformly weakly $\big(\{1\},\{2\},\{3\}\big)$-ergodic, but P is not an $(\{1\},\{2\},\{3\})$-mixing matrix (moreover, here $\nexists \Delta \in \mathrm{Par}\left(\{1,2,3\}\right)$ such that P is a Δ-mixing matrix).

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